Reducing Overparametrization in MRAC for Hyperbolic PDEs

Henrik Anfinsen, Nataša Gajić and Ole Morten Aamo

Abstract— **We construct a method of dealing with the problem of overparameterization in model reference adaptive control (MRAC) of 2×2 linear hyperbolic partial differential equations (PDEs). The method is based on linear interpolation of the uncertain parameters. The proposed method is demonstrated in simulations of the fluid mechanics of a drilling system, and compared to a previously derived, overparameterized MRAC scheme, showing improved tracking and convergence properties.**

I. Introduction

A. Background

Recently, a model reference adaptive controller (MRAC) for 2×2 coupled linear hyperbolic partial differential equations (PDE) was derived [1]. The MRAC controller manages to make the system's measured output asymptotically track a signal generated from a reference model chosen to be a simple transport delay. This is achieved from minimal knowledge of the system parameters, by transforming the system into an equivalent "canonical form" for which the MRAC problem becomes feasible. The method then introduces a set of filters, known as swapping filters, that can be used to obtain a linear parametric model, from which standard estimation techniques like the gradient method can be used.

The MRAC controller from [1] is employed in [2] for controlling the bottom hole pressure (BHP) in managed pressure drilling (MPD). MPD is a technique that enables pressure control in wells with narrow pressure windows and varying formation pressures [3]. The drilling mud is in MPD sealed from the atmosphere, and the well's pressure profile is altered by manipulating the topside choke, thus limiting the mud exiting the well. As the well is typically several kilometers long, and the choke is topside, the time-delay caused by the finite pressure-wave propagation speed may become significant. BHP pressure control is therefore typically performed in a slow manner. The backstepping-based MRAC controller presented in [1] managed to take the propagation delays into account, and hence performed better than a conventional PI controller. However, as the method in [1] employs a series of transformations to bring the system of interest into the required canonical form, the initial physical representation of the system is lost. Specifically, the only uncertain parameter, the well model's friction factor, becomes a coefficient in the observer canonical form's nonlinear, distributed parameters. The MRAC controller from [1] is not able take advantage of this relationship, and treats the uncertain parameters in the observer canonical form as completely unknown. The unknowns therefore go from single to infinite in number.

In this paper, we seek to improve the performance of the MRAC controller applied to MPD by reducing overparameterization. This is done by taking advantage of the fact that the transformations mapping the MPD model to the observer canonical form are known. By approximating the observer canonical form's distributed parameters as a linear interpolation in the friction factor, we thus reduce the number of unknowns in the canonical form from an infinite number to a single one, and hence avoiding overparameterization.

Systems of linear hyperbolic PDEs have attracted considerable attention due to the vast amount of different physical systems that can be modeled by them, ranging from the already mentioned oil wells [4], to open channel flows [5] and predator-prey systems [6]. Early control and estimation results can be found in [7], [8], [9] and more recently in [10]. For the last decade, backstepping has been used with great success for designing controllers and observers for hyperbolic PDEs, with the first result being [11], where a 1-D linear hyperbolic PDE was non-adaptively stabilized using this technique. The method was subsequently extended to systems of coupled hyperbolic PDEs in [12], [13], [14]. In 2014, the first result using backstepping for adaptive control of hyperbolic PDEs was presented in [15], where a 1-D linear hyperbolic PDE was adaptively stabilized using the aforementioned swapping technique in conjunction with backstepping. More results utilizing the backstepping method for adaptive control of linear hyperbolic PDEs quickly followed in [16], [17], [18], [19], [20], [21].

Although the problem investigated in this paper, formally stated in Section II, is motivated from BHP control in MPD, the problem is in Section III solved for a more general class of 2×2 systems of hyperbolic PDEs. The derived controller is then applied to the BHP control in MPD in section VI, and compared to the overparameterized control design previously presented in [2]. Some concluding remarks are offered in Section VII.

B. Notation

For some function f and some domain $\mathcal{D}$, such that $f : \mathcal{D} \to \mathbb{R}$, we define

$$f \in \mathcal{B}(\mathcal{D}) \Leftrightarrow |f(z)| < \infty, \ \forall z \in \mathcal{D}. \tag{1}$$

For a time varying, signal $f : [0, \infty) \to \mathbb{R}$ we define

$$f \in \mathcal{L}_2 \quad \Leftrightarrow \quad \int_0^\infty |f(t)|^2 dt < \infty \tag{2a}$$

$$f \in \mathcal{L}_\infty \quad \Leftrightarrow \quad \sup_{t \geq 0} |f(t)| < \infty. \tag{2b}$$

The authors are with the Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim N-7491, Norway. (e-mail: henrik.anfinsen@ntnu.no; natasa.gajic@ntnu.no; aamo@ntnu.no).

