

Direct Construction of Optimal Z-Complementary Code Sets for all Possible Even Length by Using Pseudo-Boolean Functions

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Abstract—Z-complementary code set (ZCCS) are well known to be used in multicarrier code-division multiple access (MC-CDMA) system to provide a interference free environment. Based on the existing literature, the direct construction of optimal ZCCSs are limited to its length. In this paper, we are interested in constructing optimal ZCCSs of all possible even lengths using Pseudo-Boolean functions. The maximum column sequence peak-to-man envelop power ratio (PMEPR) of the proposed ZCCSs is upper-bounded by two, which may give an extra benefit in managing PMEPR in an ZCCS based MC-CDMA system, as well as the ability to handle a large number of users.

Index Terms—Multicarrier code-division multiple access (MC-CDMA), generalized Boolean function (GBF), pseudo-Boolean function (PBF), Z-complementary code set (ZCCS), zero correlation zone (ZCZ), pick to mean average power ratio (PMPER).

I. INTRODUCTION

MULTICARRIER code-division multiple access (MC-CDMA) is a multiple access scheme used in orthogonal frequency division multiplexing (OFDM)-based telecommunication systems, allowing the system to support multiple users at the same time over the same frequency band. When the number of users in a channel increases, it is found that the performance of MC-CDMA degrades as a result of multi-user interference (MUI) and multipath interference (MPI). The Complete-complementary code (CCC) [1] has perfect cross- and auto-correlation characteristics, which allows for simultaneous interference-free transmission in the multi-carrier-digital mobile (MC-CDMA) system. A major disadvantage of CCC is that the number of supported users is limited by the number of row sequences in each complementary matrix. The set size of the ZCCS system is much bigger than that of the CCC system [2], which enables for a considerably greater number of users to be supported by a ZCCS-based MC-CDMA system, as opposed to a CCC-based MC-CDMA system, where the number of subcarriers is equal to the number of users.

In recent literature, research on generalized Boolean functions (GBFs) based constructions of complementary sequences has received great attention from the sequence design community, [2] [3], [4], [5], [6], [7], [8]. The GBFs based construction of CCCs were extended to optimal ZCCSs in [2] and [3]. However, GBFs based construction of optimal ZCCSs has a limitation on the sequence lengths which is in the form of power-of-two [2], [3], [7] and [9]. By extending the idea of GBFs to PBFs, recently, a direct construction of optimal ZCCSs has been introduced in [10] which is able to provide non-power-of-two length sequences but limited to the form $p2^m$,

where p is a prime number and m is a positive integer. Another direct construction like GBFs based constructions, PBFs based constructions are also known as direct constructions in the literature. Direct constructions are feasible for rapid hardware generation [2] of sequences. Besides direct constructions, many indirect constructions can be found in [11], [12], [13], [14] and [15] which are dependent on some kernel at its initial stages. The limitation on the lengths of optimal ZCCS through direct constructions in the existing literature motivates us in searching of PBFs to provide more flexibility on the lengths. In search of new ZCCS, in this paper, we have proposed a direct construction of optimal ZCCS for all possible even length using PBFs. We also have showed that, the proposed construction is able to maintain a minimum column sequence PMPER 2, unlike the existing direct construction of optimal ZCCSs of non-power-of-two lengths. The PBF reported in [10] appears as a special case of proposed construction.

The rest of this work is organised in the following way. In Section II, we will go over a few definitions. Section III offers a comprehensive description of the ZCCS's construction. Section IV of this article ends the study by comparing our findings to those of previous researchers.

II. PRELIMINARY

This section presents a few basic definitions and lemmas for use in the proposed construction. Let $\mathbf{x}_1 = [x_{1,0}, x_{1,1}, \dots, x_{1,N-1}]$ and $\mathbf{x}_2 = [x_{2,0}, x_{2,1}, \dots, x_{2,N-1}]$ be a pair of sequences whose components are complex numbers. Let τ be an integer, we define [2]

$$\Theta(\mathbf{x}_1, \mathbf{x}_2)(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} x_{1,i+\tau} x_{2,i}^*, & 0 \leq \tau < N, \\ \sum_{i=0}^{N+\tau-1} x_{1,i} x_{2,i-\tau}^*, & -N < \tau < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

and when $\mathbf{x}_1 = \mathbf{x}_2$, $\Theta(\mathbf{x}_1, \mathbf{x}_2)(\tau) = \mathcal{A}_{\mathbf{x}_1}(\tau)$. This functions Θ and \mathcal{A} are known as aperiodic cross-correlation function (ACCF) of \mathbf{x}_1 and \mathbf{x}_2 and aperiodic auto-correlation function (AACF) of \mathbf{x}_1 respectively and $*$ denotes the complex conjugate.

