One or Two Ridges? An Exact Mode Separation Condition for the Gabor Transform

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***Abstract*—In this paper, we investigate the conditions for the separation of two pure tones from the spectrogram computed with a Gaussian window. For this purpose, we put forward necessary and sufficient conditions for the existence of spectrogram ridges associated with each signal. We then show how this condition easily extends to the case of parallel linear chirps, i.e., signals with constant amplitude, linear instantaneous frequency, and same chirp rate.**

***Index Terms*—Time-frequency analysis, short-time Fourier transform, spectrogram ridges.**

I. Introduction

Analyzing the frequency content of a signal is of paramount importance in many applications, which explains why spectral analysis has become a major theory in signal processing. When these frequencies vary with time, one needs to analyze the signal jointly in time and frequency, hence the term *time-frequency analysis* [1]. The short-time Fourier or Gabor transform [2] is the simplest way of doing it, by applying a sliding (Gaussian) window. But many other linear or quadratic time-frequency representations (TFRs) have been proposed in the literature, such as the Wigner-Ville [3], the wavelet [4] or chirplet [5] transforms.

A very important aspect is the estimation of the *instantaneous frequencies* (IFs) of the modes making up a multi-component signal. Indeed, in many applications, practitioners are interested in estimating these modes and their IFs. Examples include mechanical engineering and monitoring [6], [7], seismic signal analysis [8], [9] or oceanography [10]. TFRs enable such an estimation, since the modes are known to draw so-called ridges located around the IFs [11]. To estimate these ridges, various methods have been developed in the past decades, such as [12], [13] or [14]. Once the ridges are detected, each mode can be reconstructed by various means: a direct estimation from the values on the ridge [11], a local vertical integration such as in the synchrosqueezed transform [15] or a 2D integration in a well-chosen neighborhood [16], [17]. Other methods in the literature start by decomposing the signal into its oscillating modes, and then applies the method of the analytic signal to estimate the instantaneous frequencies, such as the well-known *Empirical Mode Decomposition* (EMD) [18].

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Some of these techniques are supported by a nice theory, that give estimates of the expected transform based on first- [15] or second-order expansions of the phase [11], [19]. But they also require the modes to be well-separated in the time-frequency plane, which is a strong condition. The aim of this letter is to investigate this question for the Gabor transform. More precisely, we present a new result that characterizes when a sum of two pure tones leads to two ridges in the Gabor spectrogram. This result is the counterpart of what was already done in [20] for EMD and [21] for the wavelet synchrosqueezed transform. Additionally, we extend our result to the case of a sum of linear chirps with the same chirp-rate (i.e., IFs draw parallel lines in the TF plane).

We believe this new result to be of interest in several situations. Firstly, we will see that the result will help to characterize the appearance of some interference patterns, called *time-frequency bubbles*, following the denomination of [22]. Characterizing these patterns could help to adapt the window or wavelet width in adaptive TFRs [23] to improve mode separation. Secondly, several works have recently tackled the separation of interfering modes using the chirplet transform [24], [25], and we will explain in what way our result is complementary to these works.

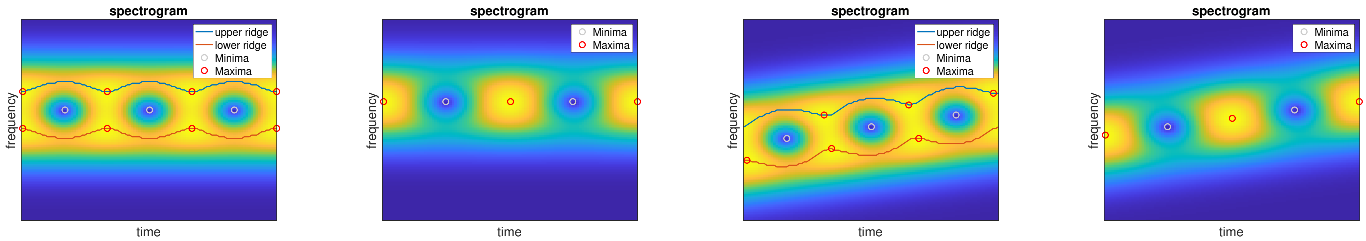
The letter is organized as follows. The results are stated in Section II for pure tones, and III for linear chirps. We discuss these results and their potential impact in the concluding Section IV, while the proofs of the main results are gathered in the Appendix.