II. PROBLEM STATEMENT

We consider systems of 2×2 coupled linear hyperbolic PDEs. They can be stated as

$$u_t(x,t) + \lambda(x)u_x(x,t) = c_1(x)v(x,t) \quad (3a)$$
$$v_t(x,t) - \mu(x)v_x(x,t) = c_2(x)u(x,t) \quad (3b)$$
$$u(0,t) = qv(0,t) \quad (3c)$$
$$v(1,t) = k_1U(t) + d \quad (3d)$$
$$u(x,0) = u_0(x) \quad (3e)$$
$$v(x,0) = v_0(x) \quad (3f)$$
$$y(t) = k_2v(0,t), \quad (3g)$$

where $u(x,t), v(x,t)$ are system states defined for $x \in [0,1],\ t \geq 0$ and y is the measurement. The parameters $\mu, \lambda, c_1, c_2, q, k_1, k_2, d$ and initial conditions u_0, v_0 are unknown, but assumed to satisfy

$$\mu, \lambda, \in C^1([0,1]), \qquad \mu(x), \lambda(x) > 0, \forall x \in [0,1] \quad (4a)$$
$$c_1, c_2 \in C^0([0,1]), \qquad q, k_1, k_2 \in \mathbb{R}\backslash\{0\} \quad (4b)$$
$$d \in \mathbb{R}, \qquad u_0, v_0 \in \mathcal{B}([0,1]). \quad (4c)$$

Although the exact profiles of λ and μ are not needed, it is assumed that the total transport delays in each direction, and the sign of the product k_1k_2, that is

$$d_\alpha = \overline{\lambda}^{-1} = \int_0^1 \frac{ds}{\lambda(s)}, \qquad d_\beta = \overline{\mu}^{-1} = \int_0^1 \frac{ds}{\mu(s)} \quad (5)$$

and

$$\text{sign}(k_1k_2) \quad (6)$$

are known quantities. In [1], an adaptive control law $U(t)$ is designed so that the system is stabilized, and the following tracking goal is achieved

$$\lim_{t\to\infty} \int_t^{t+T} (y(s) - y_r(s))^2 ds = 0 \quad (7)$$

for some bounded constant $T > 0$, where the reference signal $y_r(t)$ is generated from the reference model

$$a_t(x,t) + \overline{\lambda}a_x(x,t) = 0, \qquad a(0,t) = b(0,t)$$
$$a(x,0) = a_0(x) \quad (8a)$$
$$b_t(x,t) - \overline{\mu}b_x(x,t) = 0, \qquad b(1,t) = r(t)$$
$$b(x,0) = b_0(x), \qquad y_r(t) = b(0,t) \quad (8b)$$

where $a_0, b_0 \in \mathcal{B}([0,1])$. The signal $r(t)$ is a bounded signal of choice. The goal (7) was achieved from using the sensing (3g), and knowledge of the quantities (5), only, but all uncertain system parameters were treated as completely unknown.

III. REDUCING OVERPARAMETRIZATION

A. Observer canonical form

The controller in [1] is derived by transforming system (3) into the following *observer canonical form*

$$w_t(x,t) + \overline{\lambda}w_x(x,t) = 0 \quad (9a)$$
$$z_t(x,t) - \overline{\mu}z_x(x,t) = \overline{\mu}\theta(x)z(0,t) \quad (9b)$$
$$w(0,t) = z(0,t) \quad (9c)$$
$$\begin{aligned} z(1,t) &= \rho U(t) - r(t) \\ &\quad + \int_0^1 \kappa(\xi)(w(\xi,t) + a(\xi,t))d\xi \\ &\quad + \int_0^1 \theta(\xi)b(1-\xi,t)d\xi + \check{d}, \end{aligned} \quad (9d)$$
$$w(x,0) = w_0(x) \quad (9e)$$
$$z(x,0) = z_0(x) \quad (9f)$$
$$y(t) = z(0,t) + b(0,t). \quad (9g)$$

with $w_0, z_0 \in \mathcal{B}([0,1])$. The goal (7) is then equivalent to achieving