Let $\mathbf{B} = \{B^0, B^1, \dots, B^{M-1}\}$ be a collection of M matrices each of dimensions $K \times N$, i.e., $B^\delta = [\mathbf{b}_0^\delta, \mathbf{b}_1^\delta, \dots, \mathbf{b}_{K-1}^\delta]_{K \times N}^T$, where the notation T is used to denote the transpose of a matrix and each \mathbf{b}_i^δ is a complex-valued sequences of length N i.e., $\mathbf{b}_i^\delta = (\mathbf{b}_{i,0}^\delta, \mathbf{b}_{i,1}^\delta, \dots, \mathbf{b}_{i,N-1}^\delta)$. Suppose $B^{\delta_1}, B^{\delta_2} \in \mathbf{B}$, where $0 \leq \delta_1, \delta_2 \leq M-1$, we define the ACCF between B^{δ_1} and B^{δ_2} as, $\Theta(B^{\delta_1}, B^{\delta_2})(\tau) =$

$\sum_{i=0}^{K-1} \Theta(\mathbf{b}_i^{\delta_1}, \mathbf{b}_i^{\delta_2})(\tau)$. When the following equation holds, we refer to the set \mathbf{B} as ZCCS.

Definition 1: ([2]) Code set \mathbf{B} is called a ZCCS if

$$\Theta(B^{\delta_1}, B^{\delta_2})(\tau) = \begin{cases} KN, & \tau = 0, \delta_1 = \delta_2, \\ 0, & 0 < |\tau| < Z, \delta_1 = \delta_2, \\ 0, & |\tau| < Z, \delta_1 \neq \delta_2, \end{cases} \quad (2)$$

where Z denotes ZCZ width. With the parameter K, N, M and Z , we denote the set of matrices \mathbf{B} as a $(M, Z) - ZCCS_K^N$, which is called optimal if $M = K \lfloor \frac{N}{Z} \rfloor$ and non-optimal if $M < K \lfloor \frac{N}{Z} \rfloor$ [16]. When $K = M$ and $Z = N$, we denote \mathbf{B} by (K, K, N) -CCC.

Lemma 1: (Constrction of CCC [4])

Let $g : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ be a second-order GBF and \tilde{g} be the reversal of g i.e, $\tilde{g}(y_0, y_1, \dots, y_{m-1}) = g(1-y_0, 1-y_1, \dots, 1-y_{m-1})$. Assume that the graph $G(g)$ contains vertices denoted as $y_{\beta_0}, y_{\beta_1}, \dots, y_{\beta_{n-1}}$ such that, after executing a deletion operation on those vertices, the resultant graph is reduced to a path. We define the weight of each edge by $\frac{q}{2}$. Let the binary representation of the integer r is $\mathbf{r} = (r_0, r_1, \dots, r_{n-1})$. Define the G_r and \tilde{G}_r to be

$$\begin{cases} \left\{ g + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \mathbf{y} + v_n y_\gamma \right) : \mathbf{v} \in \{0, 1\}^n, v_n \in \{0, 1\} \right\}, \\ \left\{ \tilde{g} + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \bar{\mathbf{y}} + \bar{v}_n y_\gamma \right) : \mathbf{v} \in \{0, 1\}^n, v_n \in \{0, 1\} \right\}, \end{cases} \quad (3)$$

respectively. Denote $(\cdot) \cdot (\cdot)$ as the dot product of two real-valued vectors (\cdot) and (\cdot) , γ specifies the label for either of the path's end vertices. $\mathbf{y} = (y_{\beta_0}, y_{\beta_1}, \dots, y_{\beta_{n-1}})$, $\bar{\mathbf{y}} = (1 - y_{\beta_0}, 1 - y_{\beta_1}, \dots, 1 - y_{\beta_{n-1}})$, and $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$. Then $\{\Psi(G_r), \Psi^*(\tilde{G}_r) : 0 \leq r < 2^n\}$ forms $(2^{n+1}, 2^{n+1}, 2^m)$ -CCC, where $\Psi^*(\cdot)$ denotes the complex conjugate of $\Psi(\cdot)$.

