Geometric algebra: a computational framework for geometrical applications

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Abstract

Geometric algebra is a consistent computational framework in which to define geometric primitives and their relationships. This algebraic approach contains all geometric operators and permits specification of constructions in a totally coordinate-free manner. Since it contains primitives of any dimensionality (rather than just vectors) it has no special cases: all intersections of primitives are computed with one general incidence operator.

This paper gives an introduction to the elements of geometric algebra to aid assessment of its potential for geometric programming. It contains no really new results, but collects known elements of relevance to computer graphics.

Keywords: Geometric algebra, geometric programming.

1 Introduction

In the usual way of defining geometrical objects in fields like computer graphics, robotics and computer vision, one uses vectors to characterize the constructions. To do this effectively, the basic concept of a vector as an element of a linear space is extended by an inner product and a cross product, and some additional constructions such as homogeneous coordinates and Grassmann spaces (see [8]) to encode compactly the intersection of, for instance, offset planes in space. Many of these techniques work rather well in 3-dimensional space, although some problems have been pointed out: the difference between vectors and points [7], and the affine non-covariance of the normal vector as a characterization of a tangent line or tangent plane (i.e. the normal vector of a transformed plane is not the transform of the normal vector). These problems are then traditionally fixed by the introduction of data structures and combination rules; object-oriented programming can be used to implement this patch tidily [14].

Yet there are deeper issues in geometric programming that are still accepted as 'the way things are'. For instance, when you need to intersect linear subspaces, the intersection algorithms are split out in treatment of the various cases: lines and planes,

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planes and planes, lines and lines, et cetera, need to be treated in separate pieces of code. The linear algebra of the systems of equations with its vanishing determinants indicates changes in essential degeneracies, and finite and infinite intersections can be nicely unified by using homogeneous coordinates. But there seems no getting away from the necessity of separating the cases. After all, the outcomes themselves can be points, lines or planes, and those are essentially different in their further processing.

Yet this need not be so. If we could see subspaces as basic elements of computation, and do direct algebra with them, then algorithms and their implementation would not need to split their cases on dimensionality. For instance, $A \wedge B$ could be 'the subspace spanned by the spaces A and B', the expression $A \cdot B$ could be 'the part of B perpendicular to A'; and then we would always have the computation rule $(A \wedge B) \cdot C = A \cdot (B \cdot C)$ since computing the part of C perpendicular to the span of A and B can be computed in two steps, perpendicularity to B followed by perpendicularity to A. Subspaces therefore have computational rules of their own that can be used immediately, independent of how many vectors were used to span them (i.e. independent of their dimensionality). In this view, the split in cases for the intersection could be avoided, since intersection of subspaces always leads to subspaces. We should consider using this structure, since it would enormously simplify the specification of geometric programs.

This paper intends to convince you that subspaces form an algebra with well-defined products that have direct geometric significance. This algebra can then be used as a language for geometry, and we claim that it is a better choice than a language always reducing everything to vectors (which are just 1-dimensional subspaces). Along the way, we will see that this framework allows us to divide by vectors (in fact, we can divide by any subspace), and we will see several familiar computer graphics constructs (quaternions, normals) that fold in naturally to the framework and need no longer be considered as special cases.

It comes as a bit of a surprise that there is really one basic product between subspaces that forms the basis for such an algebra, namely the *geometric product*. The algebra is then what mathematicians call a Clifford algebra. But for applications, it is convenient to consider 'components' of this geometric product; this gives us sensible extensions, to subspaces, of the inner product (computing measures of perpendicularity), the cross product (computing measures of parallelness), and the meet and join (computing intersection and union of subspaces). When used in such an obviously geometrical way, the term *geometric algebra* is preferred to describe the field.

In this paper, we will use the basic products of geometric algebra to describe all familiar elementary constructions of basic geometric objects and their quantitative relationships. The goal is to show you that this can be done, and that it is compact, directly computational, and transcends the dimensionality of subspaces. We will not use geometric algebra to develop new algorithms for graphics; but we hope to convince you that some of the lower level algorithmic aspects can be taken care of in an automatic way, without exceptions or hidden degenerate cases by using geometric algebra as a language – instead of using only its vector algebra part as in the usual approach.

Since subspaces are the main 'objects' of geometric algebra we introduce them first, which we do by combining vectors that span the subspace in Section 2. We then introduce the geometric product, and then look at products derived from the geometric product in Section3. Some of the derived products, like the inner and outer products, are so basic that it is natural to treat them in this section also, even though the geometric product is all we really need to do geometric algebra. Other products (such meet, join, rotation and projection through 'sandwiching') are better introduced in the context of their geometrical meaning, and we develop them in later sections. This approach reduces the amount of new notation, but it may make it seem as if geometric algebra needs to invent a new technique for every new kind of geometrical operation one wants to embed. This is *not* the case: all you need is the geometric product and (anti-)commutation.

After introducing the products in Section 3, we give some examples in Section 4 of how these products can be used in elementary but important ways. In Section 5, we look at more advanced topics such as differentiation, linear algebra, and homogeneous representation spaces for affine geometry.

2 Subspaces as elements of computation

As in the classical approach, we start with a real vector space V^m that we use to denote 1-dimensional directed magnitudes. Typical usage would be to employ a vector to denote a translation in such a space, to establish the location of a point of interest. (Points are not vectors, but their locations are [7].) Another usage is to denote the velocity of a moving point. (Points are not vectors, but their velocities are.) We now want to extend this capability of indicating directed magnitudes to higher-dimensional directions such as facets of objects, or tangent planes. We will start with the simplest subspaces: the 'proper' subspaces of a linear vector space, which are lines, planes, etcetera through the origin, and develop their algebra of spanning and perpendicularity measures. Section 5.4 uses the same algebra to treat "offset" subspaces; [10] uses it for spheres.

2.1 Constructing subspaces

So we start with a real m-dimensional linear space V^m, of which the elements are called *vectors*. We will always view vectors geometrically: a vector denotes a '1-dimensional direction element', with a certain 'attitude' or 'stance' in space, and a 'magnitude', a measure of length in that direction. These properties are well characterized by calling a vector a 'directed line element', as long as we mentally associate an orientation and magnitude with it: $\mathbf{v}$ is not the same as $-\mathbf{v}$ or $2\mathbf{v}$.

Algebraic properties of these geometrical vectors are: they can be added, weighted with real coefficients, in the usual way to produce new vectors; and they can be multiplied using an *inner product*, to produce a scalar $\mathbf{a} \cdot \mathbf{b}$ (in all of this paper, we use a metric vector space with well-defined inner product).

In geometric algebra, higher-dimensional oriented subspaces are also basic elements of computation. They are called *blades*, and we use the term *k-blade* for a k-dimensional homogeneous subspace. So a vector is a 1-blade.

A common way of constructing a blade is from vectors, using a product that constructs the span of vectors. This product is called the *outer product* (sometimes the

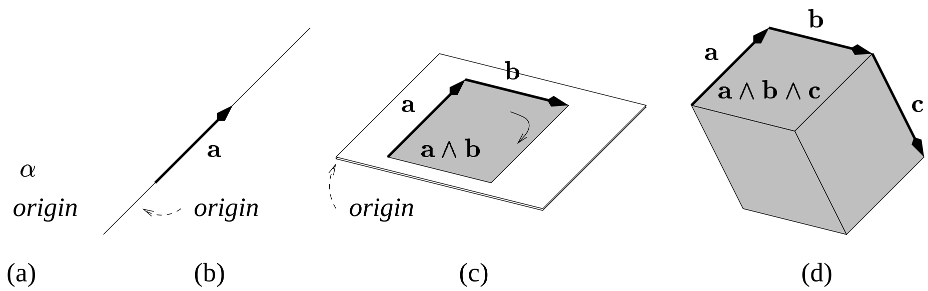

Figure 1: *Spanning proper subspaces using the outer product.*

wedge product) and denoted by $\wedge$. It is codified by its algebraic properties, which have been chosen to make sure we indeed get m-dimensional space elements with an appropriate magnitude (area element for $m = 2$, volume elements for $m = 3$; see Figure 1). As you have seen in linear algebra, such magnitudes are determinants of matrices representing the basis of vectors spanning them. But such a definition would be too specifically dependent on that matrix representation. Mathematically, a determinant is viewed as an anti-symmetric linear scalar-valued function of its vector arguments. That gives the clue to the rather abstract definition of the outer product in geometric algebra:

> The *outer product* of vectors $\mathbf{a}_1, \cdots, \mathbf{a}_k$ is anti-symmetric, associative and linear in its arguments. It is denoted $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, and called a k-blade.

The only thing that is different from a determinant is that the outer product is *not* forced to be scalar-valued; and this gives it the capability of representing the 'attitude' of a k-dimensional subspace element as well as its magnitude.

2.2 2-blades in 3-dimensional space

Let us see how this works in the geometric algebra of a 3-dimensional space V^3. For convenience, let us choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in this space, relative to which we denote any vector. Now let us compute $\mathbf{a} \wedge \mathbf{b}$ for $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. By linearity, we can write this as the sum of six terms of the form $a_1b_2\mathbf{e}_1 \wedge \mathbf{e}_2$ or $a_1b_1\mathbf{e}_1 \wedge \mathbf{e}_1$. By anti-symmetry, the outer product of any vector with itself must be zero, so the term with $a_1b_1\mathbf{e}_1 \wedge \mathbf{e}_1$ and other similar terms disappear. Also by anti-symmetry, $\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2$, so some terms can be grouped. You may verify that the final result is

$$
\begin{aligned}
&\mathbf{a} \wedge \mathbf{b} = \\
&\quad = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\
&\quad = (a_1b_2 - a_2b_1)\,\mathbf{e}_1 \wedge \mathbf{e}_2 + (a_2b_3 - a_3b_2)\,\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3b_1 - a_1b_3)\,\mathbf{e}_3 \wedge \mathbf{e}_1 \qquad (1)
\end{aligned}
$$

We cannot simplify this further. Apparently, the axioms of the outer product permit us to decompose any 2-blade in 3-dimensional space onto a basis of three elements. This '2-blade basis' (also called 'bivector basis') $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1\}$ consists of 2-blades spanned by the basis vectors. Linearity of the outer product implies that the set of 2-blades forms a linear space on this basis. We will interpret this as the space of all *plane elements* or *area elements*.

Let us show that $\mathbf{a} \wedge \mathbf{b}$ indeed has the correct magnitude for an area element. That is particularly clear if we choose a specific orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, chosen such that $\mathbf{a}$ lies in the $\mathbf{e}_1$-direction, and $\mathbf{b}$ lies in the $(\mathbf{e}_1, \mathbf{e}_2)$-plane (we can always do this). Then $\mathbf{a} = a\mathbf{e}_1$, $\mathbf{b} = b\cos\phi\,\mathbf{e}_1 + b\sin\phi\,\mathbf{e}_2$ (with ϕ the angle from $\mathbf{a}$ to $\mathbf{b}$), so that

$$\mathbf{a} \wedge \mathbf{b} = (a\,b\ \sin\phi)\,\mathbf{e}_1 \wedge \mathbf{e}_2 \tag{2}$$

This single result contains both the correct magnitude of the area $a\,b\ \sin\phi$ spanned by $\mathbf{a}$ and $\mathbf{b}$, and the plane in which it resides – for we recognize $\mathbf{e}_1 \wedge \mathbf{e}_2$ as 'the unit directed area element of the $(\mathbf{e}_1, \mathbf{e}_2)$-plane'. Since we can always adapt our coordinates to vectors in this way, this result is universally valid: $\mathbf{a} \wedge \mathbf{b}$ is an area element of the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ (see Figure 1c). Denoting the unit area element in the $(\mathbf{a}, \mathbf{b})$-plane by $\mathbf{I}$, the coordinate-free formulation of the above is

$$\mathbf{a} \wedge \mathbf{b} = (a\,b\ \sin\phi)\,\mathbf{I} \tag{3}$$

The result extends to blades of higher grades: each is proportional to the unit hypervolume element in its subspace, by a factor which is the hypervolume.