$$\lim_{t\to\infty} \int_t^{t+T} z^2(0,s)ds = 0 \quad (10)$$

The original uncertain parameters, c_1, c_2, k_1, k_2 and q are contained in the new uncertain functions $\theta(x), \kappa(x)$ and $\check{d}, \rho$, where

$$\check{d} = -k_2d, \qquad \rho = k_1k_2, \quad (11)$$

but θ and κ are nonlinear functions of the original parameters, which involve the solutions to complicated kernel equations used in backstepping transformations (these equations can be found in [1]). However, the mappings from c_1, c_2, k_1, k_2 and q to $\theta(x)$ and $\kappa(x)$ are known and can be (numerically) computed (for exact definitions of $\theta(x)$ and $\kappa(x)$, see [1]). The objective is now to take advantage of this known relationship to alleviate overparametrization.

B. Linearization

In our case, we approximate θ, κ, ρ and $\check{d}$ as

$$\theta(x) = f^T(x)\check{\nu}, \qquad \rho = h^T\check{\nu}, \quad (12a)$$
$$\kappa(x) = g^T(x)\check{\nu}, \qquad \check{d} = s^T\check{\nu}, \quad (12b)$$

where

$$\check{\nu} = \begin{bmatrix} 1 & \nu \end{bmatrix}^T \qquad \nu = \begin{bmatrix} \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix}^T, \quad (13)$$

while

$$f(x) = \begin{bmatrix} f_0(x) & f_1(x) \end{bmatrix}^T \qquad h = \begin{bmatrix} h_0 & h_1 \end{bmatrix}^T \quad (14a)$$
$$g(x) = \begin{bmatrix} g_0(x) & g_1(x) \end{bmatrix}^T \qquad s = \begin{bmatrix} s_0 & s_1 \end{bmatrix}^T \quad (14b)$$

are known $n+1$ dimensional vectors containing the scalars $f_0(x), g_0(x), h_0$ and s_0, and n-dimensional vectors $f_1(x), g_1(x), h_1$ and s_1. Inserting (12) into (9), we obtain

$$w_t(x,t) + \overline{\lambda}w_x(x,t) = 0 \quad (15a)$$
$$z_t(x,t) - \overline{\mu}z_x(x,t) = \overline{\mu}f^T(x)\check{\nu}z(0,t) \quad (15b)$$
$$w(0,t) = z(0,t) \quad (15c)$$
$$\begin{aligned} z(1,t) &= h^T\check{\nu}U(t) - r(t) + s^T\check{\nu} \\ &\quad + \int_0^1 g^T(\xi)(w(\xi,t) + a(\xi,t))d\xi\check{\nu} \\ &\quad + \int_0^1 f^T(\xi)b(1-\xi,t)d\xi\check{\nu} \end{aligned} \quad (15d)$$

$$w(x,0) = w_0(x) \tag{15e}$$
$$z(x,0) = z_0(x) \tag{15f}$$
$$y(t) = z(0,t) + b(0,t). \tag{15g}$$

C. Filter Design

As in [1], we introduce the following filters

$$\psi_t(x,t) - \overline{\mu}\psi_x(x,t) = 0, \quad \psi(1,t) = U(t)$$
$$\psi(x,0) = \psi_0(x) \tag{16a}$$
$$\phi_t(x,t) - \overline{\mu}\phi_x(x,t) = 0, \quad \phi(1,t) = y(t) - b(0,t)$$
$$\phi(x,0) = \phi_0(x) \tag{16b}$$
$$P_t(x,\xi,t) + \overline{\lambda}P_\xi(x,\xi,t) = 0, \quad P(x,0,t) = \phi(x,t)$$
$$P(x,\xi,0) = P_0(x,\xi) \tag{16c}$$

and also the following filters for the reference model states

$$M_t(x,\xi,t) - \overline{\mu}M_x(x,\xi,t) = 0, \quad M(1,\xi,t) = a(\xi,t)$$
$$M(x,\xi,0) = M_0(x,\xi) \tag{17a}$$
$$N_t(x,\xi,t) - \overline{\mu}N_x(x,\xi,t) = 0, \quad N(1,\xi,t) = b(1-\xi,t)$$
$$M(x,\xi,0) = M_0(x,\xi) \tag{17b}$$

for some initial conditions satisfying $\psi_0, \phi_0, \in \mathcal{B}([0,1])$, $P_0, M_0, N_0 \in \mathcal{B}([0,1]^2)$. Lastly, we also define the derived filters

$$p_0(x,t) = P(0,x,t), \qquad p_1(x,t) = P(1,x,t) \tag{18a}$$
$$n_0(x,t) = N(0,\xi,t), \qquad m_0(x,t) = M(0,\xi,t). \tag{18b}$$