A. Pseudo-Boolean Functions (PBFs)

A degree i monomial is a product of i distinct variables among y_0, y_1, \dots, y_{m-1} . PBFs are functions $\mathbf{F} : \{0, 1\}^m \rightarrow \mathbb{R}$, that are represented as a linear combination of monomials among $\{y_0, y_1, \dots, y_{m-1}\}$ where, y_i 's are Boolean variable and coefficients are drawn from \mathbb{R} . The highest degree of the monomials are called the degree of \mathbf{F} . As an example $\frac{4}{3}y_2y_1 + y_0$ is a 2nd order PBF of three variables y_0, y_1 and y_2 . It is clear that, when this coefficients are drawn from \mathbb{Z}_q and the range of the fuction \mathbf{F} changed to \mathbb{Z}_q the function \mathbf{F} becomes a generalized Boolean function (GBF) [10]. Let l be a positive integer and p_1, p_2, \dots, p_l be prime numbers and $\mathbf{c} = (c_1, c_2, \dots, c_l)$, where $c_i \in \{0, 1, \dots, p_i - 1\}$. Let g be a 2nd order boolean function of m variables and let $\mathbf{Y} = (y_0, \dots, y_{m+\sum_{i=1}^l s_i - 1})$. We define the following PBFs with the help of g as

$$\begin{aligned} M^{\mathbf{c}}(\mathbf{Y}) &= g(y_0, \dots, y_{m-1}) + \sum_{i=1}^l \frac{c_i q}{p_i} \sum_{k=0}^{s_i - 1} 2^k y_{m+\sum_{j=0}^{i-1} s_j + k} \\ N^{\mathbf{c}}(\mathbf{Y}) &= \tilde{g}(y_0, \dots, y_{m-1}) + \sum_{i=1}^l \frac{c_i q}{p_i} \sum_{k=0}^{s_i - 1} 2^k y_{m+\sum_{j=0}^{i-1} s_j + k} \end{aligned} \quad (4)$$

where $s_i \in \mathbb{Z}^+$ which denotes the set of all positive integer, $s_0 = 0$ and each y_i 's are Boolean variable. From (4), it is clear that both $M^{\mathbf{c}}$ and $N^{\mathbf{c}}$ are PBFs of variables $m + \sum_{i=1}^l s_i$. We chose s_i in such a way that $p_i \leq 2^{s_i} \forall i \in \{1, 2, \dots, l\}$. Let $h : \{0, 1\}^{n+1} \rightarrow \mathbb{Z}_q$ be a function and $\mathbf{r} = (r_0, r_1, \dots, r_{n-1})$ be the binary representation of the integer r where, $0 \leq r < 2^n$, and $\mathbf{c} = (c_1, c_2, \dots, c_l)$. We define the sets as,

$$\begin{aligned} \Omega_r^{\mathbf{c}} &= \left\{ M^{\mathbf{c}}(\mathbf{Y}) + h(\mathbf{v}') + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \mathbf{y} + v_n y_\gamma \right) \right. \\ &\quad \left. : \mathbf{v} \in \{0, 1\}^n, v_n \in \{0, 1\} \right\}, \\ \Lambda_r^{\mathbf{c}} &= \left\{ N^{\mathbf{c}}(\mathbf{Y}) + h(\mathbf{v}') + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \bar{\mathbf{y}} + \bar{v}_n y_\gamma \right) \right. \\ &\quad \left. : \mathbf{v} \in \{0, 1\}^n, v_n \in \{0, 1\} \right\}, \end{aligned} \quad (5)$$

