Adaptive Control of a Linear Hyperbolic PDE with Uncertain Transport Speed and a Spatially Varying Coefficient

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***Abstract*— Recently, the first result on backstepping-based adaptive control of a 1-D linear hyperbolic partial differential equation (PDE) with an uncertain transport speed was presented. The system also had an uncertain, constant in-domain coefficient, and the derived controller achieved convergence to zero in the L_∞-sense in finite time. In this paper, we extend that result to systems with a spatially varying in-domain coefficient, achieving asymptotic convergence to zero in the L_∞-sense. Additionally, for the case of having a constant in-domain coefficient, the new method is shown to have a slightly improved finite-time convergence time. The theory is illustrated in simulations.**

I. Introduction

A. Motivation and problem statement

In this paper, we consider a scalar 1–D linear hyperbolic partial differential equation (PDE) in the form

$$u_t(x,t) - \mu u_x(x,t) = \theta(x)u(0,t) \tag{1a}$$

$$u(1,t) = U(t) \tag{1b}$$

$$u(x,0) = u_0(x) \tag{1c}$$

where $u(x,t)$ is the system state defined on $\mathcal{D}_1$, where

$$\mathcal{D}_1 = \{(x,t) \mid x \in \mathcal{D}, t \geq 0\}, \quad \mathcal{D} = \{x \mid x \in [0,1]\}. \tag{2}$$

The system parameters and initial condition are unknown, but assumed to satisfy

$$\mu \in \mathbb{R},\ \mu > 0, \qquad \theta \in C^1(\mathcal{D}), \qquad u_0 \in C^1(\mathcal{D}). \tag{3}$$

The goal is to design a backstepping-based control law $U : [0,\infty) \to \mathbb{R}$ that adaptively stabilizes system (1) despite having the parameters and initial condition (3) unknown, and using boundary sensing only. The following boundary measurements are assumed available

$$y_0(t) = u(0,t), \qquad y_1(t) = u(1,t) \tag{4a}$$

$$\vartheta_0(t) = u_x(0,t), \qquad \vartheta_1(t) = u_x(1,t). \tag{4b}$$

Adaptive control of systems in the form (1) using backstepping have recently been extensively studied. Starting with results on parabolic PDEs [1], [2], [3], [4], [5], hyperbolic PDEs have been given the most attention recently. The first result on adaptive control of a hyperbolic PDE was in [6], with numerous extensions following in for instance [7], [8], [9], [10], [11].

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The second author gratefully acknowledges the support from the Research Council of Norway and Equinor ASA through project no. 255348/E30 Sensors and models for improved kick/loss detection in drilling (Semi-kidd).

However, in all the above mentioned results, the systems' transport speeds are assumed known. The first result achieving estimation involving an uncertain transport speed was presented in [12], where four different methods for estimating the transport speed were presented, three of which were proved to converge subject to some requirements of persistence of excitation, and only one of those methods relied on boundary measurements only. That method was in [13] combined with an event-triggered least-squares identifier inspired by the identifier in [14] into an event-triggered finite-time convergent estimator scheme, and then combined with a feedback law into an output-feedback control law for adaptively stabilizing a linear hyperbolic PDE with uncertain transport speed and an uncertain in-domain coefficient. The result was an algorithm that adaptively stabilized a linear hyperbolic PDE in several steps: 1) The transport speed was estimated in finite time, 2) a linear parametric form was constructed using swapping filters, 3) the in-domain parameter was estimated in finite time, and 4) the estimated parameters were used to generate an estimate of the state, which was then used in a control law achieving convergence to zero in finite time.

A similar technique was previously used in [15], combining the event-triggered finite-time convergent estimator scheme from [14] with a boundary controller, for adaptively stabilizing a parabolic PDE. The resulting control law used state feedback, however, and since parabolic PDEs cannot be finite-time controlled, the convergence was asymptotic.

In this paper, we modify the method [13], to also handle a spatially varying source term $\theta(x)$ in (1a), and propose an estimation technique for simultaneously estimating the transport speed and the endpoints of $\theta(x)$. In the case of having a constant θ, thus, the method can be modified and finite-time convergence of the system state despite having an unknown transport speed is achieved. The convergence time is slightly faster than what was achieved in [13].

As in [13], we assume a continuously differentiable initial condition u_0. We will also, as in [13], assume that the state u always will remain in $C^1(\mathcal{D})$, meaning that the boundary control law $U(t)$ must be compatible so that this is the case. This is formally stated in the following assumption:

Assumption 1: We assume $u(t) \in C^1(\mathcal{D})$.

We will achieve this property by limiting the considered solutions to those compatible with this assumption.

Regarding the uncertain parameters, we assume the following.

