# Detailed proofs of paper [1] Slepian-Bangs formula and Cramér Rao bound for circular and non-circular complex elliptical symmetric distributions

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### I. USEFUL RELATIONS AND LEMMA

# A. Useful relations

We will make use of the following well known relations which hold for any conformable matrices A, B, C and D.

$$vec(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})vec(\mathbf{B}), \tag{1}$$

1

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD},\tag{2}$$

$$Tr(\mathbf{AB}) = vec^{H}(\mathbf{A}^{H})vec(\mathbf{B}), \tag{3}$$

$$Tr(\mathbf{ABCD}) = vec^{H}(\mathbf{A}^{H})(\mathbf{D}^{T} \otimes \mathbf{B})vec(\mathbf{C}), \tag{4}$$

$$Tr(\mathbf{A} \otimes \mathbf{B}) = Tr(\mathbf{A})Tr(\mathbf{B}),$$
 (5)

$$Tr[K(A \otimes B)] = Tr(AB), \tag{6}$$

where K is the vec-permutation matrix which transforms vec(C) to  $vec(C^T)$  for any square matrix C,

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{D}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{D}\mathbf{A}^{-1},$$
(7)

where A, C and  $C^{-1} + DA^{-1}B$  are assumed invertible.

# B. Useful lemma for the proof of Result 2

Lemma 1: Let  $\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2^* & \mathbf{A}_1^* \end{pmatrix}$  and  $\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_1^* \end{pmatrix}$  be two  $2M \times 2M$  partitioned matrices with  $\mathbf{A}_1$  and  $\mathbf{B}_1$  are  $M \times M$  Hermitian matrices,  $\mathbf{A}_2$  and  $\mathbf{B}_2$  are  $M \times M$  complex symmetric matrices, and suppose that  $\mathbf{y} \sim \mathbb{C}\mathcal{N}_M(\mathbf{0}, \mathbf{I})$ . Then

$$E[(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{A}} \widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{B}} \widetilde{\mathbf{y}})] = Tr(\widetilde{\mathbf{A}})Tr(\widetilde{\mathbf{B}}) + 2Tr(\widetilde{\mathbf{A}} \widetilde{\mathbf{B}}), \tag{8}$$

where  $\widetilde{\mathbf{y}} \stackrel{\text{def}}{=} (\mathbf{y}^T, \mathbf{y}^H)^T$ . *Proof:* 

We get from (4) then (2)

$$E[(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{A}} \widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{B}} \widetilde{\mathbf{y}})] = Tr[(\widetilde{\mathbf{A}}^T \otimes \widetilde{\mathbf{B}}) E(\widetilde{\mathbf{y}}^* \widetilde{\mathbf{y}}^T \otimes \widetilde{\mathbf{y}} \widetilde{\mathbf{y}}^H)], \tag{9}$$

where from e.g. [2, Appendix B]

$$E(\widetilde{\mathbf{y}}^*\widetilde{\mathbf{y}}^T \otimes \widetilde{\mathbf{y}}\widetilde{\mathbf{y}}^H) = \mathbf{I} \otimes \mathbf{I} + \mathbf{K}(\mathbf{J}' \otimes \mathbf{J}')(\mathbf{I} \otimes \mathbf{I}) + \text{vec}(\mathbf{I})\text{vec}^T(\mathbf{I}), \tag{10}$$

where  $\mathbf{J'} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$ . Plugging (10) in (9), we get:

$$E[(\widetilde{\mathbf{y}}^{H}\widetilde{\mathbf{A}}\widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^{H}\widetilde{\mathbf{B}}\widetilde{\mathbf{y}})] = Tr[(\widetilde{\mathbf{A}}^{T} \otimes \widetilde{\mathbf{B}})(\mathbf{I} \otimes \mathbf{I})] + Tr[(\widetilde{\mathbf{A}}^{T} \otimes \widetilde{\mathbf{B}})\mathbf{K}(\mathbf{J}' \otimes \mathbf{J}')(\mathbf{I} \otimes \mathbf{I})] + Tr[(\widetilde{\mathbf{A}}^{T} \otimes \widetilde{\mathbf{B}})\operatorname{vec}(\mathbf{I})\operatorname{vec}^{T}(\mathbf{I})],$$
(11)

where we have successively

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})(\mathbf{I}\otimes\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}})\mathrm{Tr}(\widetilde{\mathbf{B}})$$

from (2) and (5),

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathbf{K}(\mathbf{J}'\otimes\mathbf{J}')(\mathbf{I}\otimes\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}})$$

from (2), (6) and  $\mathbf{J}'\widetilde{\mathbf{A}}^T\mathbf{J}'=\widetilde{\mathbf{A}}$ , and

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}})$$

from (4). Plugging these three expressions in (11), (8) follows.

# II. PROOF OF RESULT 1 AND EQ. (5) OF [1]

Since a linear transform in  $\mathbb{R}^{2M}$  is tantamount to  $\mathbb{R}$ -linear transform in  $\mathbb{C}^M$ , the definition of GCES given in [3] is equivalent to saying that<sup>1</sup>

$$\mathbf{z} = \boldsymbol{\mu} + \boldsymbol{\Psi} \mathbf{z}_0 + \boldsymbol{\Phi} \mathbf{z}_0^*, \tag{12}$$

where  $\Psi$  and  $\Phi$  are  $M \times M$  fixed complex-valued matrices and  $\mathbf{z}_0$  is a complex spherical distributed r.v. with stochastic representation  $\mathbf{z}_0 =_d \mathcal{R}\mathbf{u}$  [4, th. 3]. Since  $\mathrm{E}(\mathbf{u}\mathbf{u}^H) = \frac{1}{M}\mathbf{I}$  and  $\mathrm{E}(\mathbf{u}\mathbf{u}^T) = \mathbf{0}$  [4, lemma 1b], we get if  $\mathrm{E}(\mathcal{R}^2) < \infty$ ,