where, $\mathbf{v}' = (\mathbf{v}, v_n) = (v_0, v_1, \dots, v_n)$. Let us assume that g be the GBF as defined in Lemma 1. Let $g^{\mathbf{v}', r, h} = g + h(\mathbf{v}') + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \mathbf{y} + v_n y_\gamma \right)$ and $s^{\mathbf{v}', r, h} = \tilde{g} + h(\mathbf{v}') + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \bar{\mathbf{y}} + \bar{v}_n y_\gamma \right)$. We also assume $M^{\mathbf{v}', r, \mathbf{c}} = M^{\mathbf{c}}(\mathbf{Y}) + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \mathbf{y} + v_n y_\gamma \right)$ and $N^{\mathbf{v}', r, \mathbf{c}} = N^{\mathbf{c}}(\mathbf{Y}) + \frac{q}{2} \left((\mathbf{v} + \mathbf{r}) \cdot \bar{\mathbf{y}} + \bar{v}_n y_\gamma \right)$. As per our assumption, for any choice of $\mathbf{v}' \in \{0, 1\}^{n+1}$ and $\mathbf{r} \in \{0, 1\}^n$, the functions $g^{\mathbf{v}', r, h}$ and $s^{\mathbf{v}', r, h}$ are \mathbb{Z}_q -valued GBFs of m variables and $M^{\mathbf{v}', r, \mathbf{c}}$ and $N^{\mathbf{v}', r, \mathbf{c}}$ are PBFs of $m + \sum_{i=0}^l s_i$ variables. Let $M^{\mathbf{v}', r, \mathbf{c}, h}(\mathbf{Y}) = M^{\mathbf{v}', r, \mathbf{c}} + h(\mathbf{v}')$ and $N^{\mathbf{v}', r, \mathbf{c}, h}(\mathbf{Y}) = N^{\mathbf{v}', r, \mathbf{c}} + h(\mathbf{v}')$. We define $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h})$, the complex-valued sequence as $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h}) = (\omega_q^{M_0^{\mathbf{v}', r, \mathbf{c}, h}}, \omega_q^{M_1^{\mathbf{v}', r, \mathbf{c}, h}}, \dots, \omega_q^{M_{2^{m+\sum_{i=0}^l s_i - 1}}^{\mathbf{v}', r, \mathbf{c}, h}})$, where, $M_{r'}^{\mathbf{v}', r, \mathbf{c}, h} = M^{\mathbf{v}', r, \mathbf{c}, h}(r_0, r_1, \dots, r_{m+\sum_{i=0}^l s_i - 1})$, $0 \leq r' < 2^{m+\sum_{i=0}^l s_i}$ and the binary representation of the integer r' is $(r_0, r_1, \dots, r_{m+\sum_{i=0}^l s_i - 1})$. The r' -th component of $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h})$ is given by

$$\begin{aligned} \omega_q^{M_{r'}^{\mathbf{v}', r, \mathbf{c}, h}} &= \omega_q^{g^{\mathbf{v}', r, h}(r_0, r_1, \dots, r_{m-1})} \omega_{p_1}^{c_1 \sum_{i=0}^{s_1 - 1} 2^i r_{m+i}} \\ &\quad \omega_{p_2}^{c_2 \sum_{i=0}^{s_2 - 1} 2^i r_{m+s_1+i}} \dots \omega_{p_l}^{c_l \sum_{i=0}^{s_l - 1} 2^i r_{m+\sum_{j=1}^{l-1} s_j+i}} \\ &= \omega_q^{g_j^{\mathbf{v}', r, h}} \omega_{p_1}^{c_1(i_1)} \omega_{p_2}^{c_2(i_2)} \dots \omega_{p_l}^{c_l(i_l)}, \end{aligned} \quad (6)$$

where, $(r_0, r_1, \dots, r_{m-1})$ is the binary representation of the integer j . Since $M^{\mathbf{v}', r, \mathbf{c}}$ is a $m + \sum_{i=1}^l s_i$ variable PBF therefore the length of $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h})$ is $2^{s_1} 2^{s_2} \dots 2^{s_l} 2^m$. Any element of $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h})$ is of the form $\omega_q^{g_j^{\mathbf{v}', r, h}} \omega_{p_1}^{c_1(i_1)} \omega_{p_2}^{c_2(i_2)} \dots \omega_{p_l}^{c_l(i_l)}$, where $0 \leq i_k \leq 2^{s_k} - 1$, $0 \leq j \leq 2^m - 1$ and $0 \leq k \leq l$.

Lemma 2: ([17]) Let t and t' be two non-negative integers, where $0 \leq t \neq t' < p_i$, p_i is a prime number as defined in section-II. Then $\sum_{j=0}^{p_i - 1} \omega_{p_i}^{(t-t')j} = 0$.