2.3 Volumes as 3-blades

We can also form the outer product of *three* vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Considering each of those decomposed onto their three components on some basis in our 3-dimensional space (as above), we obtain terms of three different types, depending on how many common components occur: terms like $a_1 b_1 c_1\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_1$, like $a_1 b_1 c_2\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$, and like $a_1 b_2 c_3\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Because of associativity and anti-symmetry, only the last type survives, in all its permutations. The final result is

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_1 c_3 - a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1)\,\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

The scalar factor is the determinant of the matrix with columns $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, which is proportional to the signed volume spanned by them (as is well known from linear algebra). The term $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the denotation of which volume is used as unit: that spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The order of the vectors gives its orientation, so this is a 'signed volume'. In 3-dimensional space, there is not really any other choice for the construction of volumes than (possibly negative) multiples of this volume (see Figure 1d). But in higher dimensional spaces, the attitude of the volume element needs to be indicated just as much as we needed to denote the attitude of planes in 3-space.

2.4 The pseudoscalar as hypervolume

Forming the outer product of *four* vectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}$ in 3-dimensional space will always produce zero (since they must be linearly dependent). The highest order blade that is non-zero in an m-dimensional space is therefore an m-blade. Such a blade, representing an m-dimensional volume element, is called a *pseudoscalar* for that space (for historical reasons); unfortunately a rather abstract term for the elementary geometric concept of 'hypervolume element'.

2.5 Scalars as subspaces

To make scalars fully admissible elements of the algebra we have so far, we can define the outer product of two scalars, and a scalar and a vector, through identifying it with the familiar scalar product in the vector space we started with:

$$\alpha \wedge \beta = \alpha\, \beta \quad \text{and} \quad \alpha \wedge \mathbf{v} = \alpha\, \mathbf{v}$$

Since the scalars are constructed by the outer product of zero vectors, we can interpret them geometrically as the representation of 0-dimensional subspace elements, i.e. as a *weighted points at the origin* – or maybe you prefer ‘charged’, since the weight can be negative. We will denote scalars mostly by Greek lower case letters.

2.6 The linear space of subspaces

Collating what we have so far, we have constructed a geometrically significant algebra containing only two operations: the addition $+$ and the outer multiplication $\wedge$ (subsuming the usual scalar multiplication). Starting from scalars and a 3-dimensional vector space we have generated a 3-dimensional space of 2-blades, and a 1-dimensional space of 3-blades (since all volumes are proportional to each other). In total, therefore, we have a set of elements that naturally group by their dimensionality. Choosing some basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write what we have as spanned by the set

$$\left\{ \underbrace{1}_{\textit{scalars}}\ ,\ \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\textit{vector space}}\ ,\underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_3 \wedge \mathbf{e}_1}_{\textit{bivector space}},\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\textit{trivector space}} \right\} \tag{4}$$

Every k-blade formed by $\wedge$ can be decomposed on the k-vector basis using $+$. The ‘dimensionality’ k is often called the *grade* or *step* of the k-blade or k-vector, reserving the term *dimension* for that of the vector space that generated them. A k-blade represents a k-dimensional oriented subspace element.

If we allow the scalar-weighted addition of arbitrary elements in this set of basis blades, we get an 8-dimensional linear space from the original 3-dimensional vector space. This space, with $+$ and $\wedge$ as operations, is called the *Grassmann algebra* of 3-space. In an m-dimensional space, there are $\binom{m}{k}$ basis elements of grade k, for a total basis of 2^m elements for the Grassmann algebra. The same basis is used for the geometric algebra of the space.

3 The Products of Geometric Algebra

In this section, we describe the geometric product, the most important product of geometric algebra. The fact that the geometric product can be applied to k-blades and has an inverse considerably extends algebraic techniques for solving geometrical problems. We can use the geometric product to derive other meaningful products. The most elementary are the inner and outer products, also discussed in this section; the useful but less elementary products giving reflections, rotations and intersection are treated later in this paper.

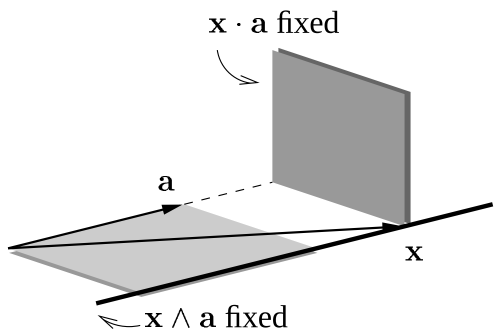

Figure 2: *Invertibility of the geometric product.*

3.1 The Geometric Product

For vectors in our metric vector space V^m, the geometric product is defined in terms of the inner and outer product as

$$\mathbf{a}\,\mathbf{b} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{5}$$

So the geometric product of two vectors is an element of mixed grade: it has a scalar (0-blade) part $\mathbf{a} \cdot \mathbf{b}$ and a 2-blade part $\mathbf{a} \wedge \mathbf{b}$. It is therefore *not* a blade; rather, it is an operator on blades (as we will soon show). Changing the order of $\mathbf{a}$ and $\mathbf{b}$ gives

$$\mathbf{b}\,\mathbf{a} \equiv \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$$

The geometric product of two vectors is therefore neither fully symmetric (unlike the inner product), nor fully anti-symmetric (unlike the outer product).

A simple drawing may convince you that the geometric product is indeed invertible, whereas the inner and outer product separately are not. In Figure 2, we have a given vector $\mathbf{a}$. We have indicated the set of vectors $\mathbf{x}$ with the same value of the inner product $\mathbf{x} \cdot \mathbf{a}$ – this is a plane perpendicular to $\mathbf{a}$. The set of all vectors with the same value of the outer product $\mathbf{x} \wedge \mathbf{a}$ is also indicated – this is the line of all points that span the same directed area with $\mathbf{a}$ (since for any point $\mathbf{p} = \mathbf{x} + \lambda\mathbf{a}$ on that line, we have $\mathbf{p} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a} + \lambda\mathbf{a} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}$ by the anti-symmetry property). Neither of these sets is a singleton (in spaces of more than 1 dimension), so the inner and outer products are not fully invertible. The geometric product provides both the plane and the line, and therefore permits determining their unique intersection $\mathbf{x}$, as illustrated in the figure. Therefore it is invertible.

Eq.(5) defines the geometric product *only for vectors*. For arbitrary elements of our algebra it is defined using linearity, associativity and distributivity over addition; and we make it coincide with the usual scalar product in the vector space, as the notation already suggests. That gives the following axioms (where α and β are scalars, $\mathbf{x}$ is a vector, A, B, C are general elements of the algebra):

$$\text{scalars} \qquad \alpha\,\beta \;\text{ and }\; \alpha\,\mathbf{x} \;\text{ have their usual meaning in } V^m \tag{6}$$

$$\text{scalars commute} \qquad \alpha\, A = A\, \alpha \tag{7}$$

$$\text{vectors} \qquad \mathbf{x}\, \mathbf{a} = \mathbf{x} \cdot \mathbf{a} + \mathbf{x} \wedge \mathbf{a} \tag{8}$$

$$\text{associativity} \qquad A\, (B\, C) = (A\, B)\, C \tag{9}$$

We have thus defined the geometric product in terms of inner and outer product; yet we claimed that it is more fundamental than either. Mathematically, it is more pure to replace eq.(8) by 'the square of a vector $\mathbf{x}$ is a scalar $Q(\mathbf{x})$'. This function Q can then actually be interpreted as the metric of the space, the same as the one used in the inner product, and it gives the same geometric algebra. Our choice was to define the new product in terms of more familiar quantities, to aid your intuitive understanding of it.

It may not be obvious that these equations give enough information to compute the geometric product of arbitrary elements. Rather than show this abstractly, let us show by example how the rules can be used to develop the geometric algebra of 3-dimensional Euclidean space. We introduce, for convenience only, an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^{3}$. Since this implies that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, we get the commutation rules

$$\mathbf{e}_i \mathbf{e}_j = \begin{cases} -\mathbf{e}_j \mathbf{e}_i & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{10}$$

In fact, the former is equal to $\mathbf{e}_i \wedge \mathbf{e}_j$, whereas the latter equals $\mathbf{e}_i \cdot \mathbf{e}_i$. Considering the unit 2-blade $\mathbf{e}_1 \wedge \mathbf{e}_2$, we find its square:

$$\begin{aligned} (\mathbf{e}_i \wedge \mathbf{e}_j)^2 &= (\mathbf{e}_i \wedge \mathbf{e}_j)\,(\mathbf{e}_i \wedge \mathbf{e}_j) = (\mathbf{e}_i\, \mathbf{e}_j)\,(\mathbf{e}_i\, \mathbf{e}_j) \\ &= \mathbf{e}_i\, \mathbf{e}_j\, \mathbf{e}_i\, \mathbf{e}_j = -\mathbf{e}_i\, \mathbf{e}_i\, \mathbf{e}_j\, \mathbf{e}_j = -1 \end{aligned} \tag{11}$$

So a unit 2-blade squares to -1 (we just computed for $\mathbf{e}_1 \wedge \mathbf{e}_2$ for convenience, but there is nothing exceptional about that particular unit 2-blade, since the basis was arbitrary). Continued application of eq.(10) gives the full multiplication for all basis elements in the Clifford algebra of 3-dimensional space. The resulting multiplication table is given in Figure 3. Arbitrary elements are expressible as a linear combination of these basis elements, so this table determines the full algebra.

3.1.1 Exponential representation

Note that the geometric product is sensitive to the relative directions of the vectors: for parallel vectors $\mathbf{a}$ and $\mathbf{b}$, the outer product contribution is zero, and $\mathbf{a}\,\mathbf{b}$ is a scalar and commutative in its factors; for perpendicular vectors, $\mathbf{a}\,\mathbf{b}$ is a 2-blade, and anti-commutative. In general, if the angle between $\mathbf{a}$ and $\mathbf{b}$ is ϕ in their common plane with unit 2-blade $\mathbf{I}$, we can write (in a Euclidean space)

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(\cos\phi + \mathbf{I}\,\sin\phi) \tag{12}$$

using a common rewriting of the inner product, and eq.(3). We have seen above that $\mathbf{I}\,\mathbf{I} = -1$, and this permits the shorthand of the exponential notation (by the usual definition of the exponential as a converging series of terms)

$$\mathbf{a}\,\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(\cos\phi + \mathbf{I}\,\sin\phi) = |\mathbf{a}|\,|\mathbf{b}|\,e^{\mathbf{I}\phi}. \tag{13}$$

$\mathcal{C}\ell_3$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{e}_{123}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Figure 3: The multiplication table of the geometric algebra of 3-dimensional Euclidean space, on an orthonormal basis. Shorthand: $\mathbf{e}_{12} \equiv \mathbf{e}_1 \wedge \mathbf{e}_2$, etcetera.

All this is reminiscent of complex numbers, but it really is different. Firstly, geometric algebra has given a straightforward real geometrical interpretation to all elements occurring in this equation, notably of $\mathbf{I}$ as the unit area element of the common plane of $\mathbf{a}$ and $\mathbf{b}$. Secondly, the math differs: if $\mathbf{I}$ were a complex scalar, it would have to commute with *all* elements of the algebra by eq.(7), but instead it satisfies $\mathbf{a}\,\mathbf{I} = -\mathbf{I}\mathbf{a}$ for vectors $\mathbf{a}$ in the $\mathbf{I}$-plane. We will use the exponential notation a lot when we study rotations, in Section 4.6.

3.1.2 Grades in the geometric product

It is a consequence of the definition of the geometric product that 'a vector squares to a scalar': the geometric product of a vector with itself is a scalar.

When you multiply two blades, the vectors in them may multiply to a scalar (if they are parallel) or to a 2-blade (if they are not). As a consequence, when you multiply two blades of grade k and ℓ, the result contains parts of grade $(k+\ell), (k+\ell-2), \cdots, (|k-\ell|+2), |k-\ell|$, just depending on how their factors align. This series of terms contains *all* information about their geometrical relationships.

