URN MODELS AND BETA-SPLINES

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1. Introduction

A well-established connection exists between discrete urn models and the standard curves and surfaces used in computer-aided geometric design (CAGD) [1],[2],[3],[4],[5]. The Bézier and B-spline blending functions both model elementary stochastic processes, and many of the geometric properties of Bézier and B-spline curves and surfaces can be derived by studying these probabilistic models [4],[5]. Recently Barsky has introduced a new type of spline into CAGD called the beta-spline [6],[7],[8]. The purpose of this paper is to try to gain some insight into the properties of beta-splines by applying the techniques of urn models.

2. Beta-Splines

Beta-splines are generalizations of B-splines. They were developed in order to replace the somewhat artificial concept of parametric continuity by the more natural notion of geometric continuity. Briefly the idea is this: Two curves $L(t)$ $t_0 \le t \le t_1$, $R(u)$ $u_0 \le u \le u_1$ are said to meet with n^{th}-order parametric continuity (C^n) if and only if

$$\left.\frac{d^kR}{du^k}\right|_{u=u_0} = \left.\frac{d^kL}{dt^k}\right|_{t=t_1} \qquad k = 0,1,\ldots,n$$

Unfortunately this definition depends on more than just the geometry of the curves $L(t),R(u)$; it also depends on the specific choice of their parametric representations. A linear change of parameter $u=\beta v$ $\beta>0$ will not change the shape of the curve $R(u)$, but by the chain rule

$$\left.\frac{d^kR}{dv^k}\right|_{v=v_0} = \beta^k \left.\frac{d^kR}{du^k}\right|_{u=u_0} = \beta^k \left.\frac{d^kL}{dt^k}\right|_{t=t_1} \neq \left.\frac{d^kL}{dt^k}\right|_{t=t_1}$$

Thus $R(\beta v)$ and $L(t)$ do not meet with n^{th}-order parametric continuity even though the curves $R(\beta v),R(u)$ are geometrically identical. To rectify this anomaly, the concept of geometric continuity is introduced.

Two curves $L(t),R(u)$ are said to meet with linear n^{th}-order geometric continuity (LG^n) if and only if there exists a constant $\beta>0$ such that

$$\left.\frac{d^kR}{du^k}\right|_{u=u_0} = \beta^k \left.\frac{d^kL}{dt^k}\right|_{t=t_1} \qquad k = 0,1,\ldots,n$$

It is easy to check that the notion of linear n^{th}-order geometric continuity is invariant under linear changes of parameter. Of course, this concept is not invariant under non-linear changes of parameter. A more general notion of geometric continuity (G^n) and more general constraint equations invariant under non-linear changes of parameter are given in [8].

Splines have typically been defined in terms of parametric continuity, and the B-splines form a convenient basis for these parametric splines. The more general notion of geometric continuity requires us to search for a new set of basis functions suitable for these new types of splines. These basis functions are called beta-splines. We shall now use urn models to construct beta-splines and study their properties.

3. An Urn Model for Beta-Splines

Consider an urn initially containing w white balls and b black balls. One ball at a time is drawn at random from the urn, its color inspected, and then returned to the urn. If the ball was the j^{th} white ball to be chosen, then $\beta^j(w+b)$ additional black balls are added to the urn; if the ball was the j^{th} black ball to be chosen, then $\beta^{-j}(w+b)$ additional white balls are added to the urn.

We now introduce the following notation:

$t = \frac{w}{w+b}$ = probability of selecting a white ball on the first trial

$\sigma_j(\beta) = 1 + \beta + \ldots + \beta^{j-1}$

$s_j^N(t) = s_j^N(\beta,t)$ = probability of selecting a white ball after selecting exactly j white balls in the first N trials

$f_j^N(t) = f_j^N(\beta,t)$ = probability of selecting a black ball after selecting exactly j white balls in the first N trials

$B_j^N(t) = B_j^N(\beta,t)$ = probability of selecting <u>exactly</u> j white balls in the first N trials

For each fixed β it can be shown that the functions $B_0^N(t),\ldots,B_N^N(t)$ are linearly independent polynomials of degree N and they satisfy the constraint equations(*)

$$(*)\quad \left.\frac{d^k B_{j-1}^N}{dt^k}\right|_{t=0} = \left.\beta^k \frac{d^k B_j^N}{dt^k}\right|_{t=1} \qquad k = 0,1,\ldots,N-1$$

The functions $B_0^N(t),\ldots,B_N^N(t)$ are the beta-spline basis functions. If $\beta=1$, these functions are the uniform B-spline basis functions and the urn model is the standard urn model for B-splines [1], [4].

Given a sequence of control points $P=(P_0,\ldots,P_M)$, we can use these beta-spline basis functions as blending functions to construct LG^{N-1} continuous beta-spline curves in much the same way that we use the uniform B-spline basis functions to define C^{N-1} continuous B-spline curves. Define the i^{th} curve segment by setting

$$B_i[\beta,P](t) = \sum_{j=0}^{N} B_j^N(\beta,t-i)P_{i+j} \qquad i \le t \le i+1$$

and define the beta-spline curve by setting

$$B[\beta,P](t) = B_i[\beta,P](t) \qquad i \le t \le i+1 \qquad 0 \le i \le M-N$$

From the constraint equations (*) it follows immediately that

$$\left.\frac{d^k B_{i+1}[\beta,P]}{dt^k}\right|_{t=i+1} = \left.\frac{\beta^k d^k B_i[\beta,P]}{dt^k}\right|_{t=i+1} \qquad k = 0,1,\ldots,N-1$$

Thus $B[\beta,P](t)$ is an LG^{N-1} continuous beta-spline curve.