$$\Sigma = \mathbf{A}\mathbf{A}^{H} = \frac{\mathrm{E}(\mathcal{R}^{2})}{N\sigma_{c}} \left( \mathbf{\Psi}\mathbf{\Psi}^{H} + \mathbf{\Phi}\mathbf{\Phi}^{H} \right) \text{ and } \mathbf{\Omega} = \mathbf{A}\mathbf{\Delta}_{\kappa}\mathbf{A}^{T} = \frac{\mathrm{E}(\mathcal{R}^{2})}{N\sigma_{c}} \left( \mathbf{\Psi}\mathbf{\Phi}^{T} + \mathbf{\Psi}\mathbf{\Phi}^{T} \right),$$
(13)

where  $\sigma_c$  is defined by  $\mathrm{E}[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^H] = \sigma_c \boldsymbol{\Sigma}$  and  $\mathrm{E}[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^T] = \sigma_c \boldsymbol{\Omega}$  whose value is  $\mathrm{E}(\mathcal{R}^2)/N$  [4, (14)]. Consequently (13) reduces to

$$\mathbf{A}\mathbf{A}^{H} = \mathbf{\Psi}\mathbf{\Psi}^{H} + \mathbf{\Phi}\mathbf{\Phi}^{H} \text{ and } \mathbf{A}\mathbf{\Delta}_{\kappa}\mathbf{A}^{T} = \mathbf{\Psi}\mathbf{\Phi}^{T} + \mathbf{\Psi}\mathbf{\Phi}^{T}. \tag{14}$$

By the one to one change of variable (because **A** is nonsingular):  $\Psi' = \mathbf{A}\Psi$  and  $\Phi' = \mathbf{A}\Phi$ , (14) is equivalent to:

$$\mathbf{I} = \mathbf{\Psi}' \mathbf{\Psi}'^{H} + \mathbf{\Phi} \mathbf{\Phi}'^{H} \text{ and } \mathbf{\Delta}_{\kappa} = \mathbf{\Psi}' \mathbf{\Phi}'^{T} + \mathbf{\Psi}' \mathbf{\Phi}'^{T}.$$
 (15)

It is clear that the solution of (15) is not unique, but we can look for solutions in real-valued diagonal form  $(\Psi, \Phi) = (\Delta_1, \Delta_2)$  with

$$\mathbf{I} = \boldsymbol{\Delta}_1^2 + \boldsymbol{\Delta}_2^2 \text{ and } \boldsymbol{\Delta}_{\kappa} = 2\boldsymbol{\Delta}_1\boldsymbol{\Delta}_2,$$
 (16)

whose solutions are  $\Delta_1 = \frac{\Delta_+ + \Delta_-}{2}$  and  $\Delta_2 = \frac{\Delta_+ - \Delta_-}{2}$  where  $\Delta_+ \stackrel{\text{def}}{=} \sqrt{\mathbf{I} + \Delta_\kappa}$  and  $\Delta_- \stackrel{\text{def}}{=} \sqrt{\mathbf{I} - \Delta_\kappa}$ . Consequently

$$\mathbf{z} =_{d} \boldsymbol{\mu} + \mathcal{R}[\boldsymbol{\Psi}\mathbf{u} + \boldsymbol{\Phi}\mathbf{u}^{*}] = \boldsymbol{\mu} + \mathcal{R}\mathbf{A}[\boldsymbol{\Delta}_{1}\mathbf{u} + \boldsymbol{\Delta}_{2}\mathbf{u}^{*}]. \tag{17}$$

If  $E(\mathcal{R}^2)$  is not finite, the scatter and pseudo-scatter matrices of  $\mathbf{z}$  given by (17) are also  $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^H$  and  $\mathbf{\Omega} = \mathbf{A}\mathbf{\Delta}_{\kappa}\mathbf{A}^T$ , respectively.

From the eigenvalue decomposition 
$$\begin{pmatrix} \mathbf{I} & \boldsymbol{\Delta}_{\kappa} \\ \boldsymbol{\Delta}_{\kappa} & \mathbf{I} \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix} \end{bmatrix} \begin{pmatrix} \mathbf{I} + \boldsymbol{\Delta}_{\kappa} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + \boldsymbol{\Delta}_{\kappa} \end{pmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix} \end{bmatrix}$$
, we deduce from  $\widetilde{\Gamma} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^* \end{pmatrix} \begin{pmatrix} \mathbf{I} & \boldsymbol{\Delta}_{\kappa} \\ \boldsymbol{\Delta}_{\kappa} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}^H & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T \end{pmatrix}$  that  $\widetilde{\Gamma}^{1/2} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^* \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta}_1 & \boldsymbol{\Delta}_2 \\ \boldsymbol{\Delta}_2 & \boldsymbol{\Delta}_1 \end{pmatrix}$ . Consequently,

<sup>&</sup>lt;sup>1</sup>Note that if  $\Phi = 0$ , **z** is C-CES distributed.

the stochastic representation  $\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R} \mathbf{A} \mathbf{v}$  is equivalent to

$$\widetilde{\mathbf{z}} =_d \widetilde{\boldsymbol{\mu}} + \mathcal{R} \widetilde{\boldsymbol{\Gamma}}^{1/2} \widetilde{\mathbf{u}} \tag{18}$$

with  $\widetilde{\mathbf{u}} \stackrel{\text{def}}{=} (\mathbf{u}^T, \mathbf{u}^H)^T$ . It follows directly  $\frac{1}{2} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\mathbf{\Gamma}}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2} \mathcal{R}^2 ||\widetilde{\mathbf{u}}||^2 = \mathcal{Q}$ .