Assumption 2: A lower bound on μ and an upper bound on the absolute value of $\theta(x)$ are known, specifically, we are in knowledge of positive parameters $\underline{\mu}$ and $\bar{\theta}$ so that

$$\underline{\mu} \leq \mu, \qquad |\theta(x)| \leq \bar{\theta}, \quad \forall x \in \mathcal{D}. \tag{5}$$

B. Notation

For two (possibly time-varying) signals $a(x), b(x)$ defined for $x \in \mathcal{D}$, the L_∞-norm is defined as

$$||a||_\infty = \max_{x\in\mathcal{D}} |a(x)| \tag{6}$$

while the operator $\equiv$ is defined as

$$a \equiv b \Leftrightarrow ||a - b|| = 0. \tag{7}$$

II. Events and event triggers

As in [13], we introduce a series of periodic time-triggered events with a fixed interval between each event. Since linear hyperbolic PDE in the form (1) is finite-time observable [16] in a time d defined as

$$d = \mu^{-1} \tag{8}$$

which is an unknown quantity, we choose an interval larger than d. We choose a time T between intervals taken as

$$T = \underline{\mu}^{-1} \tag{9}$$

where $\underline{\mu}$ is as stated in Assumption 2. Since $\underline{\mu} \leq \mu$, this means that $T \geq d$, and hence system (1) is finite-time observable within a time T.

We will let τ_i denote the ith event, which occurs at time

$$\tau_i = iT, \quad i \in \mathbb{N} \tag{10}$$

while the interval between the $(i-N)$th and ith events, $N \in \mathbb{N}^+$, is denoted

$$\mathcal{B}_i^N = [\tau_{\max\{0,i-N\}}, \tau_i], \quad i, N \in \mathbb{N}^+ \tag{11}$$

with the short-hand notation

$$\mathcal{B}_i = \mathcal{B}_i^1, \tag{12}$$

where the max operator is included to ensure that data prior to $t = 0$ is not used. Using the notation (11)–(12), means that for instance $t \in \mathcal{B}_1 \Leftrightarrow t \in [0, T]$ and $t \in \mathcal{B}_3^2 \Leftrightarrow t \in [T, 3T]$, and that (1) is observable within interval $\mathcal{B}_i^N$ for any $i, N \in \mathbb{N}^+$.

III. Estimation of the parameters μ, $\theta(0)$, $\theta(1)$

A. A linear equation in μ, $\theta(0)$, $\theta(1)$

We start by presenting a set of equations in the transport speed μ, as well as the parameter $\theta(x)$'s endpoints. Consider the following six equations in the three unknowns

$$A_i \nu = b_i \tag{13}$$

where

$$A_i = \begin{bmatrix} Q_{i,11} & -Q_{i,12} & Q_{i,12} \\ Q_{i,12} & -Q_{i,22} & Q_{i,22} \\ G_{i,11} & 0 & G_{i,12} \\ G_{i,12} & 0 & G_{i,22} \\ H_{i,11} & H_{i,12} & 0 \\ H_{i,12} & H_{i,22} & 0 \end{bmatrix} \tag{14a}$$

$$\nu = \begin{bmatrix} \mu & \theta_0 & \theta_1 \end{bmatrix}^T \tag{14b}$$

$$b_i = \begin{bmatrix} P_{i,1} & P_{i,2} & F_{i,1} & F_{i,2} & J_{i,1} & J_{i,2} \end{bmatrix}^T, \tag{14c}$$

where

$$\theta_0 = \theta(0), \qquad \theta_1 = \theta(1). \tag{15}$$

The components of A_i and b_i are generated from

$$P_i = \begin{bmatrix} P_{i,1} \\ P_{i,2} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (pq)(t,\sigma) d\sigma dt \tag{16a}$$

$$F_i = \begin{bmatrix} F_{i,1} \\ F_{i,2} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (fg)(t,\sigma) d\sigma dt \tag{16b}$$

$$J_i = \begin{bmatrix} J_{i,1} \\ J_{i,2} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (jh)(t,\sigma) d\sigma dt \tag{16c}$$

and

$$Q_i = \begin{bmatrix} Q_{i,11} & Q_{i,12} \\ Q_{i,12} & Q_{i,22} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (qq^T)(t,\sigma) d\sigma dt \tag{17a}$$

$$G_i = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,12} & G_{i,22} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (gg^T)(t,\sigma) d\sigma dt \tag{17b}$$

$$H_i = \begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,12} & H_{i,22} \end{bmatrix} = \int_{\mathcal{B}_i^N} \int_{\mathcal{B}_i^N} (hh^T)(t,\sigma) d\sigma dt, \tag{17c}$$

where

$$p(t,\sigma) = \eta(t) - \eta(\sigma) \tag{18a}$$

$$f(t,\sigma) = U(t) - U(\sigma) \tag{18b}$$

$$j(t,\sigma) = y_0(t) - y_0(\sigma) \tag{18c}$$

and

$$q(t,\sigma) = \begin{bmatrix} q_1(t,\sigma) & q_2(t,\sigma) \end{bmatrix}^T = \int_\sigma^t z(s) ds \tag{19a}$$