Let $S = \{M^{\mathbf{v}'_1, r, \mathbf{c}, h}, M^{\mathbf{v}'_2, r, \mathbf{c}, h}, \dots, M^{\mathbf{v}'_{2^{n+1}}, r, \mathbf{c}, h}\}$ where, $\mathbf{v}'_k \in \{0, 1\}^{n+1}$ and $k \in \{1, 2, \dots, 2^{n+1}\}$. We define

$$\Psi(S) = [\Psi(M^{\mathbf{v}'_1, r, \mathbf{c}, h}), \Psi(M^{\mathbf{v}'_2, r, \mathbf{c}, h}), \dots, \Psi(M^{\mathbf{v}'_{2^{n+1}}, r, \mathbf{c}, h})]^T. \quad (7)$$

Now we truncate the sequence $\Psi(M^{\mathbf{v}', r, \mathbf{c}, h})$ by reomoving all the elements of the form $\omega_q^{g_j^{\mathbf{v}', r, h}} \omega_{p_1}^{c_1(i_1)} \omega_{p_2}^{c_2(i_2)} \dots \omega_{p_l}^{c_l(i_l)}$ from

$$\begin{aligned}\Psi_{Trun}^i(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h}) &= \underbrace{(w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)}) w_q^{g_0^{\mathbf{v}, \mathbf{r}, \mathbf{d}, h}}, \dots, (w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)}) w_q^{g_{2^m-1}^{\mathbf{v}, \mathbf{r}, \mathbf{d}, h}}}_{(8)} \\ \Psi_{Trun}^i(N^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h}) &= \underbrace{(w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)}) w_q^{s_0^{\mathbf{v}, \mathbf{r}, \mathbf{d}, h}}, \dots, (w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)}) w_q^{s_{2^m-1}^{\mathbf{v}, \mathbf{r}, \mathbf{d}, h}}}_{(8)}\end{aligned}$$

$\Psi(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ if atleast one of $i_k \geq p_k$ where, $0 \leq i_k \leq 2^{s_k} - 1$, $1 \leq k \leq l$ and $0 \leq j \leq 2^m - 1$. Therefore, after the truncation we left with a sequence $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ where, each elements of $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ is of the form $w_q^{g_j^{\mathbf{v}', \mathbf{r}, h}} w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)}$ where, $0 \leq i_k \leq p_k - 1$, $0 \leq j \leq 2^m - 1$ and $1 \leq k \leq l$. Clearly we can make $p_1 p_2 \dots p_l 2^m$ number of the elements of the form $w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)} w_q^{s_j^{\mathbf{v}', \mathbf{r}, h}}$ if we vary all the i_k 's from 0 to $p_k - 1$ and j from 0 to $2^m - 1$. Hence the length of $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ is $p_1 p_2 \dots p_l 2^m$. We partition the length of $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ by $\prod_{i=1}^l p_i$ parenthesis where, each parenthesis has sequence of length 2^m . Equation (8) represents the i -th parenthesis of $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ and $\Psi_{Trun}(N^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ where, $i = i_1 + \sum_{j=2}^l i_j \prod_{b=1}^{j-1} p_b$, $0 \leq i_j \leq p_j - 1$ and $1 \leq j \leq l$.

III. PROPOSED CONSTRUCTION OF ZCCS

Theorem 1: Let $g : Z_2^m \rightarrow Z_q$ be a GBF as defined in *Lemma 1*. Let $2 \leq p_i \leq 2^{s_i}$ and $\mathbf{c} = (c_1, c_2, \dots, c_l)$ where $1 \leq i \leq l$ and $0 \leq c_i < p_i$. Then the set of codes

$$\left\{ \psi_{Trun}(\Omega_r^{\mathbf{c}}), \psi_{Trun}^*(\Lambda_r^{\mathbf{c}}) : 0 \leq r < 2^n, 0 \leq c_i \leq p_i - 1 \right\},$$

forms a $(\prod_{i=1}^l p_i 2^{n+1}, 2^m) - ZCCS_{2^{m+1}}^{2^m \prod_{i=1}^l p_i}$ if $h(\mathbf{v}') \in \{\lambda, \frac{g}{2} + \lambda\} \forall \mathbf{v}' \in \{0, 1\}^{n+1}$, where $\lambda \in Z_q$.