### III. PROOF OF RESULT 2

To prove this result, we follows the different steps of [5, sec. 3]. First, we cheek that the p.d.f.  $p(\mathbf{z}; \boldsymbol{\alpha})$  satisfies the "regularity" condition

$$E\left(\frac{\partial \log p(\mathbf{z}; \boldsymbol{\alpha})}{\partial \alpha_k}\right) = 0. \tag{19}$$

Taking the derivative of the p.d.f. [1, (1)] w.r.t.  $\alpha_k$ , yields

$$\frac{\partial \log p(\mathbf{z}; \boldsymbol{\alpha})}{\partial \alpha_k} = -\frac{1}{2} \text{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1} \widetilde{\boldsymbol{\Gamma}}_k) + \phi(\widetilde{\boldsymbol{\eta}}) \frac{\partial \widetilde{\boldsymbol{\eta}}}{\partial \alpha_k}.$$
 (20)

It follows from the definition of  $\tilde{\eta}$  that

$$\frac{\partial \tilde{\eta}}{\partial \alpha_k} = -\text{Re}\left(\tilde{\boldsymbol{\mu}}_k^H \widetilde{\boldsymbol{\Gamma}}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})\right) - \frac{1}{2} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\boldsymbol{\Gamma}}^{-1} \widetilde{\boldsymbol{\Gamma}}_k \widetilde{\boldsymbol{\Gamma}}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}), \tag{21}$$

where  $\widetilde{\mu}_k \stackrel{\text{def}}{=} \frac{\partial \widetilde{\mu}}{\partial \alpha_k}$  and  $\widetilde{\Gamma}_k \stackrel{\text{def}}{=} \frac{\partial \widetilde{\Gamma}}{\partial \alpha_k}$ . Making use of the extended stochastic representation (18), the second term of (21) is given by

$$\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \tilde{\boldsymbol{\Gamma}}^{-1} \tilde{\boldsymbol{\Gamma}}_k \tilde{\boldsymbol{\Gamma}}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2} \mathcal{Q} \tilde{\mathbf{u}}^H \tilde{\mathbf{H}}_k \tilde{\mathbf{u}}$$
(22)

where  $\widetilde{\mathbf{H}}_k \stackrel{\mathrm{def}}{=} \widetilde{\mathbf{\Gamma}}^{-1/2} \widetilde{\mathbf{\Gamma}}_k \widetilde{\mathbf{\Gamma}}^{-1/2}$ . Thus using  $\widetilde{\eta} =_d \mathcal{Q}$  [1, (5)], we get:

$$E\left(\phi(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\right) = -E\left(\mathcal{Q}^{1/2}\phi(\mathcal{Q})\operatorname{Re}(\tilde{\boldsymbol{\mu}}_k^H \tilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}})\right) - \frac{1}{2}E[\mathcal{Q}\phi(\mathcal{Q})\tilde{\mathbf{u}}^H \tilde{\mathbf{H}}_k \tilde{\mathbf{u}}]. \tag{23}$$

Since  $\mathcal{Q}$  and  $\mathbf{u}$  are independent,  $\mathcal{Q}$  and  $\tilde{\mathbf{u}}$  are also independent. It follows then from  $\mathrm{E}(\tilde{\mathbf{u}}) = \mathbf{0}$ ,  $\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H) = \frac{1}{M}\mathbf{I}$  and  $\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q})) = -M$  [5, (11)] that

$$\mathrm{E}\left(\mathcal{Q}^{1/2}\phi(\mathcal{Q})\mathrm{Re}(\widetilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\mathbf{u}})\right)=0$$

and

$$E[\mathcal{Q}\phi(\mathcal{Q})\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}] = E[\mathcal{Q}\phi(\mathcal{Q})]\operatorname{Tr}[\widetilde{\mathbf{H}}_kE(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H)] = -\operatorname{Tr}(\widetilde{\mathbf{H}}_k) = -\operatorname{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k).$$

Thus

$$E\left(\phi(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\right) = \frac{1}{2}\text{Tr}(\tilde{\Gamma}^{-1}\tilde{\Gamma}_k),\tag{24}$$

which proves (19).

Now, we evaluate the elements of the FIM. It follows from (20), using (24), that

$$[\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}]_{k,l} = \mathrm{E}\left(\frac{\partial \log p(\mathbf{z}; \boldsymbol{\alpha})}{\partial \alpha_k} \frac{\partial \log p(\mathbf{z}; \boldsymbol{\alpha})}{\partial \alpha_l}\right) = -\frac{1}{4} \mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1} \widetilde{\boldsymbol{\Gamma}}_k) \mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1} \widetilde{\boldsymbol{\Gamma}}_l) + \mathrm{E}\left(\phi^2(\widetilde{\eta}) \frac{\partial \widetilde{\eta}}{\partial \alpha_k} \frac{\partial \widetilde{\eta}}{\partial \alpha_l}\right). \tag{25}$$

It follows from (18) that  $\widetilde{\Gamma}^{-1/2}(\widetilde{\mathbf{z}}-\widetilde{\boldsymbol{\mu}})=_d\sqrt{\mathcal{Q}}\ \widetilde{\mathbf{u}}$  and hence from (21) we get

$$\phi^{2}(\tilde{\eta})\frac{\partial\tilde{\eta}}{\partial\alpha_{k}}\frac{\partial\tilde{\eta}}{\partial\alpha_{l}} =_{d} \mathcal{Q}\phi^{2}(\mathcal{Q})\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{l}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)$$

$$+ \frac{1}{2}\mathcal{Q}^{3/2}\phi^{2}(\mathcal{Q})\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{l}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\left[\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{k}\tilde{\mathbf{u}}\right] + \frac{1}{2}\mathcal{Q}^{3/2}\phi^{2}(\mathcal{Q})\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\left[\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{l}\tilde{\mathbf{u}}\right]$$

$$+ \frac{1}{4}\mathcal{Q}^{2}\phi^{2}(\mathcal{Q})\left[\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{k}\tilde{\mathbf{u}}\right]\left[\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{l}\tilde{\mathbf{u}}\right]. \tag{26}$$