$$g(t,\sigma) = \begin{bmatrix} g_1(t,\sigma) & g_2(t,\sigma) \end{bmatrix}^T = \int_\sigma^t r(s) ds \tag{19b}$$

$$h(t,\sigma) = \begin{bmatrix} h_1(t,\sigma) & h_2(t,\sigma) \end{bmatrix}^T = \int_\sigma^t m(s) ds \tag{19c}$$

with

$$\eta(t) = y_1(t) - y_0(t) \tag{20a}$$

$$z(t) = \begin{bmatrix} \omega(t) & y_0(t) \end{bmatrix}^T \tag{20b}$$

$$r(t) = \begin{bmatrix} \vartheta_1(t) & y_0(t) \end{bmatrix}^T \tag{20c}$$

$$m(t) = \begin{bmatrix} \vartheta_0(t) & y_0(t) \end{bmatrix}^T \tag{20d}$$

$$\omega(t) = \vartheta_1(t) - \vartheta_0(t) \tag{20e}$$

for some $N \in \mathbb{N}^+$. We note that all components in A_i and b_i can be generated using measured quantities.

Lemma 3: Consider system (1), and the equation (13) with the components of A_i and b_i at the ith event generated using the equations (14)–(20). Suppose the actuation signal U is taken to satisfy the differential equality

$$\dot{U}(t) = a(t)\vartheta_0(t) + b(t)y_0(t), \quad \forall t \in \mathcal{B}_i^N \tag{21}$$

for some $N \in \mathcal{N}^+$, and some bounded coefficients $a(t), b(t)$ that are non-constant and non-identical over $\mathcal{B}_i^N$. Then the Moore-Penrose inverse

$$A_i^+ = (A_i^T A_i)^{-1} A_i^T \tag{22}$$

of A_i exists.

Remark 4: The coefficients of a and b over the most recent interval $\mathcal{B}_i$ can for instance be taken as

$$a(t) = A\cos(\varpi_1 t) \qquad b(t) = B\cos(\varpi_2 t) \tag{23}$$

for some nonzero constants A, B and angular frequencies $\varpi_1, \varpi_2 \neq 0$, $\varpi_1 \neq \varpi_2$.

We will dedicate the next subsections to deriving the equations (13)–(20), and also the condition (21) on the actuation that leads to the existence of (22).

B. Dynamics of u_x

We proceed as in [13] and derive the dynamics of the spatial derivative u_x, and define the new variable v as

$$v(x,t) = u_x(x,t), \tag{24}$$

for which we derive the dynamics from (1) to obtain

$$v_t(x,t) - \mu v_x(x,t) = \theta'(x)u(0,t) \tag{25a}$$

$$v(1,t) = d\dot{U}(t) - d\theta(1)u(0,t) \tag{25b}$$

$$v(x,0) = v_0(x) \tag{25c}$$

where $v_0(x) = u_0'(x)$.

C. Obtaining parameters using Q_i

This method is based on [13]. However, we slightly modify it to use data from a longer time series, and also accommodate for the spatially varying θ. We proceed as in [13] and rewrite the definition of η in (20a) as

$$\begin{aligned}\eta(t) &= y_1(t) - y_0(t)\\ &= u(1,t) - u(0,t) = \int_0^1 u_x(x,t)dx\\ &= \int_0^1 v(x,t)dx.\end{aligned} \tag{26}$$

Differentiating (26) with respect to time, inserting the dynamics (25a) and solving the integrals, we obtain

$$\begin{aligned}\dot{\eta}(t) &= \mu\int_0^1 v_x(x,t)dx + \int_0^1 \theta'(x)dx u(0,t)\\ &= \mu(v(1,t) - v(0,t)) + (\theta_1 - \theta_0)u(0,t)\\ &= a^T z(t)\end{aligned} \tag{27}$$

where z is defined in (20b), and

$$a = \begin{bmatrix}\mu & (\theta(1) - \theta(0))\end{bmatrix}^T \tag{28}$$

Integrating (27) from σ to t, we obtain

$$p(t,\sigma) = a^T q(t,\sigma) \tag{29}$$

where p and q are defined in (18a) and (19a), respectively. Consider the cost function

$$V_i(\alpha) = \int_{\mathcal{B}_i^N}\int_{\mathcal{B}_i^N} (p(t,\sigma) - \alpha^T q(t,\sigma))^2 d\sigma dt \tag{30}$$

which equally weights all measurements in the N most recent intervals. It is clear that $\alpha = a$ minimizes (30). Evaluating $\frac{\partial V_i(\alpha)}{\partial \alpha} = 0$, we obtain

$$P_i = Q_i a \tag{31}$$

where P_i and Q_i are defined in (16a) and (17a), respectively.