3.2 The inner product

In geometric algebra, the inner product is the symmetrical part of the geometric product of two vectors:

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a})$$

Just as in the usual definition, this embodies the metric of the vector space and can be used to define distances. It also codifies the perpendicularity required in projection operators. Now that vectors are viewed as representatives of 1-dimensional subspaces, we of course want to extend this metric capability to arbitrary subspaces. The inner product is generalized to general subspaces as:

> $\mathbf{B} \cdot \mathbf{C}$ is the blade representing the largest subspace that is contained in the subspace $\mathbf{C}$ and that is perpendicular to the subspace $\mathbf{B}$; it is linear

in $\mathbf{B}$ and $\mathbf{C}$; it coincides with the usual inner product $\mathbf{b} \cdot \mathbf{c}$ of V^m when computed for vectors $\mathbf{b}$ and $\mathbf{c}$.

The above determines the inner product uniquely.[1] It turns out not to be symmetrical (as one would expect since the definition is asymmetrical) and also not associative. But we do demand linearity, to make it computable between any two elements in our linear space (not just blades).

Here we just give the rules by which to compute the resulting inner product for arbitrary blades, omitting their derivation (essentially as in [13]). In the following α, β are scalars, $\mathbf{a}$ and $\mathbf{b}$ vectors and A, B, C general elements of the algebra.

$$\begin{array}{rll} \text{scalars} & \alpha \cdot \beta = \alpha\,\beta & (14) \\ \text{vector and scalar} & \mathbf{a} \cdot \beta = 0 & (15) \\ \text{scalar and vector} & \alpha \cdot \mathbf{b} = \alpha\,\mathbf{b} & (16) \\ \text{vectors} & \mathbf{a} \cdot \mathbf{b} \text{ is the usual inner product in } V^m & (17) \\ \text{vector and element} & \mathbf{a} \cdot (\mathbf{b} \wedge B) = (\mathbf{a} \cdot \mathbf{b}) \wedge B - \mathbf{b} \wedge (\mathbf{a} \cdot B) & (18) \\ \text{distribution} & (A \wedge B) \cdot C = A \cdot (B \cdot C) & (19) \end{array}$$

As we said, linearity and distributivity over $+$ also hold, but the inner product is *not* associative.

When used on blades as $(\mathbf{A} \wedge \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$, rule eq.(19) gives the inner product its meaning of being the perpendicular part of a subspace. In words it would read: 'to get the part of $\mathbf{C}$ perpendicular to the subspace that is the span of $\mathbf{A}$ and $\mathbf{B}$, take the part of $\mathbf{C}$ perpendicular to $\mathbf{B}$; then of that, take the part perpendicular to $\mathbf{A}$'.

Figure 4a gives an example: the inner product of a vector $\mathbf{x}$ and a 2-blade $\mathbf{A}$, producing the vector $\mathbf{x} \cdot \mathbf{A}$.

3.2.1 Perpendicularity and dualization

We define the concept of perpendicularity through the inner product:

$$\mathbf{a} \text{ perpendicular to } \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{a} \cdot \mathbf{A} = 0,$$

It is then easy to prove that, for general blades $\mathbf{A}$, the construction $\mathbf{A} \cdot \mathbf{B}$ is indeed perpendicular to $\mathbf{A}$, as we suggested in the previous section. This is especially useful when $\mathbf{B}$ is taken as (a multiple of) the pseudoscalar $\mathbf{I}_m$ of the surrounding space. The orthogonal complement or *dual* $\mathbf{A}^*$ of $\mathbf{A}$ is defined using the reverse of the pseudoscalar (see Section 3.2)

$$\mathbf{A}^* \equiv \mathbf{A} \cdot \widetilde{\mathbf{I}}_m. \tag{20}$$

We observe that we may as well write $\mathbf{A}^*$ as the geometric product $\mathbf{A}\,\mathbf{I}_m$ since $\mathbf{a} \wedge \mathbf{I}_m = 0$ for any vector $\mathbf{a}$ in the blade $\mathbf{A}$. This ostentatiously invertible form is often preferred. More about the dual in Section 4.8.

[1] The resulting inner product differs slightly from the inner product commonly used in the geometric algebra literature. Our inner product has a cleaner geometric semantics, and more compact mathematical properties, and that makes it better suited to computer science. It is sometimes called the *contraction*, and denoted as $\mathbf{B} \rfloor \mathbf{C}$ rather than $\mathbf{B} \cdot \mathbf{C}$. The two inner products can be expressed in terms of each other, so this is not a severely divisive issue. They 'algebraify' the same geometric concepts, in just slightly different ways. See also [5].

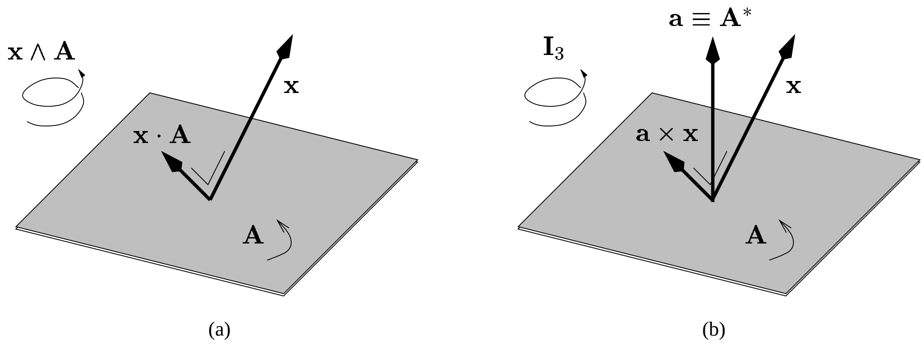

Figure 4: *(a) The inner product of blades. (b) Dual and cross product. In both figures, the corkscrew denotes the orientation of the trivector.*

3.2.2 norm/magnitude

The norm of a blade $\mathbf{A}$ is a scalar, defined in terms of the inner product of $\mathbf{A}$ with itself. In a Euclidean space, it is:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \widetilde{\mathbf{A}}}, \tag{21}$$

where $\widetilde{\mathbf{A}}$ is the *reverse* of $\mathbf{A}$, obtained by switching its spanning factors: if $\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k$, then $\widetilde{\mathbf{A}} = \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_2 \wedge \mathbf{a}_1$. The reverse of $\mathbf{A}$ differs from $\mathbf{A}$ by a sign $(-1)^{\frac{1}{2}k(k-1)}$. Using it in eq.(21) keeps norms of blades real-valued in Euclidean spaces.

For example, to compute the norm of a 2-blade $\mathbf{a} \wedge \mathbf{b}$, we find by the inner product rules: $|\mathbf{a} \wedge \mathbf{b}|^2 = (\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{b} \wedge \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{b} \wedge \mathbf{a})) = \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{b}) \wedge \mathbf{a} - \mathbf{b} \wedge (\mathbf{b} \cdot \mathbf{a})) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \phi) = (|\mathbf{a}||\mathbf{b}| \sin \phi)^2$. So the norm is indeed equal to the size of the area spanned by $\mathbf{a}$ and $\mathbf{b}$.

3.2.3 Grade of inner product

The result of an inner product $\mathbf{A} \cdot \mathbf{B}$ is an object of a lower grade than $\mathbf{B}$:

$$\mathsf{grade}\,(\mathbf{A} \cdot \mathbf{B}) = \mathsf{grade}\,(\mathbf{B}) - \mathsf{grade}\,(\mathbf{A})\,. \tag{22}$$

3.3 The outer product

We have already seen the outer product in Section 2, where it was used to construct the subspaces of the algebra. Once we have the geometric product, it is better to see the outer product as its anti-symmetric part:

$$\mathbf{a} \wedge \mathbf{b} = \tfrac{1}{2}(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a})$$

This then gives the defining properties we saw before (as before, α, β are scalars, $\mathbf{a}, \mathbf{b}$ are vectors, A, B, C : are general elements):

$$\begin{aligned} &\text{scalars} && \alpha \wedge \beta = \alpha\, \beta && (23)\\ &\text{scalar and vector} && \alpha \wedge \mathbf{b} = \alpha\, \mathbf{b} && (24)\\ &\text{anti-symmetry for vectors} && \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} && (25)\\ &\text{associativity} && (A \wedge B) \wedge C = A \wedge (B \wedge C) && (26) \end{aligned}$$

Linearity and distributivity over $+$ also hold.

3.3.1 Subspace objects without shape

We reiterate that the outer product of k-vectors gives a 'bit of k-space', in a manner that includes the attitude of the space element, its orientation (or 'handedness') and its magnitude. For a 2-blade $\mathbf{a} \wedge \mathbf{b}$, this was conveyed in eq.(3).

However, $\mathbf{a} \wedge \mathbf{b}$ is not an area element with well-defined shape, even though one is tempted to draw it as a parallelogram (as in Figure 1c). For instance, by the properties of the outer product, $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge (\mathbf{b} + \lambda \mathbf{a})$, for any λ, so $\mathbf{a} \wedge \mathbf{b}$ is just as much the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b} + \lambda \mathbf{a}$. Playing around, you find that you can move around pieces of the area elements while still maintaining the same sum $\mathbf{a} \wedge \mathbf{b}$; so really, a bivector does not really have any fixed shape or position, it is just a chunk of a precisely defined amount of 2-dimensional directed area in a well-defined plane. It follows that the 2-blades have an existence of their own, independent of any vectors that one might use to define them.

We will take these non-specific shapes made by the outer product and 'force them into shape' by cleverly chosen geometric products; this will turn out to be a powerful and flexible technique to get closed computational expressions for geometrical constructions.

3.3.2 Grade law for outer product; zero element

The grade of a k-blade is the number of vector factors that span it. It obeys the simple rule

$$\mathsf{grade}\,(\mathbf{A} \wedge \mathbf{B}) = \mathsf{grade}\,(\mathbf{A}) + \mathsf{grade}\,(\mathbf{B})\,. \tag{27}$$

Of course the outcome may be 0, so this zero element of the algebra should be seen as an element of arbitrary grade. There is then no need to distinguish separate zero scalars, zero vectors, zero 2-blades, etcetera.

3.3.3 Linear (in)dependence

Note that if three vectors are linearly dependent, they satisfy

$$\mathbf{a},\mathbf{b},\mathbf{c} \text{ linearly dependent} \quad \Longleftrightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0.$$

We interpret the latter immediately as the geometric statement that the vectors span a zero volume. This makes linear dependence a computational property rather than a predicate: three vectors can be 'almost linearly dependent'. The magnitude of $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ obviously involves the determinant of the matrix $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$, so this view corresponds with the usual computation of determinants to check degeneracy.

4 Basic geometric constructions

In the previous two sections, we showed how to construct subspaces, and introduced the geometric product and the inner and outer products. In this section, we will show how to use these products to perform geometric computations. We begin with a discussion of using the geometric product to manipulate algebraic equations involving subspaces. This algebraic manipulation typically has a strong geometric meaning, examples of which are illustrated in the remaining parts of this section.

4.1 Algebraic manipulation

The common procedure to find computational formulas for geometric objects in geometric algebra is as follows: we know certain defining properties of objects in the usual terms of perpendicularity, spanning, rotations etcetera. These give equations typically expressed using the derived products. You combine these equations algebraically, with the goal of finding an expression for the unknown object involving only the geometric product; then division (permitted by the invertibility of the geometric product) should provide the result.

Let us illustrate this by an example.

> Suppose we want to find the component $\mathbf{x}_\perp$ of a vector $\mathbf{x}$ perpendicular to a vector $\mathbf{a}$. The perpendicularity demand is clearly
>
> $$\mathbf{x}_\perp \cdot \mathbf{a} = 0.$$
>
> A second demand is required to relate the magnitude of $\mathbf{x}_\perp$ to that of $\mathbf{x}$. Some practice in 'seeing subspaces' in geometrical problems reveals that the area spanned by $\mathbf{x}$ and $\mathbf{a}$ is the same as the area spanned by $\mathbf{x}_\perp$ and $\mathbf{a}$. This is expressed using the outer product:
>
> $$\mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}.$$
>
> These two equations should be combined to form a geometric product. In this example, it is clear that just adding them works, yielding
>
> $$\mathbf{x}_\perp \cdot \mathbf{a} + \mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x}_\perp\, \mathbf{a} = \mathbf{x} \wedge \mathbf{a}.$$
>
> This one equation contains the full geometric relationship between $\mathbf{x}$, $\mathbf{a}$ and the unknown $\mathbf{x}_\perp$. Geometric algebra solves this equation through division on the right by $\mathbf{a}$:
>
> $$\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{a})/\mathbf{a} = (\mathbf{x} \wedge \mathbf{a})\, \mathbf{a}^{-1}. \tag{28}$$

We rewrote the division by $\mathbf{a}$ as multiplication by the subspace $\mathbf{a}^{-1}$ to show clearly that we mean 'division on the right'. This is an example of how the indefinite shape $\mathbf{x} \wedge \mathbf{a}$ spanned by the outer product is just the right element to generate a perpendicular to a vector $\mathbf{a}$ in its plane, through the geometric product. Figure 5a illustrates the geometry.