The first term of (26) can be further simplified as

$$\operatorname{Re}\left(\widetilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\mathbf{u}}\right)\operatorname{Re}\left(\widetilde{\boldsymbol{\mu}}_{l}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\mathbf{u}}\right) = \frac{1}{2}\operatorname{Re}\left(\widetilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\mathbf{u}}\widetilde{\mathbf{u}}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\boldsymbol{\mu}}_{l}\right) + \frac{1}{2}\operatorname{Re}\left(\widetilde{\boldsymbol{\mu}}_{k}^{T}\widetilde{\boldsymbol{\Gamma}}^{-*1/2}\widetilde{\mathbf{u}}^{*}\widetilde{\mathbf{u}}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\boldsymbol{\mu}}_{l}\right),$$

and thanks to the independence between Q and  $\tilde{\mathbf{u}}$ , the expected value of the first term of (26) is given by

$$E[\mathcal{Q}\phi^{2}(\mathcal{Q})]E\left(\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{l}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\right) = \frac{E[\mathcal{Q}\phi^{2}(\mathcal{Q})]}{2M}\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\boldsymbol{\mu}}_{l}\right) + \frac{E[\mathcal{Q}\phi^{2}(\mathcal{Q})]}{2M}\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{T}\widetilde{\boldsymbol{\Gamma}}^{-*}\mathbf{J}'\tilde{\boldsymbol{\mu}}_{l}\right) = \frac{E[\mathcal{Q}\phi^{2}(\mathcal{Q})]}{M}\operatorname{Re}\left(\tilde{\boldsymbol{\mu}}_{k}^{H}\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\boldsymbol{\mu}}_{l}\right), (27)$$

using  $E(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H) = \frac{1}{M}\mathbf{I}$  and  $E(\tilde{\mathbf{u}}^*\tilde{\mathbf{u}}^H) = \frac{1}{M}\mathbf{J}$ ,  $\widetilde{\mathbf{\Gamma}}^{-*1/2}\mathbf{J}'\widetilde{\mathbf{\Gamma}}^{-1/2} = \widetilde{\mathbf{\Gamma}}^{-*}\mathbf{J}'$  and  $\mathbf{J}'\tilde{\boldsymbol{\mu}}_l = \tilde{\boldsymbol{\mu}}_l^*$ . The expected value of the second and third terms of (26) are zero because the third-order moments of  $\mathbf{u}$  are zero. Because  $\mathbf{y} =_d \|\mathbf{y}\|\mathbf{u}$ , where  $\|\mathbf{y}\|$  and  $\mathbf{u}$  are independent when  $\mathbf{y} \sim \mathbb{C}\mathcal{N}_M(\mathbf{0},\mathbf{I})$ , we get

$$E[(\tilde{\mathbf{u}}^H \widetilde{\mathbf{H}}_k \tilde{\mathbf{u}})(\tilde{\mathbf{u}}^H \widetilde{\mathbf{H}}_l \tilde{\mathbf{u}})] = \frac{1}{E(\|\mathbf{y}\|^4)} E[(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{H}}_k \widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{H}}_l \widetilde{\mathbf{y}})].$$

Noting that  $\widetilde{\mathbf{H}}_k$  and  $\widetilde{\mathbf{H}}_l$  are structured as  $\widetilde{\mathbf{A}}$  and  $\widetilde{\mathbf{B}}$  of the Lemma 1, this lemma applies to the couples  $(\widetilde{\mathbf{H}}_k, \widetilde{\mathbf{H}}_l)$  and  $(\mathbf{I}, \mathbf{I})$  giving  $\mathrm{E}[(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{H}}_k \widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H \widetilde{\mathbf{H}}_l \widetilde{\mathbf{y}})] = \mathrm{Tr}(\widetilde{\mathbf{H}}_k)\mathrm{Tr}(\widetilde{\mathbf{H}}_l) + 2\mathrm{Tr}(\widetilde{\mathbf{H}}_k \widetilde{\mathbf{H}}_l)$  and  $\mathrm{E}[\|\widetilde{\mathbf{y}}\|^4] = 4M(M+1)$ . Consequently the expected value of the last term of (26) is given by

$$\mathbb{E}\left(\frac{1}{4}\mathcal{Q}^{2}\phi^{2}(\mathcal{Q})[\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{k}\tilde{\mathbf{u}}][\tilde{\mathbf{u}}^{H}\widetilde{\mathbf{H}}_{l}\tilde{\mathbf{u}}]\right) = \frac{\mathbb{E}(\mathcal{Q}^{2}\phi^{2}(\mathcal{Q}))}{4M(M+1)}\left(\operatorname{Tr}(\widetilde{\mathbf{H}}_{k})\operatorname{Tr}(\widetilde{\mathbf{H}}_{l}) + 2\operatorname{Tr}(\widetilde{\mathbf{H}}_{k}\widetilde{\mathbf{H}}_{l})\right) \\
= \frac{\mathbb{E}(\mathcal{Q}^{2}\phi^{2}(\mathcal{Q}))}{4M(M+1)}\left(\operatorname{Tr}(\widetilde{\Gamma}_{k}\widetilde{\Gamma}^{-1})\operatorname{Tr}(\widetilde{\Gamma}_{l}\widetilde{\Gamma}^{-1}) + 2\operatorname{Tr}(\widetilde{\Gamma}_{k}\widetilde{\Gamma}^{-1}\widetilde{\Gamma}_{l}\widetilde{\Gamma}^{-1})\right) (28)$$

Gathering (27) (28) in (25) concludes the proof.

# IV. PROOF OF Eq. (9) OF [1]

Using that [1, (4)] is a p.d.f. with  $\int_0^\infty \delta_{M,g}^{-1} \mathcal{Q}_t^{M-1} g(\mathcal{Q}_t) d\mathcal{Q}_t = 1$  and that  $E(\mathcal{Q}) = E(\mathcal{R}^2) < \infty$ , we get

$$E(\mathcal{Q}\phi(\mathcal{Q})) = \int_0^\infty \delta_{M,g}^{-1} \mathcal{Q}^M g'(\mathcal{Q}) d\mathcal{Q} = \left[\delta_{M,g}^{-1} \mathcal{Q}^M g(\mathcal{Q})\right]_0^\infty - M \int_0^\infty \delta_{M,g}^{-1} \mathcal{Q}^{M-1} g(\mathcal{Q}) d\mathcal{Q} = -M. \tag{29}$$

It follows from Cauchy-Schwarz inequality that

$$M^{2} = (\mathcal{E}(\mathcal{Q}\phi(\mathcal{Q})))^{2} \le \mathcal{E}(\mathcal{Q})\mathcal{E}(\mathcal{Q}\phi^{2}(\mathcal{Q})) = \mathcal{E}(\mathcal{Q})M\xi_{1}.$$
(30)

Next, note that

$$E(Q) = \int_0^\infty \delta_{M,g}^{-1} Q^M g(Q) dQ = \delta_{M,g}^{-1} \delta_{M+1,g} \int_0^\infty \delta_{M+1,g}^{-1} Q^M g(Q) dQ = \delta_{M,g}^{-1} \delta_{M+1,g} = M.$$
 (31)

Plugging (31) in (30) proves Eq. (9) of [1].