One can now construct non-adaptive estimates $\bar{w}$ and $\bar{z}$ of the variables w and z as

$$\bar{w}(x,t) = p_1(x,t), \quad \bar{z}(x,t) = \gamma_0(x,t) + \gamma_1^T(x,t)\nu, \tag{19}$$

where

$$\begin{aligned}\gamma_0(x,t) &= h_0\psi(x,t) - b(x,t) \\ &\quad + \int_x^1 f_0(\xi)\phi(1-(\xi-x),t)d\xi \\ &\quad + \int_0^1 g_0(\xi)(P(x,\xi,t) + M(x,\xi,t))d\xi \\ &\quad + \int_0^1 f_0(\xi)N(x,\xi,t)d\xi + s_0\end{aligned} \tag{20a}$$
$$\begin{aligned}\gamma_1(x,t) &= h_1\psi(x,t) + \int_x^1 f_1(\xi)\phi(1-(\xi-x),t)d\xi \\ &\quad + \int_0^1 g_1(\xi)(P(x,\xi,t) + M(x,\xi,t))d\xi \\ &\quad + \int_0^1 f_1(\xi)N(x,\xi,t)d\xi + s_1.\end{aligned} \tag{20b}$$

With the new form of the canonical system, the following result from [1] still holds.

Lemma 1 (Lemma 7 in [1]): Consider the system (15) and the non-adaptive state estimates (19)–(20) generated using the filters (16)–(18). Then

$$\bar{w}(t) \equiv w(t), \qquad \bar{z}(t) \equiv z(t), \tag{21}$$

for $t \geq t_F$, where

$$t_F = d_\alpha + d_\beta \tag{22}$$

with d_α and d_β defined in (5).

IV. Adaptive laws

We follow the same steps as in [1] to obtain a linear parametric model from which adaptive laws can be obtained. Following the relationship (19) and the result of Lemma 1, we have

$$\begin{aligned}y(t) &= z(0,t) + b(0,t) \\ &= \gamma_0(0,t) + b(0,t) + \gamma_1^T(0,t)\nu\end{aligned} \tag{23}$$

for $t \geq t_F$, which is a linear parametric model for which adaptive laws can be derived. We start by assuming the following.

Assumption 2: Bounds on ν are known. That is, we are in knowledge of some constants $\underline{\nu} = [\underline{\nu}_1, \underline{\nu}_2, \dots, \underline{\nu}_n]$ and $\overline{\nu} = [\overline{\nu}_1, \overline{\nu}_2, \dots, \overline{\nu}_n]$ so that

$$\underline{\nu}_i \leq \nu_i \leq \overline{\nu}_i, \quad i = 1,2,\dots,n. \tag{24}$$

Assumption 2 is no restriction, since the bounds are arbitrary. As an adaptive law, we propose the gradient method with projection.

$$\dot{\hat{\nu}}(t) = \begin{cases} 0, & t < t_F \\ \text{proj}_{\underline{\nu},\overline{\nu}}\{\tau(t), \hat{\nu}(t)\}, & t \geq t_F \end{cases}, \quad \hat{\nu}(0) = \hat{\nu}_0 \tag{25}$$

where

$$\tau(t) = \Gamma \frac{\hat{\epsilon}(0,t)\gamma_1(0,t)}{1 + |\gamma_1(0,t)|^2} \tag{26}$$

with $\Gamma = \Gamma^T > 0$ as a design matrix and

$$\hat{\epsilon}(x,t) = z(x,t) - \hat{z}(x,t). \tag{27}$$

is the prediction error, with

$$\hat{z}(x,t) = \gamma_0(x,t) + \gamma_1^T(x,t)\hat{\nu}(t), \tag{28}$$

as an estimate of the state z. The initial guess $\hat{\nu}_0 = \begin{bmatrix}\hat{\nu}_{0,1} & \hat{\nu}_{0,2} & \dots & \hat{\nu}_{0,n}\end{bmatrix}^T$ is chosen inside the feasible domain

$$\underline{\nu}_i \leq \nu_{0,i} \leq \overline{\nu}_i, \qquad i = 1,2,\dots,n, \tag{29}$$

and the projection operator is given as

$$\text{proj}_{a,b}(\tau,\omega) = \begin{cases} 0, & \text{if } \omega = a \text{ and } \tau \leq 0 \\ 0, & \text{if } \omega = b \text{ and } \tau \geq 0 \\ \tau, & \text{otherwise} \end{cases} \tag{30}$$

which for the case of vectors acts component-wise. The following lemma follows directly from [1].