Another technique is the exploitation of the non-commutativity of the geometric product, by the 'sandwiching' blades which we will use below. It is also often useful to select the required grades of expressions to reduce them to simpler, solvable expressions. We will not do so much in this paper; the technique is applied with great skill in [3].

4.2 Invertibility of the geometric product

The geometric product is invertible, so 'dividing by a vector' has a unique meaning. We will usually do this through 'multiplication by the inverse of the vector'. Since multiplication is not necessarily commutative, we have to be a bit careful: there is a 'left division' and a 'right division'. As you may verify, the unique inverse of a vector $\mathbf{a}$ is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|^2}$$

since that is the unique element that satisfies: $\mathbf{a}^{-1}\,\mathbf{a} = 1 = \mathbf{a}\,\mathbf{a}^{-1}$. So this makes eq.(28) computable. In general, a blade $\mathbf{A}$ (of which the norm should not be zero) has the inverse

$$\mathbf{A}^{-1} = \frac{\widetilde{\mathbf{A}}}{\mathbf{A} \cdot \widetilde{\mathbf{A}}} = \frac{\widetilde{\mathbf{A}}}{|\mathbf{A}|^2}$$

where $\widetilde{\mathbf{A}}$ is the reverse of Section 3.2.

4.3 Projection of subspaces

The availability of an inverse gives us an interesting of way of decomposing a vector $\mathbf{x}$ relative to a given blade $\mathbf{A}$ using the geometric product. This uses $\mathbf{x}\,\mathbf{A} = \mathbf{x} \cdot \mathbf{A} + \mathbf{x} \wedge \mathbf{A}$, a straightforward provable generalization of eq.(8). The decomposition is

$$\mathbf{x} = (\mathbf{x}\,\mathbf{A})\,\mathbf{A}^{-1} = (\mathbf{x} \cdot \mathbf{A})\,\mathbf{A}^{-1} + (\mathbf{x} \wedge \mathbf{A})\,\mathbf{A}^{-1} \qquad (29)$$

The first term is a blade fully inside $\mathbf{A}$: it is the *projection* of $\mathbf{x}$ onto $\mathbf{A}$. The second term is a vector perpendicular to $\mathbf{A}$, sometimes called the *rejection* of $\mathbf{x}$ by $\mathbf{A}$. The projection of a blade $\mathbf{X}$ of arbitrary dimensionality (grade) onto a blade $\mathbf{A}$ is given by the extension of the above, as

$$\text{projection of } \mathbf{X} \text{ onto } \mathbf{A}: \quad \mathbf{X} \;\mapsto\; (\mathbf{X} \cdot \mathbf{A})\,\mathbf{A}^{-1}$$

Geometric algebra often allows such a straightforward extension to arbitrary dimensions of subspaces, without additional computational complexity. We will see why in Section 5.2.

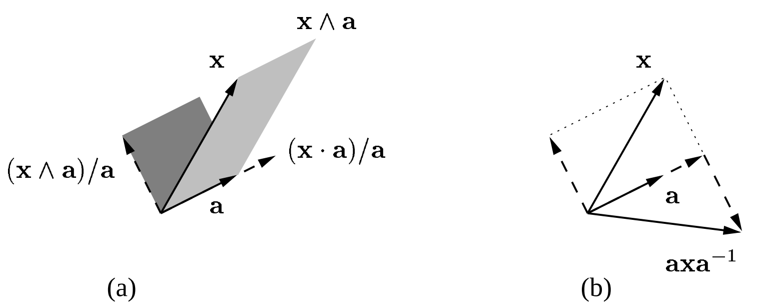

Figure 5: *(a) Projection and rejection of* $\mathbf{x}$ *relative to* $\mathbf{a}$*. (b) Reflection of* $\mathbf{x}$ *in* $\mathbf{a}$*.*

4.4 Reflection of subspaces

The *reflection* of a vector $\mathbf{x}$ relative to a fixed vector $\mathbf{a}$ can be constructed from the decomposition of eq.(29) (used for a vector $\mathbf{a}$), by changing the sign of the rejection (see Figure 5b). This can be rewritten in terms of the geometric product:

$$(\mathbf{x}\cdot\mathbf{a})\,\mathbf{a}^{-1} - (\mathbf{x}\wedge\mathbf{a})\,\mathbf{a}^{-1} = (\mathbf{a}\cdot\mathbf{x}+\mathbf{a}\wedge\mathbf{x})\,\mathbf{a}^{-1} = \mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}. \qquad (30)$$

So the reflection of $\mathbf{x}$ in $\mathbf{a}$ is the expression $\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}$, see Figure 5b; the reflection in a plane perpendicular to $\mathbf{a}$ is then $-\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}$,

We can extend this formula to the reflection of a blade $\mathbf{X}$ relative to the vector $\mathbf{a}$, this is simply

$$\text{reflection in vector } \mathbf{a}\text{:} \quad \mathbf{X} \;\mapsto\; \mathbf{a}\,\mathbf{X}\,\mathbf{a}^{-1}.$$

and even to the reflection of a blade $\mathbf{X}$ in a k-blade $\mathbf{A}$, which turns out to be

$$\text{general reflection:} \quad \mathbf{X} \;\mapsto\; -(-1)^k\,\mathbf{A}\,\mathbf{X}\,\mathbf{A}^{-1}.$$

Note that these formulas permit you to do reflections of subspaces without first decomposing them in constituent vectors. It gives the possibility of reflecting a polyhedral object by directly using a facet representation, rather than acting on individual vertices.

4.5 Vector division

With subspaces as basic elements of computation, we can directly solve equations in similarity problems such as indicated in Figure 6:

> Given two vectors $\mathbf{a}$ and $\mathbf{b}$, and a third vector $\mathbf{c}$, determine $\mathbf{x}$ so that $\mathbf{x}$ is to $\mathbf{c}$ as $\mathbf{b}$ is to $\mathbf{a}$, i.e. solve $\mathbf{x} : \mathbf{c} = \mathbf{b} : \mathbf{a}$.

In geometric algebra the problem reads

$$\mathbf{x}\,\mathbf{c}^{-1} = \mathbf{b}\,\mathbf{a}^{-1},$$

and the solution is immediate

$$\mathbf{x} = (\mathbf{b}\,\mathbf{a}^{-1})\,\mathbf{c}. \qquad (31)$$

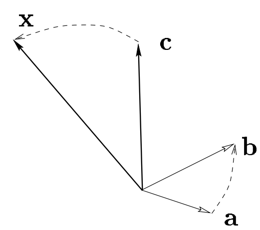

Figure 6: *Ratios of vectors*

This is a computable expression. For instance, with $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{c} = \mathbf{e}_2$ in the standard orthonormal basis, we obtain $\mathbf{x} = ((\mathbf{e}_1 + \mathbf{e}_2)\,\mathbf{e}_1^{-1})\mathbf{e}_2 = (1 - \mathbf{e}_1\,\mathbf{e}_2)\mathbf{e}_2 = \mathbf{e}_2 - \mathbf{e}_1$.

4.6 Rotations

Geometric algebra handles rotations of general subspaces in V^m, through an interesting 'sandwiching product' using geometric products. We introduce this construction gradually in this section.

4.6.1 Rotations in 2D

In the problem of Figure 6, if $\mathbf{a}$ and $\mathbf{b}$ have the same norm then we know that $\mathbf{x}$ must be related to $\mathbf{c}$ by a rotation in the plane of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We then obtain, using eq.(13)

$$\mathbf{x} = \mathsf{R}_{\mathbf{I}\phi}\mathbf{c} = (\mathbf{b}\,\mathbf{a}^{-1})\,\mathbf{c} = \frac{|\mathbf{b}|}{|\mathbf{a}|}(\cos\phi - \mathbf{I}\sin\phi)\mathbf{c} = e^{-\mathbf{I}\phi}\mathbf{c} \tag{32}$$

Here $\mathbf{I}\phi$ is the angle in the $\mathbf{I}$ plane from $\mathbf{a}$ to $\mathbf{b}$, as in eq.(13), so $-\mathbf{I}\phi$ is the angle from $\mathbf{b}$ to $\mathbf{a}$. Apparently we should interpret 'pre-multiplying by $e^{-\mathbf{I}\phi}$' as a *rotation operator* in the $\mathbf{I}$-plane. The full expression of eq.(31) denotes a rotation/dilation in the $\mathbf{I}$-plane.

> Let us consider the geometrical meaning of the terms of this rotation operation when re-expressed into its components:
>
> $$e^{-\mathbf{I}\phi}\,\mathbf{c} = \mathbf{c}\cos\phi - \mathbf{I}\mathbf{c}\sin\phi = \mathbf{c}\cos\phi + \mathbf{c}\mathbf{I}\sin\phi$$
>
> What is $\mathbf{c}\mathbf{I}$? Introduce orthonormal coordinates $\{\mathbf{e}_1, \mathbf{e}_2\}$ in the $\mathbf{I}$-plane, with $\mathbf{e}_1$ along $\mathbf{c}$, so that $\mathbf{c} \equiv c\,\mathbf{e}_1$. Then $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\,\mathbf{e}_2$. Therefore $\mathbf{c}\mathbf{I} = c\,\mathbf{e}_1\,\mathbf{e}_1\mathbf{e}_2 = c\,\mathbf{e}_2$: it is $\mathbf{c}$ *turned over a right angle*, following the orientation of the 2-blade $\mathbf{I}$ (here anti-clockwise). So $\mathbf{c}\cos\phi + \mathbf{c}\mathbf{I}\sin\phi$ is 'a bit of $\mathbf{c}$ plus a bit of its anti-clockwise perpendicular' – and those amounts are precisely right to make it equal to the rotation by ϕ, see Figure 7.

The vector $\mathbf{c}$ in the $\mathbf{I}$-plane anti-commutes with $\mathbf{I}$: $\mathbf{c}\,\mathbf{I} = -\mathbf{I}\,\mathbf{c}$. Using this to switch $\mathbf{I}$ and $\mathbf{c}$ in eq.(32), we obtain

$$\mathsf{R}_{\mathbf{I}\phi}\mathbf{c} = e^{-\mathbf{I}\phi}\mathbf{c} = \mathbf{c}\,e^{\mathbf{I}\phi}. \tag{33}$$

The planar rotation is therefore alternatively representable as a post-multiplication.

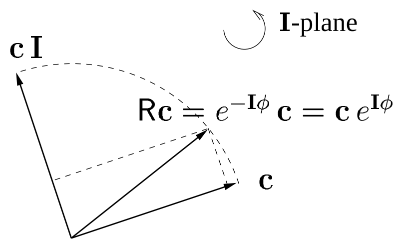

Figure 7: *Coordinate-free specification of rotation.*

4.6.2 Angles as geometrical objects

In eq.(32), the combination $\mathbf{I}\phi$ is a full indication of the angle between the two vectors: it denotes not only the magnitude, but also the plane in which the angle is measured, and even the orientation of the angle. If you would ask for the scalar magnitude of the geometrical quantity $\mathbf{I}\phi$ in the plane $-\mathbf{I}$ (the plane 'from $\mathbf{b}$ to $\mathbf{a}$' rather than 'from $\mathbf{a}$ to $\mathbf{b}$'), it is $-\phi$; so the scalar value of the angle automatically gets the right sign. The fact that the angle as expressed by $\mathbf{I}\phi$ is now a geometrical quantity independent of the convention used in its definition removes a major headache from many geometrical computations involving angles. We call this true geometric quantity the *bivector angle* (it is just a 2-blade, of course, not a new kind of element – but we use it as an angle, hence the name).