II. Pure Tones Separation from the Spectrogram

Let us first define some notation that is used throughout the paper. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and a real window $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the modified STFT of f is defined as:

$$V_f^h(t,\eta) = \int_{\mathbb{R}} f(\tau)h(\tau-t)e^{-2i\pi(\tau-t)\eta}d\tau. \qquad (1)$$

Thanks to a well localized sliding window, the spectrogram $|V_f^h(t,\eta)|^2$ estimates the spectral energy at a given time and frequency. A *pure harmonic* of the form $Ae^{i2\pi\xi t}$ is known to create a stripe in the spectrogram, centered around a *ridge* located at frequency $\eta = \xi$. In the case of a signal made of several harmonics, numerous techniques have been proposed to identify those ridges, in order to separate the modes making up the signal [4], [11], [26]. Yet, such approaches are efficient only when the modes do not interfere, which is not always the case in practice. In this section, we give a necessary and sufficient condition at which two pure waves create exactly



(a) (b) (c) (d)

Fig. 1: (a): spectrogram of two parallel pure harmonic modes with the same amplitude, when there exists a chain of LMFs associated with each mode ($A = 1$, $\xi_1 = 230$, and $\xi_2 = 260$); (b): spectrogram of the same signal as in (a) when the computation of ridges associated with each mode is no longer feasible; (c): same as (a) but when two parallel linear chirp are considered ($A = 1$, $\phi_1(t) = 230t + 100t^2$, $\phi_2(t) = 260t + 100t^2$). (d): spectrogram of the same signal as in (c) when the computation of ridges associated with each mode is no longer feasible. In each case, we also plot the zeros and the local maxima of the spectrogram.

two separated ridges, meaning that we are able to identify and separate them.

To this end, let us mathematically define the ridges as a chain of *local maxima of the spectrogram along the frequency axis* (LMFs). Such two chains can be constructed only if the spectrogram has three extrema along the frequency axis at any time t (two maxima and one minimum), and we investigate in which circumstances this is the case. In our mathematical analysis, we consider the case of the Gaussian window $h(t) = e^{-\pi\frac{t^2}{\sigma^2}}$ (though different normalizations are possible), and the extension to other types of windows is not in the scope of the present paper. Our main result is the following.

Proposition II.1. *Let $f(t) = f_1(t) + f_2(t)$ with $f_1(t) = Ae^{i2\pi\xi_1 t}$ and $f_2(t) = e^{i2\pi\xi_2 t}$, where $\xi_1 < \xi_2$ and $A > 0$. Modes f_1 and f_2 are associated with two chains of LMFs if and only if*

$$\begin{aligned} &\alpha := \sqrt{\frac{\pi}{2}}\sigma(\xi_2 - \xi_1) > 1 \textit{ and} \\ &|\log(A)| < -2\,\mathrm{arcosh}(\alpha) + 2\alpha\sqrt{\alpha^2 - 1}. \end{aligned} \tag{2}$$

The proof is detailed in Appendix A. Recalling that the spectrogram of f reads:

$$\begin{aligned} |V_f^h(t,\eta)|^2 = \sigma^2 \Big[& A^2 e^{-2\pi\sigma^2(\eta-\xi_1)^2} + e^{-2\pi\sigma^2(\eta-\xi_2)^2} \\ & + 2Ae^{-\pi\sigma^2\left[(\eta-\xi_1)^2+(\eta-\xi_2)^2\right]}\cos(2\pi(\xi_2-\xi_1)t)\Big], \end{aligned} \tag{3}$$

one remarks that $|V_f^h(.,\eta)|^2$, where . means we consider this variable, attains its maximum at $t_k = \frac{k}{\xi_2-\xi_1}, k \in \mathbb{Z}$, at which

$$|V_f^h(t_k,\eta)|^2 = \sigma^2(Ae^{-\pi\sigma^2(\eta-\xi_1)^2} + e^{-\pi\sigma^2(\eta-\xi_2)})^2. \tag{4}$$

One can then show that the conditions for the existence of two LMFs at t_k, are the same as condition (2) (see Appendix B), meaning the result of Proposition II.1 is driven by what happens at time t_k.