Without moving the control points, we can alter the shape of the beta-spline curve $B[\beta,P](t)$ simply by changing the scalar parameter β. The effect of increasing β is to move the curve closer to its control polygon and to bias the curve towards its initial control points. Thus our β corresponds to Barsky's bias parameter β_1 [7].

In table 1 we summarize those properties of beta-spline basis functions and beta-spline curves which are directly derivable from the beta-spline urn model. Many of these properties are new and are presented here for the first time.

4. Conclusion

Urn models can be used to construct beta-spline basis functions and to derive the basic properties of these blending functions and the corresponding beta-spline curves. This is only the beginning; much work remains to be done. Here we have dealt only with the simple notion of linear geometric continuity and with the most elementary beta parameter. Non-linear geometric continuity leads to additional beta parameters and to more complicated basis functions [8]. Whether urn models can give us any insight into these higher order concepts still remains to be investigated.

References

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TABLE 1 - PROPERTIES OF BETA-SPLINE BASIS FUNCTIONS AND CURVES

	Urn	Basis Functions	Curve
1.	Probability Distribution	==> $\sum B_j^N(t) = 1$	==> Coordinate Free ⇓
		==> $B_j^N(t) \geq 0 \quad 0 \leq t \leq 1$	==> Local Convex Hull Property
2.	Symmetry between white, β and black, β^{-1}	==> Symmetry Formula $B_j^N(\beta,t) = B_{N-j}^N(\beta^{-1},1-t)$	==> Curve Symmetry $B[\beta,P](t) = B[\beta^{-1},P_{Reverse}](M-N+1-t)$
3.	Counting	==> Explicit Formulas	==> Recursion Formula
		$f_j^N(t) = \frac{\beta^{N-j}(\sigma_{j+1}(\beta)-t)}{\sigma_{N+1}(\beta)}$	==> Locality of End Points
		$s_j^N(t) = \frac{\beta^{N-j}t + \sigma_{N-j}(\beta)}{\sigma_{N+1}(\beta)}$	==> Explicit Recursion Formula (see 4)
		$B_0^N(t) = \frac{\beta^{N(N-1)/2}(1-t)^N}{\prod_{j=1}^{N} \sigma_j(\beta)}$	==> P_i does not affect $\left.\frac{d^k B_i[\beta,P]}{dt^k}\right\|_{t=i+1}$ $k=0,1,\ldots,N-1$
		$B_N^N(t) = \frac{t^N}{\prod_{j=1}^{N} \sigma_j(\beta)}$	==> P_{i+N} does not affect $\left.\frac{d^k B_i[\beta,P]}{dt^k}\right\|_{t=i}$ $k=0,1,\ldots,N-1$
4.	Relationship between first N and first N+1 picks	==> Recursion Formula $B_j^{N+1}(t) = f_j^N(t)B_j^N(t) + s_{j-1}^N(t)B_{j-1}^N(t)$	==> Geometric Construction Algorithm Let $i \leq t \leq i+1$ $P_j^0(t) = P_{i+j} \quad 0 \leq j \leq N$ $P_j^L(t) = f_j^{N-L}(t-i)P_j^{L-1}(t) + s_j^{N-L}(t-i)P_{j+1}^{L-1}(t)$ Then $B[\beta,P](t) = P_0^N(t)$
5.	Recursion Formula	==> Polynomial Functions	==> Polynomial Spline
		==> Differentiability Constraints $\left.\frac{d^k B_{j-1}^N}{dt^k}\right\|_{t=0} = \beta^k \left.\frac{d^k B_j^N}{dt^k}\right\|_{t=1}$	==> LG^{N-1}
6.	Linear Independence of first N+1 σ-moments	==> Linear Independence	==> Locally Non-Degenerate
		==> Polynomial Basis	==> Local Subdivision Algorithm
7.	Limiting Conditions	Limits	Tension/Bias
	If $\beta = \infty$ at most 1 white ball can be selected.	==> $\lim_{\beta \to \infty} B_j^N(t) = (1-t)^N \quad j=0$ $= 1-(1-t)^N \quad j=1$ $= 0 \quad j \neq 0,1$	==> $\lim_{\beta \to \infty} B[\beta,P](t)$ is the polygon determined by $P_0,\ldots,P_{M-N+1}$.
	If $\beta = 0$ at most 1 black ball can be selected	==> $\lim_{\beta \to 0} B_j^N(t) = t^N \quad j=N$ $= 1-t^N \quad j=N-1$ $= 0 \quad j \neq N-1,N$	==> $\lim_{\beta \to 0} B[\beta,P](t)$ is the polygon determined by $P_{N-1},\ldots,P_M$.
8.	Adding only balls of the opposite color	==> Total Positivity	==> Variation Diminishing Property
9.	Two urns each with two colors	==> Independent Distributions $B_{ij}^{MN}(s,t) = B_i^M(s)B_j^N(t)$	==> Rectangular Tensor Product Surfaces