4.6.3 Rotations in m dimensions

The above rotates only within a plane; in general we would like to have spatial rotations. For a vector $\mathbf{x}$, the outcome of a rotation $\mathsf{R}_{\mathbf{I}\phi}$ should be:

$$\mathsf{R}_{\mathbf{I}\phi}\mathbf{x} = \mathbf{x}_\perp + \mathsf{R}_{\mathbf{I}\phi}\mathbf{x}_\|,$$

where $\mathbf{x}_\perp$ and $\mathbf{x}_\|$ are the perpendicular and parallel components of $\mathbf{x}$ relative to the rotation plane $\mathbf{I}$, respectively. We have seen that the separation of a vector into such components can be done by commutation (as in eq.(29) and eq.(30)). As you may verify, the following formula effects this separation and rotation simultaneously

$$\text{rotation over } \mathbf{I}\phi\text{:} \quad \mathbf{x} \;\mapsto\; \mathsf{R}_{\mathbf{I}\phi}\mathbf{x} = e^{-\mathbf{I}\phi/2}\,\mathbf{x}\,e^{\mathbf{I}\phi/2} \tag{34}$$

The operator $e^{-\mathbf{I}\phi/2}$, used in this way, is called a *rotor*. In the 2-dimensional rotation we treated before, $\mathbf{x}\,\mathbf{I} = -\mathbf{I}\,\mathbf{x}$, so moving either rotor 'to the other side of $\mathbf{x}$' retrieves eq.(33) if $\mathbf{x}$ is in the $\mathbf{I}$-plane.

Two successive rotations R_1 and R_2 are equivalent to a single new rotation R of which the rotor R is the geometric product of the rotors R_2 and R_1, since

$$(\mathsf{R}_2 \circ \mathsf{R}_1)\mathbf{x} = R_2\,(R_1\,\mathbf{x}\,R_1^{-1})\,R_2^{-1} = (R_2\,R_1)\,\mathbf{x}\,(R_2\,R_1)^{-1} = R\,\mathbf{x}R^{-1},$$

with $R = R_2\,R_1$. Therefore the combination of rotations is a simple consequence of the application of the geometric product on rotors, i.e. elements of the form $e^{-\mathbf{I}\phi/2} =$

$\cos\phi/2 - \mathbf{I}\sin\phi/2$, with $\mathbf{I}^2 = -1$. This is true in any dimension greater than 1 (and even in dimension 1 if you realize that any bivector there is zero, so that rotations do not exist).

> Let's see how it works in 3-space. In 3 dimensions, we are used to specifying rotations by a *rotation axis* $\mathbf{a}$ rather than by a *rotation plane* $\mathbf{I}$. The relationship between axis and plane is given by duality: $\mathbf{a} \equiv \mathbf{I}\cdot\widetilde{\mathbf{I}}_3 = -\mathbf{I}\,\mathbf{I}_3$ (you may wish to verify that this indeed gives the correct orientation). Given the axis $\mathbf{a}$, we find the plane as the 2-blade $\mathbf{I} = -\mathbf{a}\,\mathbf{I}_3^{-1} = \mathbf{a}\mathbf{I}_3 = \mathbf{I}_3\mathbf{a}$. A rotation over an angle ϕ around an axis with unit vector $\mathbf{a}$ is therefore represented by the rotor $e^{-\mathbf{I}_3\mathbf{a}\phi/2}$.
>
> To compose a rotation R_1 around the $\mathbf{e}_1$ axis of $\pi/2$ with a subsequent rotation R_2 over the $\mathbf{e}_2$ axis over $\pi/2$, we write out their rotors:
>
> $$R_1 = e^{-\mathbf{I}_3\mathbf{e}_1\pi/4} = \frac{1-\mathbf{e}_{23}}{\sqrt{2}} \quad \text{and} \quad R_2 = e^{-\mathbf{I}_3\mathbf{e}_2\pi/4} = \frac{1-\mathbf{e}_{31}}{\sqrt{2}}$$
>
> The total rotor is their product, and we rewrite it back to the exponential form to find the axis:
>
> $$\begin{aligned} R \equiv R_2\,R_1 &= \tfrac{1}{2}(1-\mathbf{e}_{23})\,(1-\mathbf{e}_{31}) = \tfrac{1}{2}(1-\mathbf{e}_{23}-\mathbf{e}_{31}-\mathbf{e}_{12}) \\ &= \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,\mathbf{I}_3\,\frac{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}{\sqrt{3}} \equiv e^{-\mathbf{I}_3\mathbf{a}\pi/3} \end{aligned}$$
>
> Therefore the total rotation is over the axis $\mathbf{a} = (\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, over the angle $2\pi/3$.

Again, geometric algebra permits straightforward generalization to the rotation of higher dimensional subspaces: a rotor can be applied immediately to an arbitrary blade through the formula

$$\text{general rotation:} \quad \mathbf{X} \;\mapsto\; R\,\mathbf{X}\,R^{-1}$$

This enables you to rotate a plane in one operation, for instance

$$R(\mathbf{e}_1\wedge\mathbf{e}_2)R^{-1} \;=\; \tfrac{1}{4}(1-\mathbf{e}_{23}-\mathbf{e}_{31}-\mathbf{e}_{12})\,\mathbf{e}_{12}\,(1+\mathbf{e}_{23}+\mathbf{e}_{31}+\mathbf{e}_{12}) = \mathbf{e}_{23}$$

There is no need to decompose the plane into its spanning vectors first.

4.7 Quaternions: based on bivectors

You may have recognized the example above as strongly similar to quaternion computations. Quaternions are indeed part of geometric algebra, in the following straightforward manner.

Choose an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$. Construct out of that a bivector basis $\{\mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}\}$. Note that these elements satisfy: $\mathbf{e}_{12}^2 = \mathbf{e}_{23}^2 = \mathbf{e}_{31}^2 = -1$, and $\mathbf{e}_{12}\,\mathbf{e}_{23} = \mathbf{e}_{13}$ (and cyclic) and also $\mathbf{e}_{12}\,\mathbf{e}_{23}\,\mathbf{e}_{31} = 1$. In fact, setting $i \equiv \mathbf{e}_{23}$, $j \equiv -\mathbf{e}_{31}$ and $k \equiv \mathbf{e}_{12}$, we find $i^2 = j^2 = k^2 = i\,j\,k = -1$ and $j\,i = k$ and cyclic. Algebraically these objects form a basis for quaternions obeying the quaternion product, commonly interpreted as some kind of '4-D complex number system'. There is nothing 'complex' about quaternions; but they are not really vectors either (as some still think) – they are just real 2-blades in 3-space, denoting elementary rotation planes, and multiplying through the geometric product. Visualizing quaternions is therefore straightforward: each is just a rotation plane with a rotation angle, and the 'bivector angle' concept of Section 4.6.2 represents that well.

So in geometric algebra, quaternions can be combined directly with vectors and other subspaces. In that algebraic combination, they are not merely a form of 'complex scalars': quaternion products are neither fully commutative nor fully anti-commutative (e.g. $i\,\mathbf{e}_1 = \mathbf{e}_1\,i$, but $i\,\mathbf{e}_2 = -\mathbf{e}_2\,i$). It all depends on the relative attitude of the vectors and quaternions, and these rules are precisely right to make eq.(34) be a rotation of a vector.

4.8 Dual representation

The dualization of eq.(20) enables manipulation of expressions involving 'spanning' to being about 'perpendicularity' and vice versa.

> A familiar example in a 3-dimensional Euclidean space is the dual of a 2-blade (or bivector). Using an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$ and the corresponding bivector basis, we write: $\mathbf{A} = b_1\mathbf{e}_2 \wedge \mathbf{e}_3 + b_2\mathbf{e}_3 \wedge \mathbf{e}_1 + b_3\mathbf{e}_3 \wedge \mathbf{e}_2$. We take the dual relative to the space with volume element $\mathbf{I}_3 \equiv \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ (i.e. the 'right-handed volume' formed by using a right-handed basis). The subspace of $\mathbf{I}_3$ dual to $\mathbf{A}$ is then
>
> $$\begin{aligned} \mathbf{A} \cdot \widetilde{\mathbf{I}}_3 &= (a_1\mathbf{e}_2 \wedge \mathbf{e}_3 + a_2\mathbf{e}_3 \wedge \mathbf{e}_1 + a_3\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1) \\ &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3. \end{aligned} \tag{35}$$
>
> This is a vector, and we recognize it (in this Euclidean space) as the *normal vector* to the planar subspace represented by $\mathbf{A}$, see Figure 4b. So we have normal vectors in geometric algebra as the duals of 2-blades, if we would want them – we will see in Section 5.2.2 why we prefer the direct representation of a planar subspace by a 2-blade rather than the indirect representation by normal vectors.

We can use either a blade or its dual to represent a subspace, and it is convenient to have some terminology. We will say that *a blade* $\mathbf{B}$ *represents a subspace* $\mathcal{B}$ if

$$\mathbf{x} \in \mathcal{B} \iff \mathbf{x} \wedge \mathbf{B} = 0 \tag{36}$$

and that *a blade* $\mathbf{B}^*$ *dually represents the subspace* $\mathcal{B}$ if

$$\mathbf{x} \in \mathcal{B} \iff \mathbf{x} \cdot \mathbf{B}^* = 0. \tag{37}$$

Switching between the two standpoints is done by the duality relation eq.(19), used for a vector $\mathbf{x}$ and a pseudoscalar $\mathbf{I}$:

$$(\mathbf{x} \wedge \mathbf{B}) \cdot \widetilde{\mathbf{I}} = \mathbf{x} \cdot (\mathbf{B} \cdot \widetilde{\mathbf{I}}), \tag{38}$$

and by a converse (but conditional) relationship which we state without proof

$$(\mathbf{x} \cdot \mathbf{B}) \cdot \widetilde{\mathbf{I}} = \mathbf{x} \wedge (\mathbf{B} \cdot \widetilde{\mathbf{I}}) \quad \text{if } \mathbf{x} \wedge \mathbf{I} = 0. \tag{39}$$

If $\mathbf{x}$ is known to be in the subspace of $\mathbf{I}$, we can write these simply as $(\mathbf{x} \wedge \mathbf{B})^* = \mathbf{x} \cdot \mathbf{B}^*$ and $(\mathbf{x} \cdot \mathbf{B})^* = \mathbf{x} \wedge \mathbf{B}^*$, which makes the equivalence of the two representations above obvious.

4.9 The cross product

Classical computations with vectors in 3-space often use the cross product, which produces from two vectors $\mathbf{a}$ and $\mathbf{b}$ a new vector $\mathbf{a} \times \mathbf{b}$ perpendicular to both (by the right-hand rule), proportional to the area they span. We can make this in geometric algebra as the dual of the 2-blade spanned by the vectors, see Figure 4b:

$$\mathbf{a} \times \mathbf{x} \equiv (\mathbf{a} \wedge \mathbf{x}) \cdot \widetilde{\mathbf{I}}_3. \tag{40}$$

You may verify that computing this explicitly using eq.(1) and eq.(35) indeed retrieves the usual expression

$$\mathbf{a} \times \mathbf{x} = (a_2 x_3 - a_3 x_2)\, \mathbf{e}_1 + (a_3 x_1 - a_1 x_3)\, \mathbf{e}_2 + (a_1 x_2 - a_2 x_1)\, \mathbf{e}_3 \tag{41}$$

Eq.(40) shows a number of things explicitly that one always needs to remember about the cross product: there is a convention involved on handedness (this is coded in the sign of $\mathbf{I}_3$); there are metric aspects since it is perpendicular to a plane (this is coded in the usage of the inner product ‘ $\cdot$ ’); and the construction really only works in three dimensions, since only then is the dual of a 2-blade a vector (this is coded in the 3-gradedness of $\mathbf{I}_3$). The vector relationship $\mathbf{a} \wedge \mathbf{x}$ does not depend on any of these embedding properties, yet characterizes the $(\mathbf{a}, \mathbf{x})$-plane just as well. In geometric algebra, we therefore have the possibility of replacing the cross product by a more elementary construction. In Section 5.2.2 we will see the advantages of doing so.

5 More advanced geometric algebra

5.1 Differentiation

Geometric algebra has an extended operation of differentiation, which contains the classical vector calculus, and much more. It is possible to differentiate with respect to a scalar or a vector, as before, but now also with respect to k-blades. This enables efficient encoding of differential geometry, in a coordinate-free manner, and gives an alternative look at differential shape descriptors like the ‘second fundamental form’ (it becomes an immediate indication of how the tangent plane changes when we slide along the surface). We show two examples of differentiation.