If $\det(Q_i) \neq 0$, a can be computed as $a = Q_i^{-1}P_i$. We will now investigate in what cases $\det(Q_i) \neq 0$, and show that choosing U to satisfy (21) over $\mathcal{B}_i$ results in $\det(Q_i) \neq 0$.

First of all, since by Assumption 1, u and hence $u(0,t)$ is assumed continuous, $Q_{i,22} = 0$ is equivalent to $q_2(t,\sigma) = 0, \quad \forall t, \sigma \in \mathcal{B}_i^N$, which in turn implies $u(0,t) = 0$ for all $t \in \mathcal{B}_i^N$, and due to the finite-time observability property of system (1), this implies $u(t) \equiv 0$ for all $t \in \mathcal{B}_i^N$, and the control objective is trivially achieved. Hence, we assume $q_2 \not\equiv 0$ for the interval. If $\det(Q_i) = 0$, we must have

$$Q_{i,11}Q_{i,22} = Q_{i,12}^2, \tag{32}$$

and since Cauchy–Schwarz' inequality holds as an equality only when the two functions q_1, q_2 are linearly dependent, we obtain the relationship

$$q_1(t,\sigma) = \lambda q_2(t,\sigma) \tag{33}$$

for some constant $\lambda \in \mathbb{R}$, and hence

$$\int_\sigma^t \omega(s)ds = \lambda\int_\sigma^t u(0,s)ds \tag{34}$$

for all σ, t in the interval, meaning that

$$\omega(t) = \lambda u(0,t), \tag{35}$$

or by the definition of ω in (20e)

$$v(1,t) = v(0,t) + \lambda u(0,t) \tag{36}$$

for all $t \in \mathcal{B}_i$. Inserting the boundary condition (25b), yields

$$\dot{U}(t) = \mu v(0,t) + (\mu\lambda + \theta(1))u(0,t) \tag{37}$$

Hence, $\det(Q_i) = 0$ can only happen if $\dot{U}$ is a linear combination of $v(0,t)$ and $u(0,t)$ for all $t \in \mathcal{B}_i^N$. The choice of $\dot{U}$ as (21) ensures that this is never the case, and thus, $\det(Q_i) \neq 0$.

D. Obtaining parameters using G_i

Consider the relationship (25b). By solving for $\dot{U}$, we obtain

$$\dot{U}(t) = \mu v(1,t) + \theta(1)u(0,t) = b^T r(t) \tag{38}$$

where r is defined in (20c), and

$$b = \begin{bmatrix}\mu & \theta(1)\end{bmatrix}^T. \tag{39}$$

Integrating (38) from σ to t, we find:

$$f(t,\sigma) = b^T g(t,\sigma) \tag{40}$$

for f and g defined in (18b) and (19b), respectively. Proceeding as in the previous method, by forming a cost function and equating its derivative to zero, we obtain

$$F_i = G_i b \tag{41}$$

for F_i and G_i defined in (16b) and (17b), respectively. Once again, we can assume $G_{i,22} \neq 0$, because $G_{i,22} = 0$ implies $u(t) \equiv 0$. If $\det(G_i) = 0$, we must have:

$$G_{i,11} G_{i,22} = G_{i,12}^2, \tag{42}$$

which implies that

$$g_1(t, \sigma) = \zeta g_2(t, \sigma) \tag{43}$$

for all $t, \sigma \in \mathcal{B}_i$, and for some constant $\zeta \in \mathbb{R}$, implying

$$v(1, t) = \zeta u(0, t) \tag{44}$$

for all $t \in \mathcal{B}_i$. Inserting the boundary condition (25b), we obtain

$$\dot{U}(t) = (\mu\zeta + \theta_1) u(0, t) \tag{45}$$

Hence: $\det(G_i) = 0$ can only happen if $\dot{U}$ is a constant times $u(0, t)$ for all $t \in \mathcal{B}_i$, which is avoided by choosing $\dot{U}(t) = a(t)u(0, t)$ for some non-constant function $a(t)$, or choosing $\dot{U}$ as (21).

E. Obtaining parameters using H_i

Consider the dynamics (1a). Evaluating at $x = 0$ gives

$$u_t(0, t) = \mu u_x(0, t) + \theta(0) u(0, t) = c^T m(t) \tag{46}$$

where m is defined in (20d), and

$$c = \begin{bmatrix} \mu & \theta(0) \end{bmatrix}^T \tag{47}$$

Integrating (38) from σ to t, we find:

$$j(t, \sigma) = c^T h(t, \sigma) \tag{48}$$

for j and h defined in (18c) and (19c), respectively. Proceeding as in the previous two methods gives

$$J_i = H_i c \tag{49}$$

for J_i and H_i defined in (16c) and (17c), respectively.

Again, it is evident that if $\det(H_i) \neq 0$, the value of c can be computed as $c = H_i^{-1} J_i$.