# V. Proof of Result 4

Because  $\xi_2 = 1$  for Gaussian distributions, we get for NC-CES distributions:

$$\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}(\boldsymbol{\alpha}_{2}) - \mathbf{I}_{\mathrm{CN}}^{\mathrm{NC}}(\boldsymbol{\alpha}_{2}) = \frac{\xi_{2} - 1}{2} \left( \frac{d \mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d \boldsymbol{\alpha}_{2}^{T}} \right)^{H} \left( (\widetilde{\boldsymbol{\Gamma}}^{-T} \otimes \widetilde{\boldsymbol{\Gamma}}^{-1}) + \frac{1}{2} \mathrm{vec}(\widetilde{\boldsymbol{\Gamma}}^{-1}) \mathrm{vec}^{H}(\widetilde{\boldsymbol{\Gamma}}^{-1}) \right) \frac{d \mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d \boldsymbol{\alpha}_{2}^{T}}$$
(32)

where  $(\widetilde{\Gamma}^{-T} \otimes \widetilde{\Gamma}^{-1}) + \frac{1}{2} \text{vec}(\widetilde{\Gamma}^{-1}) \text{vec}^H(\widetilde{\Gamma}^{-1})$  is positive definite. Replacing  $\widetilde{\Gamma}$  by  $\Gamma$ , the proof is identical for C-CES distributions.

## VI. PROOF OF RESULT 5

We note first that the general expressions of the SCRB proved here is valid for arbitrary parameterization of  $\mathbf{A}_{\theta}$  if the real-valued parameter of interest  $\boldsymbol{\theta} \in \mathbb{R}^L$  is characterized by the subspace generated by the columns of the full column rank  $M \times K$  matrix  $\mathbf{A}_{\theta}$  with K < M. It can be applied for example to near or far-field DOA modeling with scalar or vector-sensors for an arbitrary number of parameters per source  $s_{t,k}$  (with  $\mathbf{s}_t \stackrel{\text{def}}{=} (s_{t,1},...,s_{t,K})^T$  and many other modelings as the SIMO and MIMO modelings. Let us start with the circular case for which  $\mathbf{\Omega} = \mathbf{0}$  and thus  $\widetilde{\mathbf{\Gamma}} = \text{Diag}(\mathbf{\Sigma}, \mathbf{\Sigma}^*)$  where  $\mathbf{\Sigma} = \mathbf{A}_{\theta} \mathbf{R}_s \mathbf{A}_{\theta}^H + \sigma_n^2 \mathbf{I}$ . The SCRB form for this case can be then written through the compact expression of the general FIM given in Result 2, using (1) and (2), as follows:

$$\frac{1}{T} SCRB_{CES}^{-1}(\boldsymbol{\alpha}) = \left(\frac{d \text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^H \left(\xi_2(\boldsymbol{\Sigma}^{-T} \otimes \boldsymbol{\Sigma}^{-1}) + (\xi_2 - 1)\text{vec}(\boldsymbol{\Sigma}^{-1})\text{vec}^H(\boldsymbol{\Sigma}^{-1})\right) \left(\frac{d \text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right). \tag{33}$$

The SCRB of  $\theta$  alone can be deduced from (33) as follows:

$$\frac{1}{T}SCRB_{CES}^{-1}(\boldsymbol{\theta}) = \mathbf{G}^{H}\mathbf{\Pi}_{\Delta}^{\perp}\mathbf{G},$$
(34)

with  $\mathbf{G} \stackrel{\mathrm{def}}{=} \mathbf{T}_i^{1/2} (\mathbf{\Sigma}^{-T/2} \otimes \mathbf{\Sigma}^{-1/2}) \frac{\partial \mathrm{vec}(\mathbf{\Sigma})}{\partial \boldsymbol{\theta}^T}$  and  $\mathbf{\Delta} \stackrel{\mathrm{def}}{=} \mathbf{T}_i^{1/2} (\mathbf{\Sigma}^{-T/2} \otimes \mathbf{\Sigma}^{-1/2}) \frac{\partial \mathrm{vec}(\mathbf{\Sigma})}{\partial \boldsymbol{\alpha}_n^T}$  where

$$\mathbf{T}_{i} \stackrel{\text{def}}{=} \xi_{2} \mathbf{I} + (\xi_{2} - 1) \text{vec}(\mathbf{I}) \text{vec}^{T}(\mathbf{I}). \tag{35}$$

Let's further partition the matrix  $\Delta$  as  $\Delta = \mathbf{T}_i^{1/2}(\mathbf{\Sigma}^{-T/2}\otimes\mathbf{\Sigma}^{-1/2})\left[\frac{\partial\mathrm{vec}(\mathbf{\Sigma})}{\partial\rho^T}\mid\frac{\partial\mathrm{vec}(\mathbf{\Sigma})}{\partial\sigma_n^2}\right]\stackrel{\mathrm{def}}{=} [\mathbf{V}\mid\mathbf{u}_n]$ . In the sequel, the proofs presented here follow the lines of the proof presented in [6] for circular Gaussian distributed observations. It follows from [6, rel. (14)] that

$$\Pi_{\Delta}^{\perp} = \Pi_{\mathbf{V}}^{\perp} - \frac{\Pi_{\mathbf{V}}^{\perp} \mathbf{u}_{n} \mathbf{u}_{n}^{H} \Pi_{\mathbf{V}}^{\perp}}{\mathbf{u}_{n}^{H} \Pi_{\mathbf{V}}^{\perp} \mathbf{u}_{n}}.$$
(36)

Using  $\frac{\partial \text{vec}(\mathbf{\Sigma})}{\partial \sigma_n^2} = \text{vec}(\mathbf{I})$ , we obtain

$$\mathbf{u}_n = \mathbf{T}_i^{1/2} \text{vec}(\mathbf{\Sigma}^{-1}). \tag{37}$$

Consequently using (34) and (36), if  $\mathbf{g}_k$  denotes the *kth* column of  $\mathbf{G}$ , the (k,l) element of  $\mathrm{SCRB}^{-1}_{\mathrm{CES}}(\boldsymbol{\alpha})$  can be written elementwise as

$$\frac{1}{T} \left[ SCRB_{CES}^{-1}(\boldsymbol{\theta}) \right]_{k,l} = \mathbf{g}_k^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_l - \frac{\mathbf{g}_k^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u}_n \mathbf{u}_n^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_l}{\mathbf{u}_n^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u}_n}.$$
(38)