5.1.1 Scalar differentiation of a rotor

Suppose we have a rotor $R = e^{-\mathbf{I}\phi/2}$ (where $\mathbf{I}\phi$ is a function of t), and use it to produce a rotated version $\mathbf{X} = R\,\mathbf{X}_0\,R^{-1}$ of some constant blade $\mathbf{X}_0$. Scalar differentiation with respect to time gives (using chain rule and commutation rules)

$$\begin{aligned}
\tfrac{d}{dt}\mathbf{X} &= \tfrac{d}{dt}(e^{-\mathbf{I}\phi/2}\mathbf{X}_0 e^{\mathbf{I}\phi/2}) \\
&= -\tfrac{1}{2}\tfrac{d}{dt}(\mathbf{I}\phi)(e^{-\mathbf{I}\phi/2}\mathbf{X}_0 e^{\mathbf{I}\phi/2}) + \tfrac{1}{2}(e^{-\mathbf{I}\phi/2}\mathbf{X}_0 e^{\mathbf{I}\phi/2})\tfrac{d}{dt}(\mathbf{I}\phi) \\
&= \tfrac{1}{2}(\mathbf{X}\,\tfrac{d}{dt}(\mathbf{I}\phi) - \tfrac{d}{dt}(\mathbf{I}\phi)\,\mathbf{X}) \\
&= \mathbf{X}\times\tfrac{d}{dt}(\mathbf{I}\phi)
\end{aligned} \tag{42}$$

using the commutator product $\times$ defined in geometric algebra as the shorthand $A\times B \equiv \frac{1}{2}(AB - BA)$; this product often crops up in computations with Lie groups such as the rotations. The simple expression that results assumes a more familiar form when $\mathbf{X}$ is a vector $\mathbf{x}$ in 3-space, the attitude of the rotation plane is fixed so that $\frac{d}{dt}\mathbf{I} = 0$, and we introduce a scalar angular velocity $\omega \equiv \frac{d}{dt}\phi$. It is then common practice to introduce the vector dual to the plane as the angular velocity vector $\boldsymbol{\omega}$, so $\boldsymbol{\omega} \equiv \omega\mathbf{I}\cdot\widetilde{\mathbf{I}}_3 = \omega\mathbf{I}/\mathbf{I}_3$. We obtain

$$\tfrac{d}{dt}\mathbf{x} = \mathbf{x}\times\tfrac{d}{dt}(\mathbf{I}\phi) = \mathbf{x}\times(\boldsymbol{\omega}\,\mathbf{I}_3) = \mathbf{x}\cdot(\boldsymbol{\omega}\,\mathbf{I}_3) = (\mathbf{x}\wedge\boldsymbol{\omega})\,\mathbf{I}_3 = \boldsymbol{\omega}\,\boldsymbol{\times}\,\mathbf{x}$$

where $\boldsymbol{\times}$ is the vector cross product. As before when we treated other operations, we find that an equally simple geometric algebra expression is much more general; here eq.(42) describes the differential rotation of k-dimensional subspaces in n-dimensional space, rather than merely of vectors in 3-D.

5.1.2 Vector differentiation of spherical projection

Suppose that we project a vector $\mathbf{x}$ on the unit sphere by the function $\mathbf{x} \mapsto \mathsf{P}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. We compute its derivative in the $\mathbf{a}$ direction, denoted as $(\mathbf{a}\cdot\partial_{\mathbf{x}})\mathsf{P}(\mathbf{x})$ or $\mathsf{P}_{\mathbf{a}}(\mathbf{x})$, as a standard differential quotient and using Taylor series expansion. Note how geometric algebra permits compact expression of the result, with geometrical significance:

$$\begin{aligned}
(\mathbf{a}\cdot\partial_{\mathbf{x}})\frac{\mathbf{x}}{|\mathbf{x}|} &\equiv \lim_{\lambda\to 0}\frac{1}{\lambda}\left(\frac{\mathbf{x}+\lambda\mathbf{a}}{|\mathbf{x}+\lambda\mathbf{a}|} - \frac{\mathbf{x}}{|\mathbf{x}|}\right) = \lim_{\lambda\to 0}\frac{1}{\lambda}\left(\frac{\mathbf{x}+\lambda\mathbf{a}}{|\mathbf{x}|\sqrt{1+2\lambda\mathbf{a}\cdot\mathbf{x}^{-1}}} - \frac{\mathbf{x}}{|\mathbf{x}|}\right) \\
&= \lim_{\lambda\to 0}\frac{(\mathbf{x}+\lambda\mathbf{a})(1-\lambda\mathbf{a}\cdot\mathbf{x}^{-1}) - \mathbf{x}}{\lambda|\mathbf{x}|} = \frac{\mathbf{a} - \mathbf{x}(\mathbf{a}\cdot\mathbf{x}^{-1})}{|\mathbf{x}|} \\
&= \frac{(\mathbf{a}\wedge\mathbf{x})\,\mathbf{x}^{-1}}{|\mathbf{x}|}
\end{aligned}$$

We recognize the result as the rejection of $\mathbf{a}$ by $\mathbf{x}$ (Section 4.3), scaled appropriately. The sketch of Figure 8 confirms the outcome. You may verify in a similar manner that $(\mathbf{a}\cdot\partial_{\mathbf{x}})\mathbf{x}^{-1} = -\mathbf{x}^{-1}\,\mathbf{a}\,\mathbf{x}^{-1}$, and interpret geometrically.

For more advanced usage of differentiation relative to blades, the interested reader is referred to the tutorial of [3], which introduces these differentiations using examples from physics, and the application paper [12].

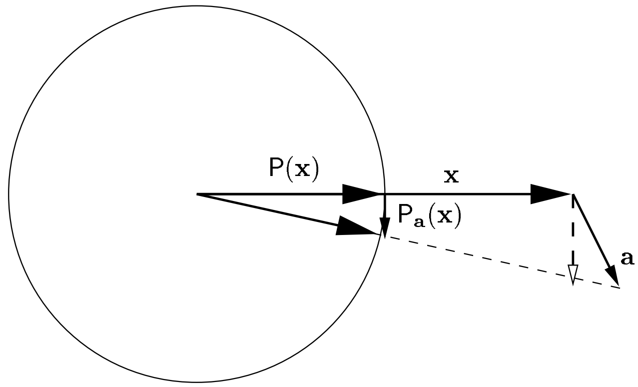

Figure 8: *The derivative of the spherical projection.*

5.2 Linear algebra

In the classical ways of using vector spaces, linear algebra is an important tool. In geometric algebra, this remains true: linear transformations are of interest in their own right, or as first order approximations to more complicated mappings. Indeed, linear algebra is an integral part of geometric algebra, and acquires much extended coordinate-free methods through this inclusion. We show some of the basic principles; much more may be found in [3] or [11].

5.2.1 Outermorphisms: spanning is linear

When vectors are transformed by a linear transformation on the vector space, the blades they span can be viewed to transform as well, simply by the rule: 'the transform of a span of vectors is the span of the transformed vectors'. This means that a linear transformation $\mathsf{f} : V^m \to V^m$ of a vector space has a natural extension to the whole geometric algebra of that vector space, as an *outermorphism*, i.e. a mapping that preserves the outer product structure:

$$\mathsf{f}(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) \equiv \mathsf{f}(\mathbf{a}_1) \wedge \mathsf{f}(\mathbf{a}_2) \wedge \cdots \wedge \mathsf{f}(\mathbf{a}_k).$$

Note that this is grade-preserving: a k-blade transforms to a k-blade. We supplement this by stating what the extension does to scalars, which is simply $\mathsf{f}(\alpha) = \alpha$. Geometrically, this means that a linear transformation leaves weighted points at the origin intact.

The fact that linear transformations are outermorphisms explains why we can generalize so many operations from vectors to general subspaces in a straightforward manner.

5.2.2 No normal vectors or cross products!

The transformation of an inner product under a linear mapping is more involved (formula given in [11]). Therefore one should steer clear of any constructions that involve the inner product, especially in the characterization of basic properties of one's objects. The practice of characterizing a plane by its normal vector – which contains the inner product in its duality, see Section 4.8 – should be avoided. Under linear transformations, *the normal vector of a transformed plane is not the transform of the normal vector of the plane!* (this is a well known fact, but always a shock to novices). The normal vector is in fact a cross product of vectors, which transforms as

$$\mathsf{f}(\mathbf{a} \times \mathbf{b}) = \bar{\mathsf{f}}^{-1}(\mathbf{a}) \times \bar{\mathsf{f}}^{-1}(\mathbf{b}) / \det(\mathsf{f}) \tag{43}$$

where $\bar{\mathsf{f}}$ is the *adjoint*, defined by $\bar{\mathsf{f}}(\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathsf{f}(\mathbf{b})$, and $\det(\mathsf{f})$ is the determinant defined by $\det(\mathsf{f}) = \mathsf{f}(\mathbf{I}_m)\mathbf{I}_m^{-1}$ with $\mathbf{I}_m$ the pseudoscalar of V^m (they are equivalent to the transpose and determinant of the matrix of f).

The right hand side of eq.(43) is usually not equal to $\mathsf{f}(\mathbf{a}) \times \mathsf{f}(\mathbf{b})$, so a linear transformation is not an 'innermorphism'. It is therefore much better to characterize the plane by a 2-blade, now that we can. *The 2-blade of the transformed plane is the transform of the 2-blade of the plane*, since linear transformations are outermorphisms preserving the 2-blade construction. Especially when the planes are tangent planes constructed by differentiation, 2-blades are appropriate: under *any* transformation f, the construction of the tangent plane is only dependent on the first order linear approximation mapping f of f. Therefore a tangent plane represented as a 2-blade transforms simply under *any* transformation (and the same applies of course to tangent k-blades in higher dimensions). Using blades for those tangent spaces should enormously simplify the treatment of object through differential geometry, especially in the context of affine transformations.

5.3 Intersecting subspaces

Geometric algebra also contains operations to determine the *union* and *intersection* of subspaces. These are the join and meet operations. Several notations exist for these in literature, causing some confusion. For this paper, we will simply use the set notations $\cup$ and $\cap$ to make the formulas more easily readable.

5.3.1 Union of subspaces: join

The join of two subspaces is their smallest superspace, i.e. the smallest space containing them both. Representing the spaces by blades $\mathbf{A}$ and $\mathbf{B}$, the join is denoted $\mathbf{A} \cup \mathbf{B}$. If the subspaces of $\mathbf{A}$ and $\mathbf{B}$ are disjoint, their join is obviously proportional to $\mathbf{A} \wedge \mathbf{B}$. But a problem is that if $\mathbf{A}$ and $\mathbf{B}$ are not disjoint (which is precisely the case we are interested in), then $\mathbf{A} \cup \mathbf{B}$ contains an unknown scaling factor that is fundamentally unresolvable due to the reshapable nature of the blades discussed in Section 2.2 (see Figure 9; this ambiguity was also observed by [15]. Fortunately, it appears that in all geometrically relevant entities that we compute this scalar ambiguity cancels.

The join is a more complicated product of subspaces than the outer product and inner product; we can give no simple formula for the grade of the result (like eq.(27)), and it cannot be characterized by a list of algebraic computation rules. Although computation of the join may appear to require some optimization process, finding the smallest superspace can actually be done in virtually constant time [1].