To illustrate the different situations just put forward, we consider the spectrogram of the sum of two pure tones with $A = 1$ in Fig. 1 (a) and Fig. 1 (b). In the former, the conditions (2) are fulfilled, and the two ridges can be easily detected, while, in the latter, the spectrogram only exhibits an oscillating pattern sometimes referred to as *time-frequency bubbles* [22]. In that paper, an equivalent definition based on the phase was proposed, but contrary to our result, the authors did not provide conditions to characterize the presence of this phenomenon. Note finally that the role played by σ for mode separation purpose was also mentioned in [27].

We believe that this result can be useful for practitioners since, in case of interference leading to time-frequency bubbles, it should be possible to separate the modes by increasing σ locally until three extrema show up, which should ease mode separation. Finally, note that other separation conditions were already proposed in the literature (e.g. Eq. (11) of [28]), but the purpose was to put forward when the modes did not overlap in the TF plane, which is not our goal here.

III. Case of Two Parallel Linear Chirps

Our goal is now to extend the previous results to parallel linear chirps. We thus consider $f(t) = f_1(t) + f_2(t)$ with $f_1(t) = Ae^{2i\pi\phi_1(t)}$ and $f_2(t) = e^{2i\pi\phi_2(t)}$, with $\phi_i(t) = a_i t + \frac{c}{2}t^2$ with c constant. As one has [28]

$$V_{f_1}^h(t,\eta) \quad = \quad f_1(t)R^{-1/2}e^{-\frac{\pi(1+ic\sigma^2)(\eta-\phi_1'(t))^2}{R^2\sigma^2}}e^{-i\frac{\theta}{2}} \tag{5}$$

with $R = \frac{\sqrt{1+c^2\sigma^4}}{\sigma^2}$ and $\theta = \arctan(-c\sigma^2)$, one obtains:

$$\begin{aligned} |V_f^h(t,\eta)|^2 = \Bigg(& A^2 e^{-2\pi\frac{(\eta-\phi_1'(t))^2}{R^2\sigma^2}} + e^{-2\pi\frac{(\eta-\phi_2'(t))^2}{R^2\sigma^2}} \\ & + 2Ae^{-\pi\frac{\left[(\eta-\phi_1'(t))^2+(\eta-\phi_2'(t))^2\right]}{R^2\sigma^2}}\cos(2\pi(\phi(t,\eta)))\Bigg)/R, \end{aligned} \tag{6}$$

with $\phi(t,\eta) = (a_2 - a_1)\left(t + \frac{c}{R^2\sigma^2}\left(\eta - ct - \frac{a_1+a_2}{2}\right)\right)$.

We now show that when t and η satisfy a specific relation, we end up with a similar expression as (3) for the spectrogram. For that purpose, let us then consider the set of TF points (t,η) such that $\cos(2\pi\phi(t,\eta)) = \lambda$. These points correspond to

$$\phi(t,\eta) = \frac{\arccos(\lambda)}{2\pi} + k \text{ for } k \in \mathbb{Z}. \tag{7}$$

Replacing $\phi(t,\eta)$ by its value, such (t,η) satisfy $t = D\eta + B_{k,\lambda}$, with

$$\begin{aligned} D &= \frac{c}{c^2 - R^2\sigma^2} \\ B_{k,\lambda} &= \frac{c(a_2^2 - a_1^2) + R^2\sigma^2(\frac{\arccos(\lambda)}{\pi} + 2k)}{2(R^2\sigma^2 - c^2)(a_2 - a_1)}. \end{aligned} \tag{8}$$