F. Proof of Lemma 3

Proof: [Proof of Theorem 3] It was shown in Sections III-C and III-D that choosing the actuation U to satisfy the differential equality (21) ensures $\det(Q_i) \neq 0$ and $\det(G_i) \neq 0$. From the equations (31) and (41), it is evident that μ, θ_0 and θ_1 therefore can be computed by solving (31) and (41). This also implies that the matrix A_i defined in (14a) at time $t = \tau_i$ has full column rank, and hence, the Moore-Penrose inverse (22) exists and is unique. ∎

Remark 5: While conditions for invertibility of Q_i and G_i are derived in Sections III-C and III-D, no conditions are offered for invertibility of H_i in Section III-E. The reason is that invertibility of Q_i and G_i is sufficient for the Moore-Penrose inverse (22) to exist. We nevertheless include H_i in A_i because it may improve the condition number of A_i.

G. Estimation theorem

Using Lemma 3, we state the following Theorem.

Algorithm 1 Estimation of ν

1) Let

$$\hat{\nu}(0) = \hat{\nu}_0. \tag{50}$$

for some initial guess

$$\hat{\nu}_0 = \begin{bmatrix} \hat{\mu}_0 & \hat{\theta}_{0,0} & \hat{\theta}_{1,0} \end{bmatrix}^T \tag{51}$$

with $\hat{\mu}_0 \geq \underline{\mu}$.

2) Select the estimation horizon $N \in \mathbb{N}^+$, and at event i, set $\hat{\nu}$ as

$$\hat{\nu}(\tau_i) = \begin{cases} A_i^+ b_i & \text{if } \operatorname{rank}(A_i) = 3 \\ \hat{\nu}(\tau_{i-1}) & \text{otherwise} \end{cases} \tag{52}$$

where A_i and b_i are generated from (14)–(20) for some $N \in \mathbb{N}^+$, and A_i^+ is the Moore-Penrose inverse given as (22).

3) For all times $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \ldots$ between events, the estimate $\hat{\nu}$ is set to the most recent event-triggered estimate, that is

$$\hat{\nu}(t) = \hat{\nu}(\tau_i), \quad \forall t \in [\tau_i, \tau_{i+1}). \tag{53}$$

Lemma 6: Consider system (1). For any $N \in \mathbb{N}^+$, if the actuation is chosen as (21) for $t \in \mathcal{B}_1$, the method of Algorithm 1 produces the correct estimate ν for $t \geq \tau_1$.

Proof: From Lemma 3, choosing U to satisfy the differential equality (21) will ensure that the Moore-Penrose inverse A_i^+ exists at each event $t = \tau_1$ and hence, will produce a nonsingular matrix $A_i^T A_i$ from $t = \tau_1$. ∎

For the remainder of the paper, we will denote the event for which ν is estimated as the i_μth event (which should equal 1), so that τ_{i_μ} is the time for which $\hat{\nu}(t) = \nu$, $\forall t \geq \tau_{i_\nu}$.

Remark 7: If it is known that θ is constant, the last two columns of A_i can be combined, yielding a simplified matrix A_i and vector ν of unknowns as

$$A_i = \begin{bmatrix} Q_{i,11} & 0 \\ Q_{i,12} & 0 \\ G_{i,11} & G_{i,12} \\ G_{i,12} & G_{i,22} \\ H_{i,11} & H_{i,12} \\ H_{i,12} & H_{i,22} \end{bmatrix}, \qquad \nu = \begin{bmatrix} \mu & \theta \end{bmatrix}^T. \tag{54}$$

The vector b_i remains unchanged.

IV. Adaptive control

A. Spatially varying $\theta(x)$

Now that μ has been estimated using the method of the previous section, we can proceed by introducing a swapping-based adaptive control law. This adaptive control method was first introduced in [6] for the case $\mu = 1$. The slight extension to non-unity μ was first presented in [17] for solving an MRAC problem. We introduce the filters

$$\phi_t(x, t) - \hat{\mu}(t)\phi_x(x, t) = 0, \qquad \phi(1, t) = y_0(t)$$

$$\phi_0(x,0) = \phi_0(x) \tag{55a}$$
$$\psi_t(x,t) - \hat{\mu}(t)\psi_x(x,t) = 0, \qquad \psi(1,t) = U(t)$$
$$\psi_0(x,0) = \psi_0(x) \tag{55b}$$

for some initial conditions $\phi_0, \psi_0 \in C^1(\mathcal{D})$ of choice, and the adaptive law

$$\hat{\theta}_t(x,t) = \begin{cases} 0, & t \in [0, t_{i_\mu}) \\ \text{proj}_{\bar{\theta}}\left\{f(x,t), \hat{\theta}(x,t)\right\}, & t \geq t_{i_\mu} \end{cases} \tag{56a}$$
$$\hat{\theta}(x,0) = \hat{\theta}_0(x) \tag{56b}$$