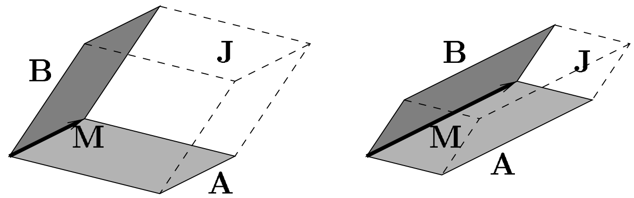

Figure 9: *The ambiguity of scale for* meet $\mathbf{M}$ *and* join $\mathbf{J}$ *of two blades* $\mathbf{A}$ *and* $\mathbf{B}$. *Both figures are acceptable solutions.*

5.3.2 Intersection of subspaces: meet

The meet of two subspaces $\mathbf{A}$ and $\mathbf{B}$ is their largest common subspace. Given the join $\mathbf{J} \equiv \mathbf{A} \cup \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$, we can compute their meet $\mathbf{A} \cap \mathbf{B}$ by the property that its dual (with respect to the join) is the outer product of their duals (this is a not-so-obvious consequence of the required 'containment in both'). In formula, this is

$$(\mathbf{A} \cap \mathbf{B}) \cdot \widetilde{\mathbf{J}} = (\mathbf{B} \cdot \widetilde{\mathbf{J}}) \wedge (\mathbf{A} \cdot \widetilde{\mathbf{J}}) \ \text{ or } \ (\mathbf{A} \cap \mathbf{B})^* = \mathbf{B}^* \wedge \mathbf{A}^*$$

with the dual taken with respect to the join $\mathbf{J}$.[2] This leads to a formula for the meet of $\mathbf{A}$ and $\mathbf{B}$ relative to the chosen join (use eq.(38)):

$$\mathbf{A} \cap \mathbf{B} = (\mathbf{B} \cdot \widetilde{\mathbf{J}}) \cdot \mathbf{A}. \tag{44}$$

Let us do an example.

> The intersection of two planes represented by the 2-blades $\mathbf{A} = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_2 + \mathbf{e}_3)$ and $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$. Note that we have normalized them (this is not necessary, but convenient for a point we want to make later). These are planes in general position in 3-dimensional space, so their join is proportional to $\mathbf{I}_3$. It makes sense to take $\mathbf{J} = \mathbf{I}_3$. This gives for the meet:
>
> $$\begin{aligned} \mathbf{A} \cap \mathbf{B} &= \tfrac{1}{2}\left((\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1)\right) \cdot \left((\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_2 + \mathbf{e}_3)\right) \\ &= \tfrac{1}{2}\mathbf{e}_3 \cdot \left((\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3\right) \\ &= -\tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) = -\frac{1}{\sqrt{2}}\left(\frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}\right) \end{aligned} \tag{45}$$
>
> (the last step expresses the result in normalized form). Figure 10 shows the answer; as in [15] the sign of $\mathbf{A} \cap \mathbf{B}$ is the right-hand rule applied to the turn required to make $\mathbf{A}$ coincide with $\mathbf{B}$, in the correct orientation.

Classically, one computes the intersection of two planes in 3-space by first converting them to normal vectors, and then taking the cross product. We can see that this gives

[2] The somewhat strange order means that the join $\mathbf{J}$ can be written using the meet $\mathbf{M}$ in the factorization $\mathbf{J} = (\mathbf{A}\mathbf{M}^{-1}) \wedge \mathbf{M} \wedge (\mathbf{M}^{-1}\mathbf{B})$, and it corresponds to [15] for vectors.

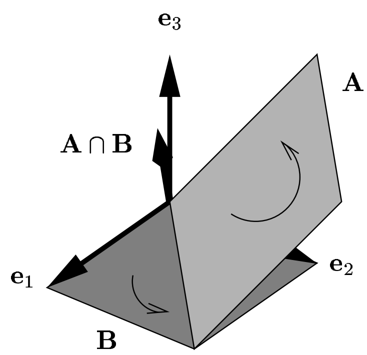

Figure 10: *An example of the* meet

the same answer in this non-degenerate case in 3-space, using our previous equations eq.(39), eq.(38), and noting that $\widetilde{\mathbf{I}}_3 = -\mathbf{I}_3$:

$$\begin{aligned}(\mathbf{A} \cdot \widetilde{\mathbf{I}}_3) \times (\mathbf{B} \cdot \widetilde{\mathbf{I}}_3) &= \Big((\mathbf{A} \cdot \widetilde{\mathbf{I}}_3) \wedge (\mathbf{B} \cdot \widetilde{\mathbf{I}}_3)\Big) \cdot \widetilde{\mathbf{I}}_3 = \Big((\mathbf{B} \cdot \widetilde{\mathbf{I}}_3) \wedge (\mathbf{A} \cdot \widetilde{\mathbf{I}}_3)\Big) \cdot \mathbf{I}_3 \\ &= (\mathbf{B} \cdot \widetilde{\mathbf{I}}_3) \cdot \Big((\mathbf{A} \cdot \widetilde{\mathbf{I}}_3) \cdot \mathbf{I}_3\Big) = (\mathbf{B} \cdot \widetilde{\mathbf{I}}_3) \cdot \mathbf{A} = \mathbf{A} \cap \mathbf{B}.\end{aligned}$$

So the classical result is a special case of eq.(44), but that formula is much more general: it applies to the intersection of subspaces of *any* grade, within a space of *any* dimension. Again we see the power of geometric algebra in compact expressions valid for any grade or dimension.

The norm of the meet gives an impression of the 'strength' of the intersection. Between normalized subspaces in Euclidean space, the magnitude of the meet is the sine of the angle between them. From numerical analysis, this is a well-known measure for the 'distance' between subspaces in terms of their orthogonality: it is 1 if the spaces are orthogonal, and decays gracefully to 0 as the spaces get more parallel, before changing sign. This numerical significance is very useful in applications.

5.4 Caseless computations through blades

So far we have been treating only homogeneous subspaces of the vector spaces, i.e. subspaces containing the origin. We have spanned them, projected them, and rotated them, but we have not moved them out of the origin to make more interesting geometrical structures such as lines floating in space. We construct them now. by extending the thoughts behind 'homogeneous coordinates' to geometric algebra. It turns out that these elements of geometry can also be represented by blades, in a representational space with an extra dimension. The geometric algebra of this space then turns out to give us precisely what we need. In this view, more complicated geometrical objects do not require new operations or techniques in geometric algebra, merely the standard computations in a higher dimensional space.

5.4.1 The affine model

The affine model in geometric algebra extends the usual geometric model from linear algebra, which works merely on vectors. That model is often described as augmenting a 3-dimensional vector $\mathbf{v}$ with coordinates $(v_1, v_2, v_3)^T$ to a 4-vector $(v_1, v_2, v_3, 1)^T$. That extension makes non-linear operations such as translations implementable as linear mappings.

Following the approach of this paper, we have to give the $(m+1)$-dimensional space into which we embed our m-dimensional Euclidean space a full geometric algebra. We will call this space 'homogeneous space' (the affine model is sometimes called the 'homogeneous model' though recently that term has been preferred for an even more homogeneous representation of Euclidean space [10]). Let the unit vector for the extra dimension be denoted by e. This vector must be perpendicular to all regular vectors in the Euclidean space E^m, so $e \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in E^m$. We let e denote 'the point at the origin'. A point at any other location $\mathbf{p}$ is made by translation of the point at the origin over the Euclidean vector $\mathbf{p}$. This is done by adding $\mathbf{p}$ to e. This construction therefore gives the representation of the point $\mathcal{P}$ at location $\mathbf{p}$ as the vector p in $(m+1)$-dimensional space:

$$p = e + \mathbf{p}$$

This is just a regular vector in the $(m+1)$-space, now interpretable as a point. It is of course no more than the usual 'homogeneous coordinates' method in disguise. We will denote points of the m-dimensional Euclidean space in script, the vectors in the corresponding vector space in **bold**, and vectors in the $(m+1)$-dimensional homogeneous space in *italic*. You can visualize this construction as in Figure 11a (necessarily drawn for $m = 2$).

These vectors in $(m+1)$-dimensional space can be multiplied using the products in geometric algebra. Let us consider in particular the outer product, and form blades. We compute

$$p \wedge q = (e + \mathbf{p}) \wedge (e + \mathbf{q}) = e \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge \mathbf{q}$$

We recognize the vector $\mathbf{q} - \mathbf{p}$, and the area spanned by $\mathbf{p}$ and $\mathbf{q}$. Both are elements that we need to describe an element of the directed line through the points $\mathcal{P}$ and $\mathcal{Q}$. The former is the *direction vector* of the directed line, the latter is an area that we will call the *moment* of the line through p and q. It denotes the distance to the origin, for we can rewrite it to a rectangle spanned by the direction $(\mathbf{q} - \mathbf{p})$ and any vector on the line, such as $\mathbf{p}$ or $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ or the *perpendicular support vector* $\mathbf{d}$:

$$\mathbf{p} \wedge \mathbf{q} = \mathbf{p} \wedge (\mathbf{q} - \mathbf{p}) = \tfrac{1}{2}(\mathbf{p} + \mathbf{q}) \wedge (\mathbf{q} - \mathbf{p}) = \mathbf{d}\,(\mathbf{q} - \mathbf{p}) \tag{46}$$

where $\mathbf{d} = (\mathbf{p} \wedge \mathbf{q})(\mathbf{q} - \mathbf{p})^{-1}$ is the rejection of $\mathbf{p}$ by $\mathbf{q} - \mathbf{p}$.

So the 2-blade $p \wedge q$ can be interpreted as a *directed line element for the line* $\mathcal{PQ}$. However, note that $p \wedge q$ is not a line *segment*: neither p nor q can be retrieved from $p \wedge q$. The 2-blade is just a line element of specified direction and length, somewhere

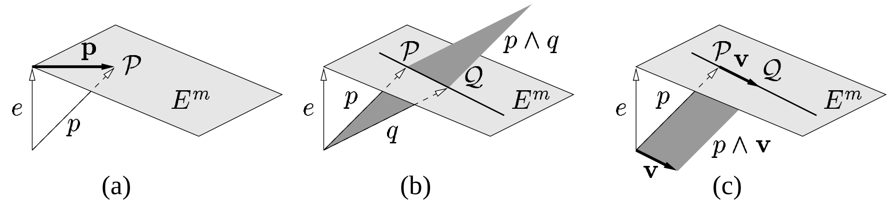

Figure 11: *Representing offset subspaces of E^m in $(m+1)$-dimensional space. In (c),* $\mathbf{v} \equiv \mathbf{q} - \mathbf{p}$.

along the line through $\mathcal{P}$ and $\mathcal{Q}$ (in that order). Geometrically, a point $\mathcal{X}$ lies on the line through $\mathcal{P}$ and $\mathcal{Q}$ if the vector x in the affine model lies in the plane spanned by p and q, so satisfies $x \wedge p \wedge q = 0$. This is depicted in Figure 11b,c.

This way of making offset planar subspaces extends easily. An element of the oriented plane through the points $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ is represented by the 3-blade $p \wedge q \wedge r$, and so on for higher dimensional 'offset' subspaces – if the space has enough dimensions to accommodate them. The blades we construct in this way can always be rewritten in the form $A = d\mathbf{A}$, where $\mathbf{A}$ is purely Euclidean, and d is of the form $e + \mathbf{d}$, with $\mathbf{d}$ a Euclidean vector. We should interpret $\mathbf{A}$ as the direction element, and its grade therefore denotes the dimensionality of the flat subspace represented by A. The vector d represents the closest point to the origin (so that $\mathbf{d}$ is the perpendicular support vector).

But the affine model also contains other blades, namely the purely Euclidean k-blades of the form $\mathbf{A}$. These should be interpret as k-dimensional directions (think of a tangent space at an arbitrary point), or alternatively (but less usefully) as k-spaces at infinity. Even a 0-blade (i.e. a scalar) is useful: it is the representation of a scalar distance in the Euclidean space (with a sign, but without a direction), as we will see in the next section. Such distances are of course elements of geometry as well, so it is satisfying to find them on a par with position vectors, direction vectors and other elements of higher dimensionality as just another case of a representing blade in the affine model of a flat Euclidean subspace.

Having such a unified representation for the various geometrical elements implies that computations using them are unified as well: they have just become operations on blades in $(m+1)$-space, blissfully ignorant of what different geometrical situations these computations might represent.

5.4.2 Application: universal incidence computations

The meet and join operations that we introduced in the context of a general geometric algebra in Section 5.3 can be applied immediately to blades in the affine model. This is a universal and simple computation, independent of what those blades represent, and there is no need to consider different cases in performing the calculation. Yet as we interpret the result, we find that we may have to render the various outcomes differently, and this is where the cases appear.