Proof: From (8) it can be observed that the i -th parenthesis of $\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ can be expressed as $w_{p_1}^{c_1(i_1)} w_{p_2}^{c_2(i_2)} \dots w_{p_l}^{c_l(i_l)} \Psi(g^{\mathbf{v}', \mathbf{r}, h})$ where, $i = i_1 + \sum_{j=2}^l i_j \prod_{b=1}^{j-1} p_b$, $0 \leq i_j \leq p_j - 1$ and $1 \leq j \leq l$. From (5), (8), *Lemma 1* and *Lemma 2* the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})$ for $\tau = 0$ can be derived as

$$\begin{aligned}\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})) &(0) \\ &= \sum_{\mathbf{v}'} \Theta(\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h}), \Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}', \mathbf{c}', h})) (0) \\ &= \sum_{\mathbf{v}'} \Theta(\Psi(g^{\mathbf{v}', \mathbf{r}, h}), \Psi(g^{\mathbf{v}', \mathbf{r}', h})) (0) \prod_{d=1}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{(c_d - c'_d)\alpha} \\ &= \Theta(\Psi(G_r), \Psi(G_{r'})) (0) \prod_{d=1}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{(c_d - c'_d)\alpha} \\ &= \begin{cases} p_1 p_2 \dots p_l 2^{m+n+1}, & r = r', \mathbf{c} = \mathbf{c}', \\ 0, & r = r', \mathbf{c} \neq \mathbf{c}', \\ 0, & r \neq r', \mathbf{c} = \mathbf{c}', \\ 0, & r \neq r', \mathbf{c} \neq \mathbf{c}'. \end{cases}\end{aligned}\quad (9)$$

Now, Using (5), (8), *Lemma 1* and the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})$ for $0 < |\tau| < 2^m$ can be derived as,

$$\begin{aligned}\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})) &(\tau) \\ &= \Theta(\Psi(G_r), \Psi(G_{r'})) (\tau) \prod_{d=1}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{(c_d - c'_d)\alpha} \\ &+ \Theta(\Psi(G_r), \Psi(G_{r'})) (\tau - 2^m) \sum_{\alpha=0}^{p_1-2} \omega_{p_1}^{c_1(\alpha+1) - c'_1\alpha} \\ &\prod_{d=2}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{c_d(\alpha) - c'_d(\alpha)} + \Theta(\psi(G_r), \psi(G_{r'})) (\tau - 2^m) \\ &\sum_{f=1}^{l-2} \prod_{d=1}^f \omega_{p_d}^{c_d(0) - c'_d(p_d-1)} \sum_{\alpha=0}^{p_{f+1}-2} \omega_{p_{f+1}}^{c_{f+1}(\alpha+1) - c'_{f+1}(\alpha)} \prod_{k=f+2}^l \\ &\sum_{\alpha=0}^{p_k-1} \omega_{p_k}^{c_{p_k}(\alpha) - c'_{p_k}(\alpha)} + \Theta(\psi(G_r), \psi(G_{r'})) (\tau - 2^m) \\ &\prod_{d=1}^{l-1} \omega_{p_d}^{c_d(0) - c'_d(p_d-1)} \sum_{\alpha=0}^{p_l-2} \omega_{p_l}^{c_l(\alpha+1) - c'_l\alpha}.\end{aligned}\quad (10)$$

From *Lemma 1*, we have, $\Theta(\Psi(G_r), \Psi(G_{r'})) (\tau) = 0, \forall \tau, 0 < |\tau| < 2^m$. Therefore, from the above we can say,

$$\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})) (\tau) = 0, 0 < |\tau| < 2^m. \quad (11)$$

From (9) and (11) We have,

$$\begin{aligned}\theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})) &(\tau) \\ &= \begin{cases} p_1 p_2 \dots p_l 2^{m+n+1}, & r = r', \mathbf{c} = \mathbf{c}', \tau = 0, \\ 0, & r = r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} = \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m. \end{cases}\end{aligned}\quad (12)$$

Similarly, it can be shown that

$$\begin{aligned}\theta(\Psi_{Trun}^*(\Lambda_r^{\mathbf{c}}), \Psi_{Trun}^*(\Lambda_{r'}^{\mathbf{c}'})) &(\tau) \\ &= \begin{cases} p_1 p_2 \dots p_l 2^{m+n+1}, & r = r', \mathbf{c} = \mathbf{c}', \tau = 0, \\ 0, & r = r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} = \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m. \end{cases}\end{aligned}\quad (13)$$