Lemma 3 (Lemma 9 in [1]): The gradient adaptive law guarantees that

$$\underline{\nu}_i \leq \hat{\nu}_i(t) \leq \overline{\nu}_i, \quad \forall t \geq 0,\ i = 1,2,\dots,n \tag{31a}$$
$$\frac{\hat{\epsilon}(0,t)}{\sqrt{1 + |\gamma_1(0,t)|^2}}, \dot{\hat{\nu}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \tag{31b}$$

V. Control Law

The MRAC control law from [1] can straightforwardly be combined with the results of Lemma 3, to obtain the following result.

Theorem 4 (Theorem 10 in [1]): Consider system (3), the reference model (8) and the filters (16)–(18). Suppose $r(t)$ is bounded. Then, the control law

$$\begin{aligned} U(t) = \frac{1}{\hat{h}(t)}\Bigg(& r(t) - \int_0^1 \hat{g}(\xi,t)(p_1(\xi,t)+a(\xi,t))d\xi \\ & - \int_0^1 \hat{f}(\xi,t)b(1-\xi,t)d\xi - \hat{d}(t) \\ & + \int_0^1 \hat{k}(1-\xi,t)\hat{z}(\xi,t)d\xi\Bigg), \end{aligned} \tag{32}$$

where $\hat{h}(t) = h_0(t) + h_1^T(t)\hat{\nu}(t)$, $\hat{g}(\xi,t) = g_0(\xi,t) + g_1^T(\xi,t)\hat{\nu}(t)$, $\hat{f}(\xi,t) = f_0(\xi,t)+f_1^T(\xi,t)\hat{\nu}(t)$, $\hat{d}(t) = s_0(t)+ s_1^T(t)\hat{\nu}(t)$ and $\hat{z}(\xi,t)$ is generated using (28) and $\hat{k}(x,t)$ is the online solution to the Volterra integral equation

$$\hat{k}(x,t) = \int_0^x \hat{k}(x-\xi,t)\hat{f}(\xi,t)d\xi - \hat{f}(x,t) \tag{33}$$

with $\hat{\nu}$ generated from adaptive law (25), guarantee (7). Moreover, all additional variables in the closed loop system are bounded in the L_2-sense.

Remark 5: The projection operator used in the adaptive law (25) must ensure, by an appropriate choice of the bounds (24), that the estimated value $\hat{h}$ does not cross zero, as this will result in a division by zero in (32). This is incorporated in the projection operator used in [1], and is the reason for the requirement (6).

VI. Application to the Control of Downhole Pressure in Oil Well Drilling

A. The Oil Well Model

The following is a model for an MPD system [4]

$$p_t(z,t) = -\frac{\beta}{A}\bar{q}_z(z,t) \tag{34a}$$

$$\bar{q}_t(z,t) = -\frac{A}{\bar{\rho}}p_z(z,t) - \frac{F}{\bar{\rho}}\bar{q}(z,t) - Ag \tag{34b}$$

$$p(l,t) = p_l(t), \quad p(z,0) = p_0(z) \tag{34c}$$

$$\bar{q}(0,t) = q_{bit}, \quad \bar{q}(z,0) = \bar{q}_0(z) \tag{34d}$$

$$y(t) = p(0,t) \tag{34e}$$

where l is depth of the well, $z \in [0,l], t \geq 0, p(z,t)$ is pressure, $\bar{q}(z,t)$ is volumetric flow, β is the bulk modulus of the mud, $\bar{\rho}$ is the density of the mud, A is the cross sectional area of the annulus, F is the friction factor, g is the acceleration of gravity and $q_{bit}(t)$ is the flow from the drill string into the annulus around the drill string. The initial conditions satisfy $p_0, \bar{q}_0 \in \mathcal{B}([0,1])$. We assume that the choke is operated by a fast pressure controller so that we can treat the pressure at the top of the well, $p_l(t)$, rather than choke opening, as our manipulated variable. The control objective is to have the downhole pressure follow the reference $y_r(t)$, that is

$$p(0,t) = y_r(t). \tag{35}$$

with y_r generated using a reference model on the form (8). The only uncertain parameter in the oil well model (34) is the scalar friction factor F.