where

$$f(x,t) = \gamma(x)\frac{\hat{e}(0,t)\phi(1-x,t)}{1 + ||\phi(t)||^2} \tag{57}$$

with the projection operator defined as

$$\text{proj}_a(\tau,\omega) = \begin{cases} 0 & \text{if } \omega = -a \text{ and } \tau \leq 0 \\ 0 & \text{if } \omega = a \text{ and } \tau \geq 0 \\ \tau & \text{otherwise.} \end{cases} \tag{58}$$

for some initial guess $\hat{\theta}_0 \in C^1(\mathcal{D})$, with

$$\hat{u}(x,t) = \psi(x,t) + \frac{1}{\hat{\mu}(t)}\int_x^1 \hat{\theta}(\xi,t)\phi(1-\xi+x,t)d\xi \tag{59a}$$
$$\hat{e}(x,t) = u(x,t) - \hat{u}(x,t). \tag{59b}$$

Consider also the adaptive control law

$$U(t) = \int_0^1 \hat{k}(1-\xi,t)\hat{u}(\xi,t)d\xi \tag{60}$$

where $\hat{k}$ is the on-line solution to the Volterra integral equation

$$\hat{\mu}(t)\hat{k}(x,t) = \int_0^x \hat{k}(x-\xi,t)\hat{\theta}(\xi,t)d\xi - \hat{\theta}(x,t). \tag{61}$$

Theorem 8: Consider system (1), the filters (55) and adaptive law (56), with the estimate $\hat{\mu}$ of μ generated using Algorithm 1. The control law (60) guarantees

$$||u||, ||\psi||, ||\phi||, ||u||_\infty, ||\psi||_\infty, ||\phi||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{62a}$$
$$||u||, ||\psi||, ||\phi||, ||u||_\infty, ||\psi||_\infty, ||\phi||_\infty \to 0. \tag{62b}$$

Proof: By Lemma 6, $\hat{\mu}(t) = \mu$ for $t \geq \tau_{i_\mu}$. The result then follows immediately from for instance [11, Theorem 5.1]. ■

Remark 9: It should be possible to improve the convergence time by utilizing the fact that the endpoint values of $\theta(x)$ are known from the estimation method described in Algorithm 1. However, this possibility is not investigated in this paper.

Remark 10: We have not payed attention to ensuring that solutions stay in $C^1(\mathcal{D})$, which is an assumption stated in Assumption 1, and is assumed the mathematical analysis. A modification to the parameter update laws would be needed to render $U(t)$ sufficiently smooth for Assumption 1 to hold. Such a modification would only serve to hide the main aspects of the design in unnecessary technical details.

B. Constant θ

If it is known beforehand that the parameter θ is a constant, finite-time convergence of the state can be achieved, as in [13]. The convergence time can be slightly improved by the method presented in this paper. The method of Algorithm 1 will then produce an estimate $\hat{\nu}$, where the latter two components of $\hat{\nu}$ are identical estimates of the parameter θ.

We introduce the state observer

$$\hat{u}_t(x,t) - \hat{\mu}(t)\hat{u}_x(x,t) = \hat{\theta}(t)y_0(t) \tag{63a}$$
$$\hat{u}(1,t) = U(t) \tag{63b}$$
$$\hat{u}(x,0) = \hat{u}_0(x) \tag{63c}$$

and the control law

$$U(t) = -\frac{\hat{\theta}(t)}{\hat{\mu}(t)}\int_0^1 \exp\left(\frac{\hat{\theta}(t)}{\hat{\mu}(t)}(1-\xi)\right)\hat{u}(\xi,t)d\xi. \tag{64}$$

Theorem 11: Consider system (1) where it is known that the parameter θ is a constant. Consider also the state observer (63) with the estimate $\hat{\mu}$ of μ, and estimate $\hat{\theta}$ generated using the method of Algorithm 1. Then the control law (64) guarantees

$$u(t), \hat{u}(t) \equiv 0, \text{for } t \geq \tau_{i_\nu} + 2d \tag{65}$$

Proof: Since, by Lemma 6, under the assumption of having θ constant, we have $\hat{\mu}(t) = \mu$ and $\hat{\theta}_0(t) = \hat{\theta}_1(t) = \theta$ for $t \geq \tau_{i_\nu}$, if follows from [11, Theorem 3.2] that $\hat{u}(t) \equiv u(t)$ for $t \geq \tau_{i_\nu} + d$. It then follows from [11, Theorem 3.3] that $u(t) \equiv 0$ $t \geq \tau_{i_\nu} + d$, and since $\hat{u}(t) \equiv u(t)$ for $t \geq \tau_{i_\nu} + d$, $\hat{u}(t) \equiv 0$ also follows. The explicit form of the controller gain in (60) is derived in e.g. [11, Example 3.1]. ■

Assuming $\tau_{i_\nu} = T$, the achievement of having $u(t) \equiv 0$ for $t \geq \tau_{i_\nu} + 2d = T + 2d$ is a slight improvement over the convergence time in [13], which was $3T + d$.