Let us proceed now to determine the expression of  $\mathbf{g}_k$ . Letting  $\mathbf{A}'_{\theta_k} \stackrel{\text{def}}{=} \frac{\partial \mathbf{A}_{\theta}}{\partial \theta_k}$ , we get

$$\frac{\partial \mathbf{\Sigma}}{\partial \theta_k} = \mathbf{A}'_{\theta_k} \mathbf{R}_s \mathbf{A}_{\theta}^H + \mathbf{A}_{\theta} \mathbf{R}_s \mathbf{A}_{\theta_k}^{'H}, \tag{39}$$

Hence, using (1), the kth column of G in (38) is given by

$$\mathbf{g}_k = \mathbf{T}_i^{1/2} \operatorname{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H) \text{ where } \mathbf{Z}_k \stackrel{\text{def}}{=} \mathbf{\Sigma}^{-1/2} \mathbf{A}_{\theta} \mathbf{R}_s \mathbf{A}_{\theta_k}^{'H} \mathbf{\Sigma}^{-1/2}.$$
 (40)

Next, we determine V and then  $\Pi_{\mathbf{V}}^{\perp}$ . Since  $\mathbf{R}_s$  is a Hermitian matrix, it can be then factorized as

$$vec(\mathbf{R}_s) = \mathbf{J}\boldsymbol{\rho} \tag{41}$$

where  $\bf J$  is a  $K^2 \times K^2$  constant nonsingular matrix. It follows, using (1), that  $\bf V$  can be be expressed as

$$\mathbf{V} = \mathbf{T}_i^{1/2} (\mathbf{\Sigma}^{-T/2} \mathbf{A}_{\theta}^* \otimes \mathbf{\Sigma}^{-1/2} \mathbf{A}_{\theta}) \mathbf{J} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2} \mathbf{W} \mathbf{J}.$$

Note from (38) that the SCRB depends on V only via  $\Pi_{\mathbf{V}}^{\perp}$ , that can be expressed as

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{V}(\mathbf{V}^{H}\mathbf{V})^{-1}\mathbf{V}^{H} = \mathbf{I} - \mathbf{T}_{i}^{1/2}\mathbf{W}(\mathbf{W}^{H}\mathbf{T}_{i}\mathbf{W})^{-1}\mathbf{W}^{H}\mathbf{T}_{i}^{1/2}.$$
(42)

After some algebraic manducation, using (1) and (2), we obtain

$$\mathbf{W}^{H}\mathbf{T}_{i}\mathbf{W} = \xi_{2}(\mathbf{U}^{*} \otimes \mathbf{U}) + (\xi_{2} - 1)\operatorname{vec}(\mathbf{U})\operatorname{vec}^{H}(\mathbf{U}),$$

where  $\mathbf{U} \stackrel{\text{def}}{=} \mathbf{A}_{\theta}^H \mathbf{\Sigma}^{-1} \mathbf{A}_{\theta}$  is a  $K \times K$  Hermitian nonsingular matrix. It follows from matrix inverse lemma (given by (7)), that its inverse can be expressed as

$$(\mathbf{W}^H \mathbf{T}_i \mathbf{W})^{-1} = \frac{1}{\xi_2} (\mathbf{U}^{-*} \otimes \mathbf{U}^{-1}) - \eta \text{vec}(\mathbf{U}^{-1}) \text{vec}^H(\mathbf{U}^{-1})$$

where  $\eta \stackrel{\text{def}}{=} \frac{\xi_2 - 1}{\xi_2^2 (1 + \frac{\xi_2 - 1}{\xi_2} \text{vec}^H(\tilde{\mathbf{U}})(\tilde{\mathbf{U}}^{-*} \otimes \tilde{\mathbf{U}}^{-1}) \text{vec}(\tilde{\mathbf{U}}))}$  can be simplified, using (4), as  $\eta \stackrel{\text{def}}{=} \frac{\xi_2 - 1}{\xi_2^2 (1 + \frac{\xi_2 - 1}{\xi_2} K)}$ . Thus, using (1) and (2), we obtain

$$\mathbf{W}(\mathbf{W}^H \mathbf{T}_i \mathbf{W})^{-1} \mathbf{W}^H = \frac{1}{\xi_2} (\mathbf{H}_1^* \otimes \mathbf{H}_1) - \eta \operatorname{vec}(\mathbf{H}_1) \operatorname{vec}^H(\mathbf{H}_1) \stackrel{\text{def}}{=} \mathcal{B}, \tag{43}$$

where  $\mathbf{H}_1 \stackrel{\text{def}}{=} \mathbf{\Sigma}^{-1/2} \mathbf{A}_{\theta} \mathbf{U}^{-1} \mathbf{A}_{\theta}^H \mathbf{\Sigma}^{-1/2}$ . Therefore, (42) becomes

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{T}_i^{1/2} \mathcal{B} \mathbf{T}_i^{1/2}.\tag{44}$$

Now let us show that  $\mathbf{u}_n^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_k = 0$ . It follows from (37) and (40), using (44), that

$$\mathbf{u}_{n}^{H} \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_{k} = \operatorname{vec}^{H}(\mathbf{\Sigma}^{-1}) \mathbf{T}_{i} \operatorname{vec}(\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H}) - \operatorname{vec}^{H}(\mathbf{\Sigma}^{-1}) \mathbf{T}_{i} \mathcal{B} \mathbf{T}_{i} \operatorname{vec}(\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H}).$$
(45)