B. Feasibility of design

To use the estimation theory on this problem, we need to transform the system into the form (3). Using the invertible transformation

$$\begin{aligned} u(x,t) = \frac{1}{2}\Bigg(& (\bar{q}(xl,t) - q_{bit}) \\ & + \frac{A}{\sqrt{\bar{\rho}\beta}}(p(xl,t) + \frac{Flq_{bit}}{A} + \bar{\rho}glx)\Bigg)e^{\frac{lF}{2\sqrt{\bar{\rho}\beta}}x} \end{aligned} \tag{36a}$$

$$\begin{aligned} v(x,t) = \frac{1}{2}\Bigg(& (\bar{q}(xl,t) - q_{bit}) \\ & - \frac{A}{\sqrt{\bar{\rho}\beta}}(p(xl,t) + \frac{Flq_{bit}}{A} + \bar{\rho}glx)\Bigg)e^{-\frac{lF}{2\sqrt{\bar{\rho}\beta}}x}, \end{aligned} \tag{36b}$$

we obtain (3) with system parameters

$$\lambda(x) = \mu(x) = \frac{1}{l}\sqrt{\frac{\beta}{\bar{\rho}}} \tag{37a}$$

$$c_1(x) = -\frac{1}{2}\frac{F}{\bar{\rho}}e^{\frac{lF}{\sqrt{\bar{\rho}\beta}}x}, \qquad c_2(x) = -\frac{1}{2}\frac{F}{\bar{\rho}}e^{-\frac{lF}{\sqrt{\bar{\rho}\beta}}x} \tag{37b}$$

$$k_1 = \frac{1}{2}e^{\frac{lF}{2\sqrt{\bar{\rho}\beta}}}, \qquad k_2 = -2\frac{\sqrt{\bar{\rho}\beta}}{A} \tag{37c}$$

$$d = -\frac{q_{bit}lF}{2\sqrt{\bar{\rho}\beta}}e^{-\frac{lF}{\sqrt{\bar{\rho}\beta}}}, \qquad q = -1, \tag{37d}$$

For more details on this, see the similar transformation in [22].

Now that we have a system in the proper form, we can see that most of the system parameters (37) depend non-linearly on the uncertain parameter F. It is possible to use the method proposed in [1] directly, which was what was done in [2]. However, treating the coefficients (37) as completely unknown functions of x leads to severe overparameterization. Thus, we propose to use the approximation (12).

C. Linearization

We assume that bounds on F are given as

$$\underline{F} \leq F \leq \overline{F}. \tag{38}$$

As a result of the aforementioned continuous mapping of the MPD system (34) into the canonical from (15), there exists continuous mappings, m_θ and m_κ such that $\theta(x) = m_\theta(F)$ and $\kappa(x) = m_\kappa(F)$. Given that $F \in [\underline{F}, \overline{F}]$, we approximate $\theta(x)$ by a linear interpolation between $\underline{\theta}(x) = m_\theta(\underline{F})$ and $\overline{\theta}(x) = m_\theta(\overline{F})$, and $\kappa(x)$ by a linear interpolation between $\underline{\kappa}(x) = m_\kappa(\underline{F})$ and $\overline{\kappa}(x) = m_\kappa(\overline{F})$, that is, we select $n = 1$, and

$$f_0(x) = \underline{\theta}(x), \qquad f_1(x) = \overline{\theta}(x) - \underline{\theta}(x) \tag{39a}$$

$$g_0(x) = \underline{\kappa}(x), \qquad g_1(x) = \overline{\kappa}(x) - \underline{\kappa}(x) \tag{39b}$$

with $\nu = \frac{F - \underline{F}}{\overline{F} - \underline{F}}$ for $F \in [\underline{F}, \overline{F}]$. This gives $\nu \in [0, 1]$. Furthermore, for ρ and $\check{d}$, we have

$$h_0 = -\frac{\sqrt{\rho \beta}}{A} e^{\frac{l F}{\sqrt{\rho \beta}}} \tag{40a}$$

$$h_1 = -\frac{\sqrt{\rho \beta}}{A} e^{\frac{l \overline{F}}{\sqrt{\rho \beta}}} + \frac{\sqrt{\rho \beta}}{A} e^{\frac{l \underline{F}}{\sqrt{\rho \beta}}} \tag{40b}$$

$$s_0 = -\frac{l \underline{F}}{2A} q_{bit} e^{-\frac{l \underline{F}}{\sqrt{\rho \beta}}} \tag{40c}$$

$$s_1 = -\frac{l \overline{F}}{2A} q_{bit} e^{-\frac{l \overline{F}}{\sqrt{\rho \beta}}} + \frac{l \underline{F}}{2A} q_{bit} e^{-\frac{l \underline{F}}{\sqrt{\rho \beta}}}. \tag{40d}$$