V. Simulations

A. Spatially varying $\theta(x)$

System (1) is implemented in MATLAB using the system parameters and initial condition

$$\mu = 2 \qquad \theta(x) = 1 + x, \qquad u_0(x) = x \tag{66}$$

System (1) with parameters (66) is open-loop unstable. The known lower bound on μ is set as

$$\underline{\mu} = 1 \tag{67}$$

which results in $T = 1$, and hence an event every second.

The parameter estimation method of Algorithm 1 was also implemented using $N = 4$, and so was the control law of Theorem 8. The actuation signal was chosen to satisfy (23) in Remark 4, with

$$A = B = 10, \qquad \varpi_1 = \frac{2\pi}{T}, \qquad \varpi_2 = \frac{\sqrt{2}\pi}{T}, \tag{68}$$

so that the Moore-Penrose inverse is by Lemma 3 guaranteed to exist at the first event. The initial conditions ϕ_0, ψ_0 of the

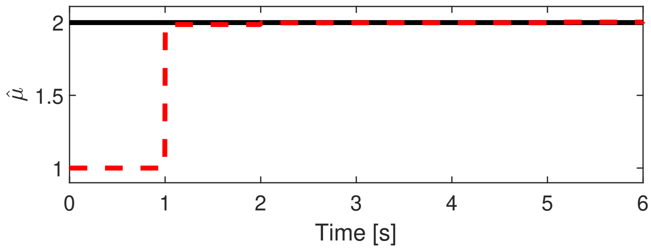

Fig. 1: Actual (solid black) and estimated (dashed red) μ for Case 1.

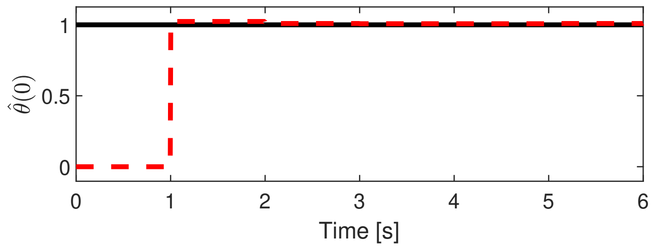

Fig. 2: Actual (solid black) and estimated (dashed red) $\theta(0)$ for Case 1.

filters as well as the initial guess $\hat{\theta}_0$ were all set identically to zero.

Due to numerical issues when implementing on a computer, the requirements of Algorithm 1 will not always produce a well-conditioned Moore-Penrose inverse A^+. The inverse is thus only computed if the absolute value of the determinant $A_i^T A_i$ is above a certain threshold, that is: $|\det(A_i^T A_i)| \geq \epsilon_0$ for a small number ϵ_0 chosen in these simulations to be 10^{-3}. The simulation results are found in Figures 1–5. It is observed from Figures 1–3 that Algorithm 1 produces fairly accurate estimates of $\mu, \theta(0), \theta(1)$ at the first event, and that these estimates' accuracy is improved at the second event. This is probably due to numerical issues, and the improved accuracy is due to the fact that data from more than one interval is used from the second event onward. The state norm converges asymptotically to zero, as seen from Figure 4, while the actuation signal stays bounded, as seen in Figure 5.

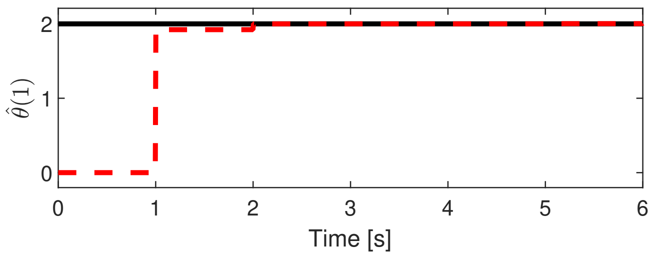

Fig. 3: Actual (solid black) and estimated (dashed red) $\theta(1)$ for Case 1.

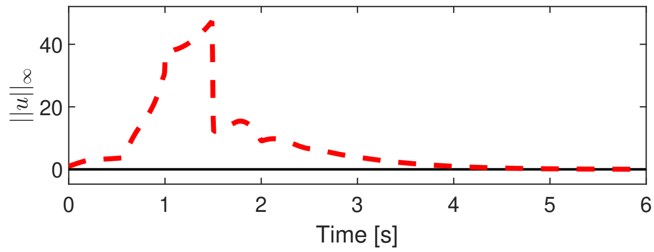

Fig. 4: System norm $||u||_\infty$ for Case 1.

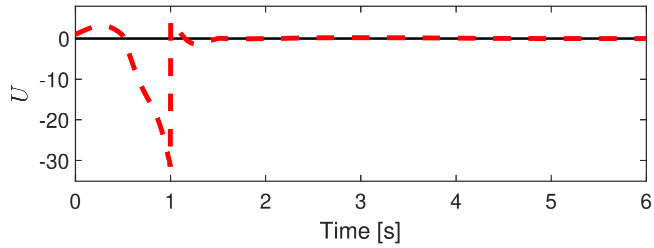

Fig. 5: Actuation signal U for Case 1.