It follows, after some algebraic manipulation, using (1), (3) and (43) that

$$\mathbf{T}_{i}\mathcal{B}\mathbf{T}_{i} = \xi_{2}(\mathbf{H}_{1}^{*}\otimes\mathbf{H}_{1}) - \xi_{2}^{2}\eta\operatorname{vec}(\mathbf{H}_{1})\operatorname{vec}^{H}(\mathbf{H}_{1}) + (\xi_{2} - 1)(1 - K\eta\xi_{2})\left(\operatorname{vec}(\mathbf{I})\operatorname{vec}^{H}(\mathbf{H}_{1}) + \operatorname{vec}(\mathbf{H}_{1})\operatorname{vec}^{T}(\mathbf{I})\right) + \frac{(\xi_{2} - 1)^{2}K}{\xi_{2}}(1 - K\eta\xi_{2})\operatorname{vec}(\mathbf{I})\operatorname{vec}^{T}(\mathbf{I}),$$

$$(46)$$

using  $\mathbf{H}_1^2 = \mathbf{H}_1$  and  $\text{Tr}(\mathbf{H}_1) = K$ . Using the definition (35) for  $\mathbf{T}_i$  and (3), the first term of (45) can be expressed as

$$\operatorname{vec}^{H}(\mathbf{\Sigma}^{-1})\mathbf{T}_{i}\operatorname{vec}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H}) = \xi_{2}\operatorname{Tr}(\mathbf{\Sigma}^{-1}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H})) + (\xi_{2}-1)\operatorname{Tr}(\mathbf{\Sigma}^{-1})\operatorname{Tr}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H})$$

$$= 2\xi_{2}\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta}^{'H})) + 2(\xi_{2}-1)\operatorname{Tr}(\mathbf{\Sigma}^{-1})\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta}^{'H})) (47)$$

using  $\operatorname{Tr}(\mathbf{\Sigma}^{-1}(\mathbf{Z}_k + \mathbf{Z}_k^H)) = 2\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_{\theta}\mathbf{R}_s\mathbf{A}_{\theta_k}^{'H}))$  and  $\operatorname{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H) = 2\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_s\mathbf{A}_{\theta_k}^{'H}))$ . After simple algebraic manipulations, using (46), (1) and (3), and that  $\operatorname{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H) = \operatorname{Tr}((\mathbf{Z}_k + \mathbf{Z}_k^H)\mathbf{H}_1) = \operatorname{Tr}(\mathbf{H}_1(\mathbf{Z}_k + \mathbf{Z}_k^H)\mathbf{H}_1) = 2\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_s\mathbf{A}_{\theta_k}^{'H}))$  and  $\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_1^2) = \operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_1)$ , the second term of (45) can be simplified as

$$\operatorname{vec}^{H}(\mathbf{\Sigma}^{-1})\mathbf{T}_{i}\mathcal{B}\mathbf{T}_{i}\operatorname{vec}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H})$$

$$=\xi_{2}\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_{1}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H})\mathbf{H}_{1})+(\xi_{2}-1)\operatorname{Tr}(\mathbf{\Sigma}^{-1})\operatorname{Tr}(\mathbf{Z}_{k}+\mathbf{Z}_{k}^{H})$$

$$=2\xi_{2}\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{U}^{-1}\mathbf{A}_{\theta}^{H}\mathbf{\Sigma}^{-2}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))+2(\xi_{2}-1)\operatorname{Tr}(\mathbf{\Sigma}^{-1})\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))$$

$$=2\xi_{2}\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))+2(\xi_{2}-1)\operatorname{Tr}(\mathbf{\Sigma}^{-1})\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H})),$$

$$(48)$$

where the first term in the last line is obtained using  $\mathbf{A}_{\theta}\mathbf{U}^{-1}\mathbf{A}_{\theta}^{H}\mathbf{\Sigma}^{-2}\mathbf{A}_{\theta} = \mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}$ . It follows, therefore, from (45), (47) and (48) that

$$\mathbf{u}_n^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_k = 0.$$

This identity together with (40) and (44) allows us to rewrite the individual elements of (38) as

$$\frac{1}{T} \left[ SCRB_{CES}^{-1}(\boldsymbol{\theta}) \right]_{k,l} = \mathbf{g}_{k}^{H} \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_{l} 
= vec^{H} (\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H}) \mathbf{T}_{i} vec(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H}) - vec^{H} (\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H}) \mathbf{T}_{i} \mathcal{B} \mathbf{T}_{i} vec(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H}). (49)$$

After simple algebraic manipulations, using the definition (35) for  $T_i$ , (1) and (3), the first term in (49) can be simplified as

$$\operatorname{vec}^{H}(\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H})\mathbf{T}_{i}\operatorname{vec}(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H}) = \xi_{2}\operatorname{Tr}((\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H})(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H})) + (\xi_{2} - 1)\operatorname{Tr}(\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H})\operatorname{Tr}(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H})$$

$$= 2\xi_{2}\left[\operatorname{Re}(\operatorname{Tr}((\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{l}}^{'H})(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\right]$$

$$+ \operatorname{Re}(\operatorname{Tr}((\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}^{'}\mathbf{R}_{s}\mathbf{A}_{\theta})(\mathbf{\Sigma}^{-1}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\right]$$

$$+ 4(\xi_{2} - 1)\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\operatorname{Re}(\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))$$
(50)

Similarly, after some algebraic manipulations, using (46), (1) and (4), the second term in (49) can be simplified as

$$\operatorname{vec}^{H}(\mathbf{Z}_{k} + \mathbf{Z}_{k}^{H})\mathbf{T}_{i}\operatorname{vec}(\mathbf{Z}_{l} + \mathbf{Z}_{l}^{H}) = 2\xi_{2}\left[\operatorname{Tr}(\operatorname{Re}((\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{l}}^{'H})(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\right] + \operatorname{Tr}(\operatorname{Re}((\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{U}^{-1}\mathbf{A}^{H}\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta_{l}}^{'}\mathbf{R}_{s}\mathbf{A}_{\theta_{l}}^{H})(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\right] + 4(\xi_{2} - 1)\operatorname{Tr}(\operatorname{Re}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H}))\operatorname{Tr}(\operatorname{Re}(\mathbf{\Sigma}^{-1}\mathbf{A}_{\theta}\mathbf{R}_{s}\mathbf{A}_{\theta_{k}}^{'H})). (51)$$