D. Simulation

The MPD model (34) and the controller of Theorem 4 were implemented in MATLAB, using the system parameters

$$\bar{\beta} = 7317 \cdot 10^5 \ Pa, \qquad \bar{\rho} = 1250 \ kg/m^3, \tag{41a}$$

$$l = 2500 \ m, \qquad A_1 = 0.024 \ m^2, \tag{41b}$$

$$A_2 = 0.02 \ m^2, \qquad g = 9.81 \ m/s^2. \tag{41c}$$

The uncertain friction factor F was set to

$$F = \begin{cases} 1000 & \text{for } t < 140 \ kg/m^3 s \\ 800 & \text{for } t \geq 140 \ kg/m^3 s \end{cases} \tag{42}$$

The friction factor is set to change at $t = 140 \ s$ to demonstrate the adaptive properties of the method. The adaptation gain was set to

$$\Gamma = 100, \tag{43}$$

while the lower and upper values for F were set to

$$\underline{F} = 700, \qquad \bar{F} = 1200. \tag{44}$$

The method from [2] was also implemented for comparison, using the same adaptation gains. The initial parameter estimates for the method from [2] were set to values corresponding to the initial value of ν, that is $\nu = \nu_0$.

1) Case 1: The initial guess for ν was in this case set to 1.0, corresponding to a friction factor $F = \bar{F}$. It can be observed from the tracking objective displayed in Figure 1 and the tracking error shown in Figure 2, that both methods manage to track the reference signal. However, the old method oscillates and heavily overshoots during set point changes, and especially when the friction factor changes at $t = 140$.

The initial transient seen for both methods are due to the initialization of the swapping filters, and the new method is also a bit slower at the initial tracking, but this is due to the initial slow adaptation, as seen from Figure 3. However, the rate of adaptation can be improved by increasing the adaptation gain.

2) Case 2: For this case the initial guess for ν is set to 0.0, corresponding to a friction factor $F = \underline{F}$. Heavy oscillations are observed in Figures 4 and 5 for the old method. Such oscillations are not present for the new method. The oscillations for the old method die out however, and the old method manages to track the reference reasonably well when the friction factor changes at $t = 140$. The friction

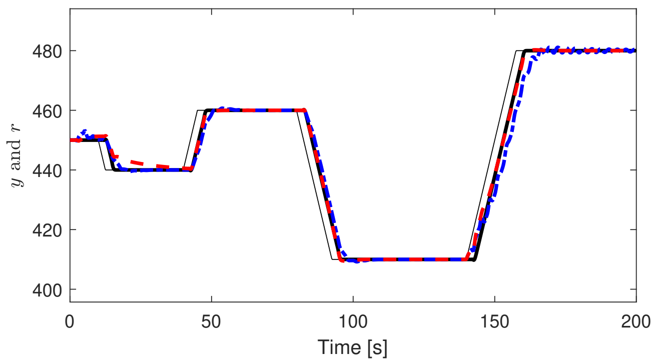

Fig. 1: Case 1: Reference signal y_r (thick black) and measured downhole pressure for the new (red) and old (blue) methods. Also plotted is the input r to the reference model (thin black).

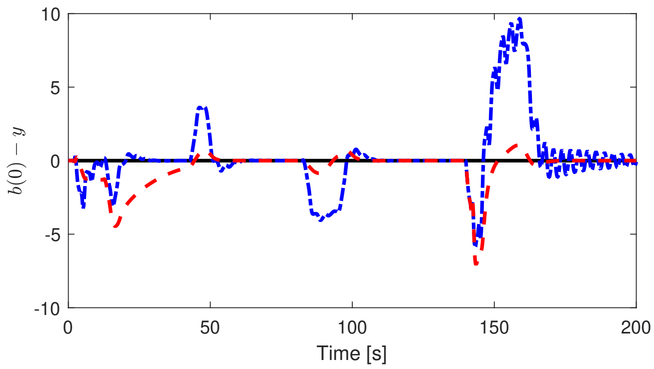

Fig. 2: Case 1: Tracking error for the old (blue) and new (red) methods.

factor is estimated quite well by the new method in both cases, as seen from Figures 3 and 6. An estimate of the friction factor is not provided by the old method.