B. Constant θ

In this simulation, a constant θ was used, and we chose

$$\theta = 3, \tag{69}$$

while the value of μ and u_0 were the same as in Case V-A. Again, the system parameters constitute an unstable system. The Algorithm 1 with A_i designed according to Remark 7, the control law of Theorem (11) were also implemented. The simulation results are found in Figures 6–10. Figures 6 and 7 show that the transport speed μ and in-domain parameter θ are correctly estimated at the first event. The system norm is displayed in Figure 8, with a close up in Figure 9. It is seen from the latter that the system norm converges to zero after a time $2T + d = 2.5$ sec as predicted by Theorem 11. The actuation signal remains bounded, as seen in Figure 10.

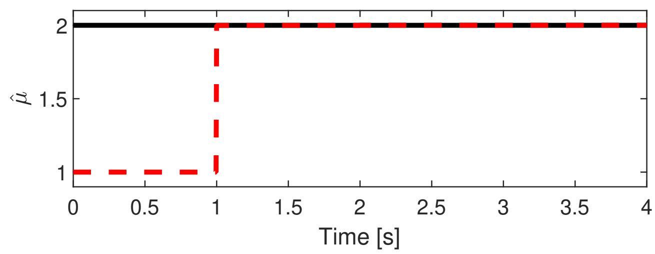

Fig. 6: Actual (solid black) and estimated (dashed red) μ for Case 2.

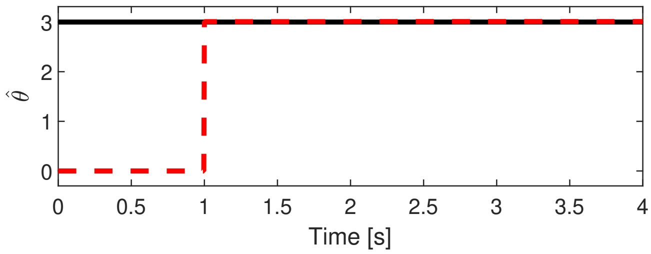

Fig. 7: Actual (solid black) and estimated (dashed red) θ for Case 2.

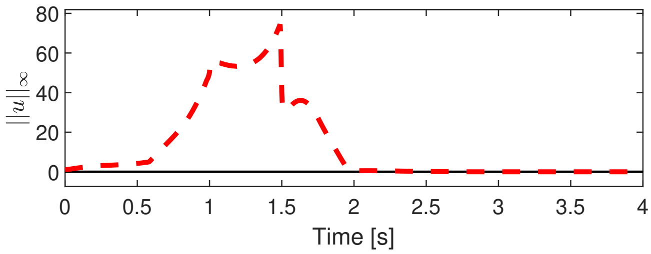

Fig. 8: System norm $||u||_\infty$ for Case 2.

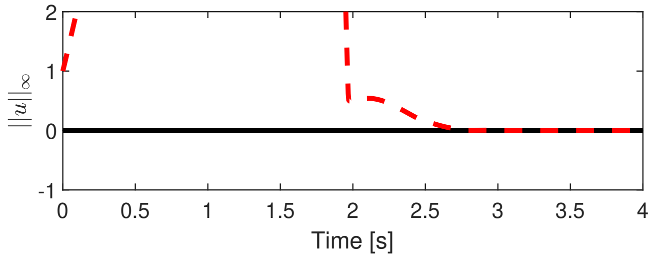

Fig. 9: System norm $||u||_\infty$ (zoomed) for Case 2.

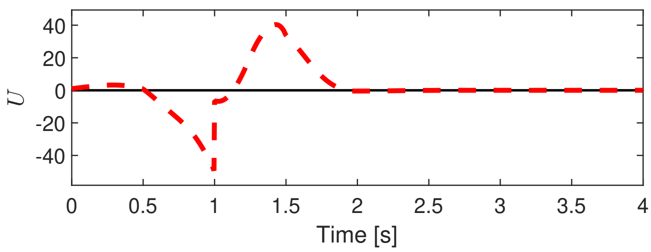

Fig. 10: Actuation signal U for Case 2.

VI. Conclusions

We have in this paper slightly modified a recently derived trigger-based estimation scheme [13] for adaptive control of a type of scalar, linear hyperbolic PDEs, into also handling systems with a spatially varying in-domain coefficient. The resulting controller achieves asymptotic convergence of the state to zero in the L_∞-sense. In the special case of having a constant in-domain parameter, the presented method achieves a slightly improved convergence time compared to what was achieved in [13]. Also, a condition on the actuation signal is derived, so that the method is always guaranteed to produce the required estimates at the first event. Such a condition was not given in [13]. The theory was illustrated in numerical examples.

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