On such a straight line, setting $\tilde{\sigma} = \frac{1-cD}{R\sigma}$, $\tilde{\xi}_1 = \frac{a_1 + cB_{k,\lambda}}{1-cD}$, and $\tilde{\xi}_2 = \frac{a_2 + cB_{k,\lambda}}{1-cD}$, (6) becomes:

$$\begin{aligned} |V_f^h(D\eta + B_{k,\lambda}, \eta)|^2 &= \Big(A^2 e^{-2\pi\tilde{\sigma}^2(\eta - \tilde{\xi}_1)^2} \\ +e^{-2\pi\tilde{\sigma}^2(\eta-\tilde{\xi}_2)^2} &+ 2Ae^{-\pi\tilde{\sigma}^2\left[(\eta-\tilde{\xi}_1)^2 + (\eta-\tilde{\xi}_2)^2\right]}\lambda\Big)/R, \end{aligned} \tag{9}$$

which is the same expression as (3), replacing σ, ξ_1 and ξ_2 by $\tilde{\sigma}$, $\tilde{\xi}_1$ and $\tilde{\xi}_2$, and thus one can deduce the following

Proposition III.1. *Let $f_1 = Ae^{2i\pi\phi_1(t)}$ and $f_2 = e^{2i\pi\phi_2(t)}$, with $\phi_i(t) = a_i t + \frac{c}{2}t^2$, f_1 and f_2 are associated with two chains of maxima of the spectrogram along the lines $t = D\eta + B_{k,\lambda}$, $k \in \mathbb{Z}$, $\lambda \in [-1, 1]$ iff*

$$\begin{aligned} \alpha_c = \sqrt{\frac{\pi}{2}}\frac{a_2 - a_1}{R\sigma} > 1, \ \textit{and} \\ |\log(A)| < -2\,\mathrm{arcosh}(\alpha_c) + 2\alpha_c\sqrt{\alpha_c^2 - 1}. \end{aligned} \tag{10}$$

Similarly to the case of two pure tones, the existence of two separated ridges is here related to what happens when $\lambda = 1$. Let us denote by (t_k, η_k) a local maximum on the line $t = D\eta + B_{k,1}$ and then a neighboring point (t, η) of (t_k, η_k) in the direction of vector $(1, c)$. For such a point, one has $\eta - \phi_1'(t) = \eta_k - \phi_1'(t_k)$ and $\eta - \phi_2'(t) = \eta_k - \phi_2'(t_k)$ and, as $\cos(2\pi\phi(t,\eta)) < 1$, one gets that (t_k, η_k) is also a maximum in the direction of the vector $(1, c)$, and thus a local maximum of the bidimensional function $|V_f^h(t,\eta)|^2$. An illustration of the two cases put forward in Proposition III.1 is displayed in Fig. 1 (c) and (d), corresponding respectively to the existence and the absence of a chain of maxima associated with each mode. In [25], the separation of close modes was also investigated using linear chirp approximations for the different modes and then considering the essential support of each mode in the time-frequency plane, which they call a *ribbon*. The separation condition in that paper was related to when the modes lead to separate ribbons, which is essential for mode reconstruction. But, even when ribbons overlap, one can still have different ridges, which is what we study here.

IV. Conclusion

In this paper, we first put forward necessary and sufficient conditions for the existence of ridges associated with each of the modes of a two-tone signal. We noticed that these conditions were very similar when dealing with parallel linear chirps, except that the modulation has to be taken into account. The novel results shown in this paper pave the way for the detection of interference pattern in the TF plane and would probably be useful to improve adaptive time-frequency representation in which the window size varies with time. This will be the topic for future works.