From *Lemma 1*, we have $\Theta(\Psi(G_r), \Psi^*(\bar{G}_{r'})) (\tau) = 0, |\tau| < 2^m$. Therefore, from *Lemma 1*, (5), (8) the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi_{Trun}^*(\Lambda_{r'}^{\mathbf{c}'})$ for $\tau = 0$ can be derived as,

$$\begin{aligned}
& \Theta(\Psi_{Trun}(\Omega_r^c), \Psi_{Trun}^*(\Lambda_r^c))(0) \\
&= \sum_{\mathbf{v}'} \Theta(\Psi_{Trun}(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h}), \Psi_{Trun}^*(N^{\mathbf{v}', \mathbf{r}', \mathbf{c}', h}))(0) \\
&= \sum_{\mathbf{v}'} \Theta(\Psi(g^{\mathbf{v}', \mathbf{r}, h}), \Psi^*(s^{\mathbf{v}', \mathbf{r}', h}))(0) \prod_{d=1}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{(c_d+c'_d)\alpha} \\
&= \omega_q^{2\lambda} \Theta(\Psi(G_r), \Psi^*(\bar{G}_{r'}))(0) \prod_{d=1}^l \sum_{\alpha=0}^{p_d-1} \omega_{p_d}^{(c_d+c'_d)\alpha} \\
&= 0.
\end{aligned} \tag{14}$$

By the similar calculation as in (10), we have

$$\Theta(\Psi_{Trun}(\Omega_r^c), \Psi_{Trun}^*(\Lambda_r^c))(\tau) = 0, \forall 0 < |\tau| < 2^m. \tag{15}$$

Hence by (12), (13), (14) and (15) we conclude the set

$$\left\{ \psi_{Trun}(\Omega_r^c), \psi_{Trun}^*(\Lambda_r^c) : 0 \leq r < 2^n, 0 \leq c_i \leq p_i - 1 \right\}, \tag{16}$$

forms a $(\prod_{i=1}^l p_i 2^{n+1}, 2^m) - ZCCS_{2^{n+1}}^{2^m \prod_{i=1}^l p_i}$. ■

Corollary 1: Our construction gives optimal ZCCS of length of this form $(p_1 p_2 \dots p_l) 2^m$, where p_i 's are any prime number. From the fundamental theorem of arithmetic [18] any number can be expressed as product of prime numbers therefore the optimal ZCCS obtained by using our suggested construction gives all possible length of this form $n 2^m$, where n is any positive integer greater than or equal 1. If $m = 1$ we get all possible even length optimal ZCCS.

Remark 1: For $l = 1$, the proposed result in Theorem 1 reduces to $(p 2^{n+1}, 2^m) - ZCCS_{2^{n+1}}^{p 2^m}$ as in [10]. Therefore, the proposed construction is a generalization of [10]

Corollary 2: ([7]) Let us assume that $G(h)$ is a path where, the edges have the identical weight of $\frac{q}{2}$. Then $h(\mathbf{v}')$ can be expressed as

$$h(v_0, v_1, \dots, v_n) = \frac{q}{2} \sum_{\alpha=0}^{n-1} v_{\pi(\alpha)} v_{\pi(\alpha+1)} + \sum_{\alpha=0}^n u_{\alpha} v_{\alpha} + u,$$

where $u, u_0, u_1, \dots, u_n \in \mathbb{Z}_q$. From (5), it is clear that the i -th column of $\psi(\Omega_r^c)$ is obtained by fixing \mathbf{Y} at $\mathbf{i} = (i_0, i_1, \dots, i_m, \dots, i_{m+\sum_{i=1}^l s_i-1})$, in the expression of $M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h}$ where $(i_0, i_1, \dots, i_m, \dots, i_{m+\sum_{i=1}^l s_i-1})$ is the binary representation of i . Because the i -th column sequence of $\Psi(\Omega_r^c)$ is derived from a GBF whose graph is a path over $n+1$ vertices, hence from [6] the i -th column sequence of $\Psi(\Omega_r^c)$ is a q -ary Golay sequence. Thus each column of $\Psi_{Trun}(\Omega_r^c)$ is Golay sequence. Thus the PMPER of each column $\Psi_{Trun}(\Omega_r^c)$ is bounded by 2. Similarly the PMPER of each column of $\Psi_{Trun}^*(\Lambda_r^c)$ is bounded by 2.

Remark 2: From (6), it can be observed that $w_q^{M^{\mathbf{v}', \mathbf{r}, h}}$ is a root of the polynomial: $z^\sigma - 1$, where σ , denotes a positive integer given by the least common multiple (lcm) of p_1, p_2, \dots, p_l and q . Therefore, the components of $\Psi(M^{\mathbf{v}', \mathbf{r}, \mathbf{c}, h})$ are given by the roots of the polynomial: $z^\sigma - 1$.