It follows then from (50) and (51) that (49) can be simplified as

$$\frac{1}{T} \left[ SCRB_{CES}^{-1}(\boldsymbol{\theta}) \right]_{k,l} = 2\xi_{2} Re \left( Tr \left[ (\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \mathbf{A} \mathbf{U}^{-1} \mathbf{A}^{H} \boldsymbol{\Sigma}^{-1}) (\mathbf{A}_{\theta_{l}}^{'} \mathbf{R}_{s} \mathbf{A}_{\theta}^{H} \boldsymbol{\Sigma}^{-1} \mathbf{A}_{\theta} \mathbf{R}_{s} \mathbf{A}_{\theta_{k}}^{'H}) \right] \right)$$

$$= \frac{2\xi_{2}}{\sigma_{n}^{2}} Re \left( Tr \left[ (\boldsymbol{\Pi}_{\mathbf{A}_{\theta}}^{\perp}) (\mathbf{A}_{\theta_{l}}^{'} \mathbf{R}_{s} \mathbf{A}_{\theta}^{H} \boldsymbol{\Sigma}^{-1} \mathbf{A}_{\theta} \mathbf{R}_{s} \mathbf{A}_{\theta_{k}}^{'H}) \right] \right)$$

$$= \frac{2\xi_{2}}{\sigma_{n}^{2}} Re \left( Tr \left[ \boldsymbol{\Pi}_{\mathbf{A}_{\theta}}^{\perp} \mathbf{A}_{\theta_{l}}^{'} \mathbf{H} \mathbf{A}_{\theta_{k}}^{'H} \right] \right), \tag{52}$$

where the second equality is obtained using  $\Sigma^{-1} - \Sigma^{-1} \mathbf{A} \mathbf{U}^{-1} \mathbf{A}^H \Sigma^{-1} = \frac{1}{\sigma_n^2} \mathbf{\Pi}_{\mathbf{A}}^{\perp}$  thanks to  $\mathbf{A} \mathbf{U}^{-1} \mathbf{A}^H \Sigma^{-1} = \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$ . Using (4), we can write (52) in matrix form as is shown in Result 5.

In the noncircular case, the proof follows the similar above steps by replacing  $\mathbf{T}_i$  by  $\widetilde{\mathbf{T}}_i \stackrel{\text{def}}{=} \frac{\xi_2}{2}\mathbf{I} + \frac{\xi_2 - 1}{4} \text{vec}(\mathbf{I}) \text{vec}^T(\mathbf{I})$ , and  $\Sigma$  by  $\widetilde{\Gamma}$  where (39) is replaced by  $\frac{\partial \widetilde{\Gamma}}{\partial \theta_k} = \widetilde{\mathbf{A}}_{\theta_k}' \mathbf{R}_{\tilde{s}} \widetilde{\mathbf{A}}_{\theta}^H + \widetilde{\mathbf{A}}_{\theta} \mathbf{R}_{\tilde{s}} \widetilde{\mathbf{A}}_{\theta_k}^H$  with  $\widetilde{\mathbf{A}}_{\theta} \stackrel{\text{def}}{=} \frac{\partial \widetilde{\mathbf{A}}_{\theta}}{\partial \theta_k}$ .

# VII. PROOF OF RESULT 6

The proof of this result follows similar steps as the proof of Result 5 based on [7, th. 1] by replacing  $\Sigma$  by  $\widetilde{\Gamma} = \widetilde{\mathbf{A}}_{\omega} \mathbf{R}_r \widetilde{\mathbf{A}}_{\omega}^H + \sigma_n^2 \mathbf{I}$ ,  $\mathbf{A}_{\theta}$  by  $\widetilde{\mathbf{A}}_{\omega} = \begin{pmatrix} \mathbf{A}_{\theta} \mathbf{\Delta}_{\phi} \\ \mathbf{A}_{\theta}^* \mathbf{\Delta}_{\phi}^* \end{pmatrix}$  where  $\boldsymbol{\omega} \stackrel{\text{def}}{=} (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T$  with  $\boldsymbol{\phi} \stackrel{\text{def}}{=} (\phi_1, ..., \phi_K)^T$ , and also by pointing out that  $\mathbf{R}_r \in \mathbb{R}^{K \times K}$  is symmetric which lead us to replace  $\mathbf{J}$  in (41) by  $\mathbf{D}_{\rho}$  defined in [7, th. 1] to get  $\text{vec}(\mathbf{R}_r) = \mathbf{D}_{\rho} \boldsymbol{\rho}$ . Thus,  $\mathbf{V}$  becomes  $\mathbf{V} = \widetilde{\mathbf{T}}_i^{1/2} \mathbf{W} \mathbf{D}_{\rho}$  with  $\mathbf{W} = (\widetilde{\mathbf{\Gamma}}^{-T/2} \widetilde{\mathbf{A}}_{\omega}^* \otimes \widetilde{\mathbf{\Gamma}}^{-1/2} \widetilde{\mathbf{A}}_{\omega})$ . Hence  $\mathbf{\Pi}_{\mathbf{V}}^{\perp}$  in [7, th. 1] takes here the following key form expression:  $\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \widetilde{\mathbf{T}}_i^{1/2} \mathcal{B} \widetilde{\mathbf{T}}_i^{1/2}$  with  $\boldsymbol{\mathcal{B}} = \frac{2}{\xi_2} \mathbf{W} (\mathbf{U}^{-1} \otimes \mathbf{U}^{-1}) \mathbf{N}_K \mathbf{W}^H - \widetilde{\boldsymbol{\eta}} \text{vec}(\mathbf{H}_1) \text{vec}^H(\mathbf{H}_1)$  where  $\mathbf{U} \stackrel{\text{def}}{=} \widetilde{\mathbf{A}}_{\omega}^H \widetilde{\mathbf{\Gamma}}^{-1} \widetilde{\mathbf{A}}_{\omega}$ ,  $\mathbf{N}_K$  is defined in [7, th. 1] and  $\widetilde{\boldsymbol{\eta}} \stackrel{\text{def}}{=} \frac{\xi_2 - 1}{\xi_3^2 (1 + \frac{\xi_2 - 1}{2\varepsilon_2} K)}$ . The rest of the proof follows the same lines of arguments as that of the proof of Result 5.

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