Appendix

A. Proof of Proposition II.1

Lemma. *α being defined as in Proposition II.1, and put $\gamma = \cos(2\pi(\xi_2 - \xi_1)t)$, $|V_f^h(t,.)|^2$ has three extrema iff $\alpha > \sqrt{\frac{1+\gamma}{2}}$ and $|\log(A)| < -\mathrm{arcosh}(X_2) + 2\alpha^2\frac{\sqrt{X_2^2-1}}{X_2+\gamma}$, with $X_2 = \gamma(\alpha^2 - 1) + \alpha\sqrt{\gamma^2(\alpha^2-2)+2}$, and a unique extremum otherwise.*

Proof. Consider $l(\eta) = A^2 e^{-2\pi\sigma^2(\eta-\xi_1)^2} + e^{-2\pi\sigma^2(\eta-\xi_2)^2} + 2Ae^{-\pi\sigma^2\left[(\eta-\xi_1)^2+(\eta-\xi_2)^2\right]}\cos(2\pi(\xi_2-\xi_1)t)$, and then put $\gamma = \cos(2\pi(\xi_2 - \xi_1)t)$. Setting $\eta = \nu + \frac{\xi_1+\xi_2}{2}$ and putting $\xi = \frac{\xi_2 - \xi_1}{2}$, one may define $l_1(\nu) = l(\nu + \frac{\xi_1+\xi_2}{2}) = e^{-2\pi\sigma^2(\nu^2+\xi^2)}(A^2e^{-4\pi\sigma^2\xi\nu} + e^{4\pi\sigma^2\xi\nu} + 2A\gamma)$. Putting $\nu = \frac{\log(A)+\mu}{4\pi\sigma^2\xi}$, one derives: $l_1(\nu) = l_2(\mu) = e^{-2\pi\sigma^2\left[\left(\frac{\log(A)+\mu}{4\pi\sigma^2\xi}\right)^2+\xi^2\right]}2A(\cosh(\mu)+\gamma)$. The derivative of l_2 reads:

$$\begin{aligned} l_2'(\mu) =& e^{-2\pi\sigma^2\left[\left(\frac{\log(A)+\mu}{4\pi\sigma^2\xi}\right)^2+\xi^2\right]} \\ & 2A\left[-\frac{\log(A)+\mu}{4\pi\sigma^2\xi^2}(\cosh(\mu)+\gamma) + \sinh(\mu)\right] \end{aligned}$$

which has the sign of (assuming $\gamma \neq -1$): $g(\mu) = -\frac{\log(A)+\mu}{2\alpha^2} + \frac{\sinh(\mu)}{\cosh(\mu)+\gamma}$. Differentiating g we get $g'(\mu) = -\frac{1}{2\alpha^2} + \frac{1+\gamma\cosh(\mu)}{(\cosh(\mu)+\gamma)^2}$ which has the same sign as

$$\begin{aligned} g_1(\cosh(\mu)) &:= -\frac{(\cosh(\mu)+\gamma)^2}{2\alpha^2} + 1 + \gamma\cosh(\mu) \\ &= \frac{1}{\alpha^2}(-\frac{\cosh(\mu)^2}{2} + \gamma(\alpha^2-1)\cosh(\mu) + \alpha^2 - \frac{\gamma^2}{2}), \end{aligned}$$

which is a second order polynomial in $\cosh(\mu)$ whose discriminant reads: $\Delta = \gamma^2(\alpha^2 - 2) + 2 > 0$, and whose roots are denoted by X_1 and X_2 with $X_1 < X_2$, and $g_1(X) > 0$ if $X \in]X_1, X_2[$ and negative otherwise. To locate $\cosh(\mu)$ with respect to X_1 and X_2, one computes $g_1(1) = (\gamma+1)(1 - \frac{\gamma+1}{2\alpha^2})$ which has the sign of $\alpha^2 - \frac{\gamma+1}{2}$. Assuming $\alpha \leq \sqrt{\frac{\gamma+1}{2}}$, $g_1(1) \leq 0$ and 1 belongs to $]-\infty, X_1]$ or $[X_2, \infty[$. As $\frac{X_1+X_2}{2} = \gamma(\alpha^2-1) < 1$, 1 belongs to $]X_2, +\infty[$. Finally, as $\cosh(\mu) \geq 1$, $g_1(\cosh(\mu))$ and thus $g'(\mu)$ are negative, which leads to the following table of variations:

μ	$-\infty$		μ_1		$+\infty$
$g(\mu)$	$+\infty$	$\searrow$		$\searrow$	$-\infty$
$l_2'(\mu)$		$+$	0	$-$	
$l_2(\mu)$		$\nearrow$		$\searrow$	

and $l(\eta)$ has a unique extremum (maximum) when $\alpha \leq \sqrt{\frac{\gamma+1}{2}}$. If $\alpha > \sqrt{\frac{\gamma+1}{2}}$, then $g_1(1) > 0$ and thus 1 belongs to $]X_1, X_2[$. As $\cosh(\mu) \in]X_1, X_2[$ is equivalent to $|\mu| < \mathrm{arcosh}(X_2)$, $g_1(\cosh(\mu)) < 0$ if $\mu \in]-\infty, -\mathrm{arcosh}(X_2)[\bigcup]\mathrm{arcosh}(X_2), +\infty[$, and $g_1(\cosh(\mu))$ is positive otherwise, leading to the following table of variations:

μ	$-\infty$		$-\operatorname{arcosh}(X_2)$		$\operatorname{arcosh}(X_2)$		$+\infty$
$g'(\mu)$		$-$	0	$+$	0	$-$	
$g(\mu)$	$+\infty$	$\searrow$		$\nearrow$		$\searrow$	$-\infty$

$g(\mu)$ vanishes with a change of sign only once at some $\mu = \mu_2$ if and only if $g(-\operatorname{arcosh}(X_2)) \geq 0$ or $g(\operatorname{arcosh}(X_2)) \leq 0$. In this case, we deduce that

μ	$-\infty$		μ_2		$+\infty$
$l_2'(\mu)$		$+$	0	$-$	
$l_2(\mu)$		$\nearrow$		$\searrow$	

meaning $l(\eta)$ has a unique extremum (which is a maximum). If $g(-\operatorname{arcosh}(X_2)) < 0$ and $g(\operatorname{arcosh}(X_2)) > 0$, $g(\mu)$ vanishes and changes signs three times at some $\mu = \mu_3$, μ_4 and μ_5 leading to the following table of variations for l_2:

μ	$-\infty$		μ_3		μ_4		μ_5		$+\infty$
$g(\mu)$		$+$	0	$-$	0	$+$	0	$-$	
$l_2(\mu)$		$\nearrow$		$\searrow$		$\nearrow$		$\searrow$	

In such case, $l(\eta)$ has 2 maxima and a mimimum. Finally, as

$$\begin{aligned} & g(\operatorname{arcosh}(X_2)) > 0 \\ \Leftrightarrow & \log(A) < -\operatorname{arcosh}(X_2) + 2\alpha^2 \frac{\sqrt{X_2^2 - 1}}{X_2 + \gamma} \\ & \text{and } g(-\operatorname{arcosh}(X_2)) < 0 \\ \Leftrightarrow & -\log(A) < -\operatorname{arcosh}(X_2) + 2\alpha^2 \frac{\sqrt{X_2^2 - 1}}{X_2 + \gamma}, \end{aligned}$$

$l(\eta)$ has three extrema if and only if $\alpha > \sqrt{\frac{1+\gamma}{2}}$ and then if $|\log(A)| < -\operatorname{arcosh}(X_2) + 2\alpha^2 \frac{\sqrt{X_2^2-1}}{X_2+\alpha}$ with $X_2 = \gamma(\alpha^2 - 1) + \alpha\sqrt{\gamma^2(\alpha^2-2)+2}$. ■