Let us see how the incidence of two lines in a 3-dimensional Euclidean space with pseudoscalar $\mathbf{I}_3$ would be treated using geometric algebra. These lines are encoded in the affine model as the 2-blades $p \wedge \mathbf{u}$ and $q \wedge \mathbf{v}$ in $(m+1)$-dimensional space. Their meet is the element

$$\mathbf{M} \equiv (p \wedge \mathbf{u}) \cap (q \wedge \mathbf{v})$$

and if we wish we can compute further with it, and with the corresponding join. There are *no* cases in this outcome or the continued computation – for whether the outcome is a scalar, vector or 2-blade is immaterial,

However, if we desire, at this point, to *draw* the meet (or the join), the drawing routines are different depending on how $\mathbf{M}$ is interpreted, and this depends its grade and stance in the representation space. The grade can be 0, 1 or 2, and the stance may be special relative to the exceptional e-direction. As we consider these, we obtain all recognizable situations from the traditional case-based computations. As a nice surprise, the meet and join also contain numerical stability measures, in the same computational effort.

- *skew lines*:
  If the lines are in general position, then $(p \wedge \mathbf{u}) \wedge (q \wedge \mathbf{v}) \neq 0$ (read this as: $\mathcal{P}$ is not in the plane $\mathbf{u} \wedge q \wedge \mathbf{v}$ spanned by $\mathbf{u}$ and $\mathbf{v}$ from $\mathcal{Q}$). In this case, that 4-blade can be used as the join in homogeneous space. Using $p \wedge q = \frac{1}{2}(p+q) \wedge (q-p) = e(\mathbf{q} - \mathbf{p})$, this can be rewritten to $e\,((\mathbf{p} - \mathbf{q}) \wedge \mathbf{u} \wedge \mathbf{v})$, so this is proportional to $e\,\mathbf{I}_3$, a split into purely Euclidean and representational parts of unit magnitude. Using that as the join, we find the meet by a fairly straightforward computation. The outcome is:

  $$\begin{aligned} \mathbf{J} &= e\,\mathbf{I}_3 \quad (\text{proportional to } p \wedge \mathbf{u} \wedge q \wedge \mathbf{u} = e\,((\mathbf{p} - \mathbf{q}) \wedge \mathbf{u} \wedge \mathbf{v}))\ ) \\ \mathbf{M} &= ((\mathbf{p} - \mathbf{q}) \wedge \mathbf{u} \wedge \mathbf{v})^* \end{aligned}$$

  (with duality relative to $\mathbf{I}_3$). This scalar meet is proportional to the *perpendicular signed distance between the two lines*, weighted by a factor $|\mathbf{u} \wedge \mathbf{v}|$. The latter can be interpreted as denoting the numerical significance of the outcome.

- *intersecting lines*:
  When $\mathbf{u} \wedge \mathbf{v} \neq 0$ but $(p \wedge \mathbf{u}) \wedge (q \wedge \mathbf{v}) = 0$, the lines intersect. Their join is now proportional to the common plane $\mathbf{v} \wedge q \wedge \mathbf{u} = q \wedge \mathbf{u} \wedge \mathbf{v}$. Let the unit blade representing this be I. Their meet is then explicitly (after some rewriting)

  $$\begin{aligned} \mathbf{J} &= I \quad (\text{proportional to } q \wedge \mathbf{u} \wedge \mathbf{v}) \\ \mathbf{M} &= e\,(\mathbf{u} \wedge \mathbf{v})^* + (\mathbf{q} \wedge \mathbf{v})^* \mathbf{u} - (\mathbf{p} \wedge \mathbf{v})^* \mathbf{v} + \tfrac{1}{2}((\mathbf{p} + \mathbf{q}) \wedge \mathbf{u} \wedge \mathbf{v})^* \end{aligned}$$

  with duality relative to $\mathbf{I}_2$, the unit blade of $\mathbf{u} \wedge \mathbf{v}$. This vector in homogeneous space is interpreted as a weighted point in Euclidean space, since it can be written in the form $d \wedge \boldsymbol{\alpha} = (e + \mathbf{d}) \wedge \boldsymbol{\alpha}$ with

  $$\mathbf{d} = \mathbf{u}\,\frac{\mathbf{q} \wedge \mathbf{v}}{\mathbf{u} \wedge \mathbf{v}} + \mathbf{v}\,\frac{\mathbf{p} \wedge \mathbf{u}}{\mathbf{v} \wedge \mathbf{u}} + \tfrac{1}{2}\,\frac{(\mathbf{p} + \mathbf{q}) \wedge \mathbf{u} \wedge \mathbf{v}}{\mathbf{u} \wedge \mathbf{v}}$$

  Note how the result is constructed in a coordinate-free manner from an amount of $\mathbf{u}$, of $\mathbf{v}$ and of a vector perpendicular to both. The proportionality factor

$(\mathbf{u} \wedge \mathbf{v})^*$ of e in the expression for $\mathbf{M}$ is the 'weight' of the point at $\mathbf{d}$. It indicates the numerical stability of the intersection, and may be used as a measure of the significance of the interpretation as an intersection point.

- *parallel lines*:
  If the lines are parallel but not coincident, then $\mathbf{u} \wedge \mathbf{v} = 0$ while $p \wedge \mathbf{u} \wedge q \neq 0$. The join is then the representation of the Euclidean plane containing them. We get (after some rewriting)

$$\begin{aligned} \mathbf{J} &= e\,\mathbf{I}_2 \quad \text{(proportional to } p \wedge \mathbf{u} \wedge q = e((\mathbf{p} - \mathbf{q}) \wedge \mathbf{u})\text{)} \\ \mathbf{M} &= ((\mathbf{q} - \mathbf{p}) \wedge \mathbf{u})^* \mathbf{v}. \end{aligned}$$

  The meet is again a vector in $(m+1)$-space, but now purely Euclidean. This is interpretable as the common directional part $\mathbf{v}$ weighted by a scalar magnitude proportional to the distance of the lines, with proportionality factor $|\mathbf{u}|$.

- *coincident lines*:
  If the lines are coincident, then $\mathbf{u} \wedge \mathbf{v} = 0$ and $p \wedge \mathbf{u} \wedge q = 0$. The join is therefore one of the lines, say $q \wedge \mathbf{v}$ – no need to normalize to a unit blade. The computation of the meet is straightforward

$$\begin{aligned} \mathbf{J} &= q \wedge \mathbf{v} \\ \mathbf{M} &= p \wedge \mathbf{u} \end{aligned}$$

  In this case the intersection is therefore interpretable as the original line.

We reiterate that in all situations, *the cases occur in the interpretation, not in the computation*, which is simply to invoke the meet and join operators. This unification at the computational level is an enormous advantage of the geometric algebra approach to incidence. The computational caselessness should lead to new methods that take full advantage of it, leaving the interpretation (and with it, the occurrence of cases) to the very last step of any calculation.

The directional outcomes are accompanied by numerical factors relating to the numerical significance of the computation. We showed that they are an intrinsic part of the computation of the object, not just secondary aspects that need to be thought of separately (with the danger of being *ad hoc*) or that need be computed separately (costing time).[3]

5.5 The homogeneous model

An embedding of Euclidean space into a representational space of 2 extra dimensions and its geometric algebra has been recently shown to be very powerful and simplifying [10]. This *homogeneous model* of Euclidean space gives a useful semantics to the derived product of vectors representing points.

[3]Although to separate the outcomes into scalar geometric measures such as distance and into the numerical significance measures, one also needs to compute the meet of the directional parts separately (in the examples above, $(\mathbf{u} \wedge \mathbf{v})^*$).

- The inner product provides the Euclidean distance: $p \cdot q = -\frac{1}{2}|\mathbf{q} - \mathbf{p}|^2$. That means the space has a rather special metric, since it follows that $p \cdot p = 0$ for any vector p.
- The outer product constructs k-spheres: a k-blade represents a Euclidean $(k-1)$-sphere. As a consequence, $p \wedge q$ is the ordered point pair $(\mathcal{P}, \mathcal{Q})$, $p \wedge q \wedge r$ the circle through $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. The dual of the $(m+1)$-blade representing an m-sphere in E^m is a homogeneous vector which immediately provides center and radius of the sphere. Flat subspaces are represented as spheres through infinity, and this is possible because one of the two extra representational dimensions is a vector representing the point at infinity (the other, as in the affine model, represents the point at the origin).
- The sandwiching by the geometric product gives not only rotations, but all conformal mappings, including translation and inversion. (This is why the homogeneous model is sometimes called the *conformal model*.)
- The meet of two blades is interpretable as intersecting k-spheres in m-space, and again reduces the cases that would need to be distinguished.

This model looks appropriate for many computer graphics applications, and we are currently looking at developing it further for practical usage.

6 Conclusion

This introduction of geometric algebra intends to alert you to the existence of a small set of products that appears to generate all geometric constructions in one consistent framework. Using this framework can simplify the set of data structures representing objects since it inherently encodes all relationships and symmetries of the geometrical primitives in those operators (an example was eq.(46)). While there are many interesting facets to geometric algebra, we would like to highlight the following:

- *Division by subspaces*; having a geometric product with an inverse allows us to divide by subspaces, increasing our ability to manipulate algebraic equations involving vectors.
- *Subspaces are basic elements of computation*; thus, no special representations are needed for subspaces of dimension greater than 1 (e.g., tangent planes), and we can manipulate them like we manipulate vectors.
- *Generalization*; expressions for operations on subspaces are often as simple as those for vectors (especially true for linear operations), and as easy to compute.
- *Caseless computation*; degenerate cases are computed automatically, and the computation allows us to test the numerics of the solution.
- *Quaternions*; in geometric algebra, quaternions are subsumed and become a natural part of the algebra, with no need to convert between representations to perform rotations.

This paper only covers some of what we felt to be the most important or useful ideas of geometric algebra as it relates to computer graphics. Many topics have been left out, including a description of more geometries (the affine model implements and generalizes the Grassmann spaces of [8], the homogeneous model implements and generalizes projective spaces); the fact that the homogeneous model requires the use of negative signatures (with some unit vectors squaring to -1) which have some peculiarities; and a lot more can be said about differentiation and coordinate-free differential geometry. The reader may be able to glean the connections from the suggested further reading (see below), but an accessible explanation for computer graphics of such issues still needs to be given.

That such a system exists is a happy surprise to all learning about it. Whether it is also the way we should structure our programming is at the moment an open question. Use of the affine model would require representing the computations on the Euclidean geometry of a 3-dimensional space on a basis of $2^{3+1} = 16$ elements, rather than just 3 basis vectors (plus 1 scalar basis). It seems a hard sell. But you often have to construct objects representing higher order relationships between points (such as lines, planes and spheres) anyway, even if you do not encode them on such a 'basis'. Also, our investigations show that perhaps all one needs to do all of geometry are blades and operators composed of products of vectors; the product combinations of this limited subset can be optimized in time and space requirements, with very little overhead for their membership of the full geometric algebra. That automatic membership enables us to compute directly with lines and planes (and in the homogeneous model of [10], circles and spheres) and their intersections without needing to worry about special or degenerate cases, which should eliminate major headaches and bugs. We also find the coordinate-free specification of the operations between objects very attractive; relegating the use of coordinates purely to the input and output of geometric objects banishes them from the body of the programs and frees the specification of algorithms from details of the data structures used to implement them. Such properties make geometric programs so much more easy to verify, and – once we have learned to express ourselves fluently in this new language – to construct.

Further reading

There is a growing body of literature on geometric algebra. Unfortunately much of the more readable writing is not very accessible, being found in books rather than journals. Little has been written with computer science in mind, since the initial applications have been to physics. No practical implementations in the form of libraries with algorithms yet exist (though there are packages for Maple [2] and Matlab [6] that can be used as a study-aid or for algorithm design). We would recommend the following as natural follow-ups on this paper:

- GABLE: a Matlab package for geometric algebra, accompanied by a tutorial [6].
- The introductory chapters of 'New Foundations of Classical Mechanics' [9].
- An introductory course intended for physicists [3].

- An application to a basic but involved geometry problem in computer vision, with a brief introduction into geometric algebra [12].
- Papers showing how linear algebra becomes enriched by viewing it as a part of geometric algebra [4, 11].

Read them in approximately this order. We are working on texts more specifically suited for a computer graphics audience; these may first appear as SIGGRAPH courses.

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