VII. Conclusion

We have proposed a method for reducing overparametrization for model reference adaptive control (MRAC) of linear hyperbolic PDEs. The method is based on linearization of the distributed, uncertain parameters, and expressing them as a linear combination of a finite number of parameters. The method was shown in simulations on a managed pressure drilling system to perform better than the previously derived MRAC controller which uses a heavily overparameterized scheme.

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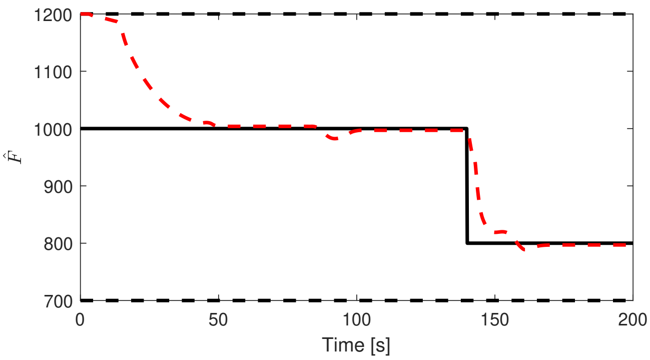

Fig. 3: Case 1: Estimated (red) and actual (black) friction factor F.

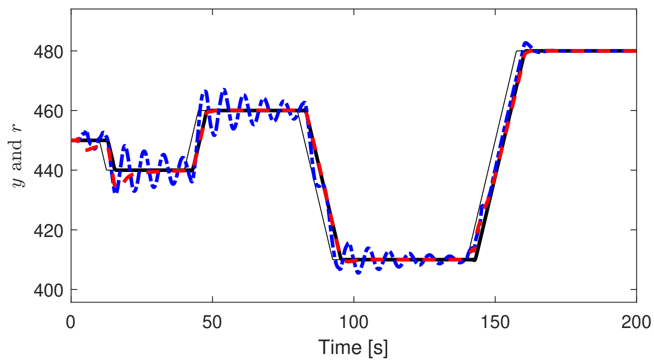

Fig. 4: Case 2: Reference signal y_r (thick black) and measured downhole pressure for the new (red) and old (blue) methods. Also plotted is the input r to the reference model (thin black).

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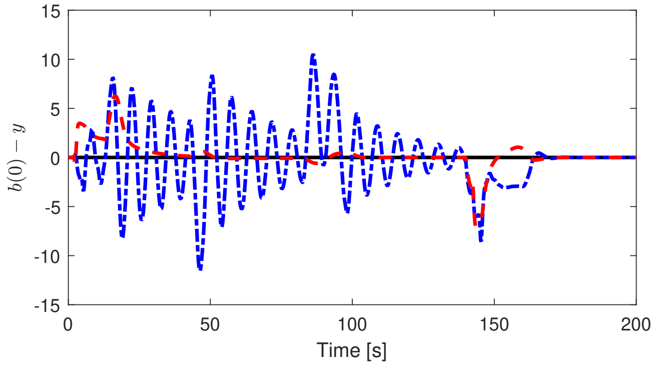

Fig. 5: Case 1: Tracking error for the old (blue) and new (red) methods.

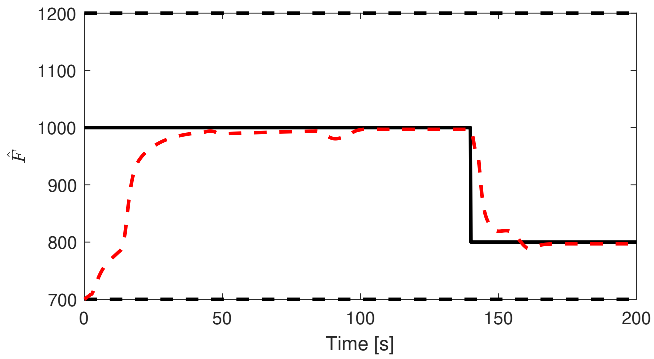

Fig. 6: Case 2: Estimated (red) and actual (black) friction factor F.

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