To prove Proposition II.1, assume that $-1 < \gamma \leq 1$, since when $\gamma = -1$, one can easily check that the spectrogram has three extrema. $l_2'(\mu)$ has the same sign as $g(\mu, \gamma) = -\frac{\log(A)+\mu}{2\alpha^2} + \frac{\sinh(\mu)}{\cosh(\mu)+\gamma}$. From the lemma, l_2 has three extrema iff $\alpha > \sqrt{\frac{1+\gamma}{2}}$, $g(-\operatorname{arcosh}(X_2), \gamma) < 0$ and $g(\operatorname{arcosh}(X_2), \gamma) > 0$, which is equivalent to: $\exists\ y_0 < 0\ \ g(y_0, \gamma) < 0$ and $y_0' > 0\ \ g(y_0', \gamma) > 0$. Indeed, this condition is a necessary condition since $y_0 = -\operatorname{arcosh}(X_2)$ and $y_0' = \operatorname{arcosh}(X_2)$ satisfy it. Conversely, if there exist $y_0 < 0$ and $y_0' > 0$ such that $g(y_0, \gamma) < 0$ and $g(y_0', \gamma) > 0$, as $\lim\limits_{\mu \to +\infty} g(\mu, \gamma) = -\infty$ and $\lim\limits_{\mu \to -\infty} g(\mu, \gamma) = +\infty$, $l_2'(\mu)$ changes signs three times, respectively on $]-\infty, y_0[$, $]y_0, y_0'[$, and $]y_0', +\infty[$, thus l_2 has three extrema. Finally, we show that if l_2 has three extrema for $\gamma = 1$, it has three extrema for $\gamma \in]-1, 1]$. in the latter case, there exist $y_0 < 0$ and $y_0' > 0$ such that $g(y_0, 1) < 0$ and $g(y_0', 1) > 0$. Then, as $\gamma \in]-1, 1]$ one has $\forall \mu \in \mathbb{R}\ \frac{1}{\cosh(\mu)+\gamma} \geq \frac{1}{\cosh(\mu)+1}$, which means that $\forall \mu \geq 0$, $\frac{\sinh(\mu)}{\cosh(\mu)+\gamma} \geq \frac{\sinh(\mu)}{\cosh(\mu)+1}$ and $\forall \mu \leq 0$, $\frac{\sinh(\mu)}{\cosh(\mu)+\gamma} \leq \frac{\sinh(\mu)}{\cosh(\mu)+1}$. One deduces that $\forall \mu \geq 0$, $g(\mu, \gamma) \geq g(\mu, 1)$ and $\forall \mu \leq 0$, $g(\mu, \gamma) \leq g(\mu, 1)$, meaning in particular that $g(y_0', \gamma) \geq g(y_0', 1)$, and that $g(y_0, \gamma) \leq g(y_0, 1)$, implying that $g(y_0', \gamma) > 0$ and $g(y_0, \gamma) < 0$. Thus, whatever $\gamma \in]-1, 1]$, the function g satisfies the condition for l_2 to have three extrema.

B. Existence of LMFs at time t_k

Consider the function $(Ae^{-\pi\sigma^2(\eta-\xi_1)^2} + e^{-\pi\sigma^2(\eta-\xi_2)^2})^2$ having the same extrema as $l(\eta) = Ae^{-\pi\sigma^2(\eta-\xi_1)^2} + e^{-\pi\sigma^2(\eta-\xi_2)^2}$. Defining $\xi = \frac{\xi_2 - \xi_1}{2}$ and making the change of variables $\eta = \nu + \frac{\xi_1+\xi_2}{2}$, one gets $q(\nu) := l(\nu + \frac{\xi_1+\xi_2}{2}) = Ae^{-\pi\sigma^2(\nu+\xi)^2} + e^{-\pi\sigma^2(\nu-\xi)^2}$. Differentiating q, one obtains after some simplifications:

$$\begin{aligned} q'(\nu) = \ & 2\pi\sigma^2 e^{-\pi\sigma^2\nu^2} e^{-\pi\sigma^2\xi^2} \xi (Ae^{-2\pi\sigma^2\xi\nu} + e^{2\pi\sigma^2\xi\nu}) \\ & \left(-\frac{\nu}{\xi} + \tanh(2\pi\sigma^2\xi\nu - \frac{\log(A)}{2})\right). \end{aligned}$$