TABLE I
COMPARISON OF THE PROPOSED CONSTRUCTION WITH [2], [7], [9]–[11], [19]

Source	Based On	Parameters	Conditions	Optimal
[2]	Direct	$(2^{k+p+1}, 2^{m-p}) - ZCCS_{2^{p+1}}^{2^m}$	$k+p \leq m$	yes
[11]	Indirect	$(K, M) - ZCCS_M^K$	$K, M \geq 2$	yes
[7]	Direct	$(2^{n+p}, 2^{m-p}) - ZCCS_{2^n}^{2^m}$	$p \leq m$	yes
[9]	Direct	$(2^{k+v}, 2^{m-v}) - ZCCS_{2^v}^{2^m}$	$v \leq m, k \leq m-v$	yes
[10]	Direct	$(p 2^{k+1}, 2^m) - ZCCS_{2^{k+1}}^{2^m}$	$m \geq 2, k \leq m, p$ prime	yes
[19]	Direct	$(q^{n+1}, q^{m-v}) - ZCCS_q^{2^m}$	$v \leq m$	yes
[11]	Indirect	$(K, M^{N+1}) - ZCCS_M^{M^{N+1}P}$	$K, M \geq 2$	yes
Theorem 1	Direct	$(k 2^{n+1}, 2^m) - ZCCS_{2^{n+1}}^{2^m}$	$k, m, n \in \mathbb{Z}^+$	yes

Example 1: Let us assume that $q=2, p_1=3, p_2=2, p_3=2, m=3, n=1$ and $s_1=2, s_2=1$ and $s_3=1$. Let us take the GBF $f: \{0, 1\}^2 \rightarrow \mathbb{Z}_2$ as follows: $f = y_1 y_2 + y_0$, where $G(f|_{y_0=0})$ and $G(f|_{y_0=1})$ give a path with y_1 as one of the end vertices. Let $h: \{0, 1\}^2 \rightarrow \mathbb{Z}_2$ defined by $h(v_0, v_1) = v_0 v_1$. From (4) we have,

$$\begin{aligned}
M^c &= y_1 y_2 + y_0 + \frac{2c_1}{3}(y_3 + 2y_4) + c_2 y_5 + c_3 y_6, \\
N^c &= \bar{y}_1 \bar{y}_2 + \bar{y}_0 + \frac{2c_1}{3}(y_3 + 2y_4) + c_2 y_5 + c_3 y_6,
\end{aligned} \tag{17}$$

where $c_1 = 0, 1, 2, c_2 = 0, 1$ and $c_3 = 0, 1$. From (5), we have

$$\begin{aligned}
\Omega_r^c &= \{M^c + v_0 v_1 + v_0 y_0 + r_0 y_0 + v_1 y_1 : v_0, v_1 \in \{0, 1\}\}, \\
\Lambda_r^c &= \{N^c + v_0 v_1 + v_0 \bar{y}_0 + r_0 \bar{y}_0 + \bar{v}_1 y_1 : v_0, v_1 \in \{0, 1\}\},
\end{aligned} \tag{18}$$

where (r_0) is the binary representation of the integer r and $0 \leq r < 2$. Therefore,

$$\left\{ \Psi_{Trun}(\Omega_r^c), \Psi_{Trun}^*(\Lambda_r^c) : 0 \leq r \leq 1, 0 \leq c_1 \leq 2, \right. \\
\left. 0 \leq c_2 \leq 1, 0 \leq c_3 \leq 1 \right\},$$

forms an optimal $(48, 8) - ZCCS_4^{96}$ and the maximum column sequence PMPER is at most 2.

Remark 3: Our proposed construction also have some advantages over [19].

- 1) In [19] multivariable functions is used which is less feasible for hardware generation of sequences as their domains contain the domains of PBF as subset.
- 2) Our construction has more flexibility on the phases of sequences.
- 3) Also in [19] the length of ZCCs is of the form q^m where $m \geq 2, q \in \mathbb{Z}^+$ which, may not produce all even lengths, for example 6.

IV. CONCLUSION

In this work, we have proposed a direct construction of optimal ZCCSs for all possible even lengths using PBFs. The maximum column sequence PMPER of the proposed ZCCSs is upper-bounded by 2 which can be useful in MC-CDMA system to control high PMPER problem. The proposed construction also provides more flexible parameter as compared to the existing PBFs based constructions of optimal ZCCS.

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