So has the sign of $g(\nu) = -\frac{\nu}{\xi} + \tanh(2\pi\sigma^2\xi\nu - \frac{\log(A)}{2})$. Then:

$$g'(\nu) = \frac{-1 + 2\pi\sigma^2\xi^2\left(1 - \tanh^2(2\pi\sigma^2\xi\nu - \frac{\log(A)}{2})\right)}{\xi},$$

which is negative if $2\pi\sigma^2\xi^2 \leq 1$. In such a case, g is decreasing and as $\lim\limits_{\nu \to -\infty} g(\nu) = +\infty$ and $\lim\limits_{\nu \to +\infty} g(\nu) = -\infty$, q' annihilates once, and l has a unique extremum which is a maximum. Now when $2\pi\sigma^2\xi^2 > 1$, g' has two zeros, and one has the following table of variations:

ν	$-\infty$		ν_1		ν_2		$+\infty$
$g'(\nu)$		$-$	0	$+$	0	$-$	
$g(\nu)$	$+\infty$	$\searrow$	$g(\nu_1)$	$\nearrow$	$g(\nu_2)$	$\searrow$	$-\infty$

If $g(\nu_1) \geq 0$, then $g(\nu)$ and hence $q'(\nu)$ annihilates once and changes signs at some ν_2' in $]\nu_2, +\infty[$, at which q admits a maximum which is the unique extremum. If $g(\nu_2) \leq 0$, $g(\nu)$ and hence $q'(\nu)$ annihilates and changes signs once for a certain ν_1' in $]-\infty, \nu_1[$ and then q has a maximum at $\nu = \nu_1'$ which is its unique extremum. If $g(\nu_1) < 0$ and $g(\nu_2) > 0$, $g(\nu)$ and $q'(\nu)$ annihilate and change sign 3 times, once in $]-\infty, \nu_1[$, once in $]\nu_1, \nu_2[$, and once in $[\nu_2, +\infty[$, and thus q has three extrema. Now, Remember that ν_1 and ν_2 are the roots of g' thus also of $1 - \frac{1}{2\pi\sigma^2\xi^2} = \tanh^2(2\pi\sigma^2\xi\nu - \frac{\log(A)}{2})$ and therefore

$$\begin{aligned} \nu_1 &= \frac{1}{2\pi\sigma^2\xi}\left(\frac{\log(A)}{2} + \operatorname{artanh}\left(-\sqrt{1 - \frac{1}{2\pi\sigma^2\xi^2}}\right)\right) \\ \nu_2 &= \frac{1}{2\pi\sigma^2\xi}\left(\frac{\log(A)}{2} + \operatorname{artanh}\left(\sqrt{1 - \frac{1}{2\pi\sigma^2\xi^2}}\right)\right), \end{aligned}$$

and thus as $\alpha = \sqrt{\frac{\pi}{2}}\sigma(\xi_2 - \xi_1) = \sqrt{2\pi}\sigma\xi$, we may write:

$$\begin{aligned} g(\nu_1) &= -\frac{\log(A)}{2\alpha^2} - \frac{\log(\frac{\alpha-\sqrt{\alpha^2-1}}{\alpha+\sqrt{\alpha^2-1}})}{2\alpha^2} - \frac{\sqrt{\alpha^2-1}}{\alpha} \\ g(\nu_2) &= -\frac{\log(A)}{2\alpha^2} - \frac{\log(\frac{\alpha+\sqrt{\alpha^2-1}}{\alpha-\sqrt{\alpha^2-1}})}{2\alpha^2} + \frac{\sqrt{\alpha^2-1}}{\alpha}. \end{aligned}$$

From this we deduce that $g(\nu_1) < 0$ and $g(\nu_2) > 0$, q has three extrema, if and only if: $|\log(A)| < 2\alpha\sqrt{\alpha^2-1} - \log(\frac{\alpha+\sqrt{\alpha^2-1}}{\alpha-\sqrt{\alpha^2-1}}) = 2\alpha\sqrt{\alpha^2-1} - 2\operatorname{arcosh}(\alpha)$.

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