Smooth attitude stabilisation in prescribed time of a rigid body despite uncertainties in inertia and additive disturbances

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Abstract— **We consider the problem of stabilisation in predefined, finite time of the attitude of a rigid body. Inspired by classical laws in missile guidance, the proposed control law is time-varying but smooth as opposed to classical sliding-mode-based laws that are generally discontinuous, or continuous at best, and suffer from robustness issues. Through a backstepping design, and the explicit construction of a strict Lyapunov function, we are able to ensure stabilisation to a desired attitude even in the presence of uncertainties in the inertia matrix and under the effect of additive disturbances. Numerical simulations show the efficiency of the proposed controller.**

I. Introduction

Aerospace vehicles such as satellites, launchers and drones are constantly required to be more efficient while performing under challenging conditions. This translates into designing robust control algorithms that can respond successfully to various mission scenarios. Typically, the principal objectives that need to be attained by these vehicles are precise attitude stabilisation and/or tracking.

This is the general attitude control problem which has drawn the attention of engineers and researchers for decades and is still in the focus of a large number of publications. Seminal papers such as [1], [2], [3] and [4] have shed light to the geometric and analytic properties of the underlying problems and proposed various solutions. As there are different types of available attitude parametrisation [5], such as rotation matrices, quaternion, Euler angles and the (Modified) Rodrigues vector, one can select which is appropriate to model the underlying problem at hand and eventually, hint on its solution. However, in general, it is preferable to work with minimal and global parametrisations.

A minimal and almost global representation, that is widely used in the aerospace industry, is the Modified Rodrigues vector [5]. Some of the most impactful original works that adopted this representation in order to solve the attitude cotrol problem were [6] and [7]. The authors applied the sliding mode control methodology and established attitude stabilisation (in finite time) with a discontinuous law and showed that this law was optimal with respect to a performance index that is quadratic with respect to the orientation and angular velocity errors. The attractive nature of such a SMC design, is that it exploits the additional inherent property of sliding mode control of rejecting bounded (with known bound), additive perturbations.

Since these seminal works, sliding-mode control has been rapidly evolving as a field of research and applied to various aerospace vehicles [8]. These advances try to remedy the robustness issues of SMC that are well known (chattering), the constructive gain selection and the involved stability and robustness analyses. One crucial point with regards to SMC is that the final convergence time is dependent on initial conditions, which is a limitation in the context of aerospace and defence applications. Some recent developments on SMC in this direction were presented in [9].

The particularly tight performance requirements imposed on aerospace vehicles necessitate that the control design guarantees stabilisation in a finite horizon even in the presence of unknown uncertainties and perturbations acting on the vehicle. Hinging upon the sliding mode framwork, a large number of works, for example see the recent works [10] and [11], have tackled the problem of attitude stabilisation (and the more general one for tracking) in fixed/finite-time while rejecting bounded disturbances. However, these solutions suffer from the common robustness issues of SMC and are only valid for bounded disturbances. Furthermore, they do not consider the possible variations of the inertias. On the other hand, the effect of time-variations of the inertia matrix, although important in applications (e.g. fuel depletion, mass displacement), has been poorly studied. An exception is the work [12] which however does not account for additive disturbances nor a fixed-time convergence.

This paper proposes a smooth, time-varying control law that ensures fixed-time stabilisation of a desired orientation in a prescribed time, which is independent of initial conditions, and despite uncertainties in the (time-varying) inertia matrix and additive, non-vanishing, (matched) disturbances. In addition, the stability proof is established by a constructive Lyapunov analysis. The design follows the recent paradigm for prescribed-time stabilisation proposed in [13], [14] that hinges upon the backstepping methodology [15] and applied to autonomous nonlinear systems that are in strict-feedback form. Our control design differs from the one in [13], [14] in certain ways: a) We do not transform the system in the (partially linear) strict-feedback form therein and tailor the design to the specific rigid body equations; b) We define differently the scaled error in the angular velocity, which allows us to simplify the design and analysis; c) We establish the prescribed-time stability claim without resorting to small-gain arguments but through a direct Lyapunov analysis. To the author's knowledge this is the first work that establishes prescribed-time stabilisation for the (fully actuated) attitude control problem.

The organisation of the paper is as follows. Section II introduces the model of the rigid body, the necessary defi-

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nitions of prescribed-time stability and the working assumptions. Section III presents the proposed control law and the corresponding Lyapunov-based stability analysis. Numerical simulations of the closed-loop system then follow in Section IV. We conclude with some remarks and future perspectives.

II. Problem Formulation

A. Notation

Depending on the context, we will denote by $|x|$ the Euclidean norm of the vector x or the $\mathcal{L}_2$ norm of the signal $x(t)$. Similarly, $||A||$ will represent the matrix induced 2-norm for any matrix A. Also, I will denote the identity matrix of appropriate dimensions and $\lambda(A)$ will denote the spectrum of the matrix A. Throughout, for any $x \in \mathbb{R}^3$ we will define by

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{1}$$

the matrix representation of the linear map $y \mapsto x \times y$, with $y \in \mathbb{R}^3$, and ' $\times$ ' denoting the usual cross product in $\mathbb{R}^3$. We will interchangeably use both notations, *i.e.* $x \times y$ and $S(x)y$, to simplify the presentation.

B. Dynamic model

We consider a moving rigid body subjected to the angular velocity $\omega \in \mathbb{R}^3$ (in body axes). Its orientation will be represented by the Modified Rodrigues Parameters (MRPs) that allow for eigenaxis rotations up to 360 degrees [5].

The kinematic model is represented by the differential equations [5]

$$\dot{\sigma} = F(\sigma)\omega \tag{2}$$

$$\frac{d}{dt}(J(t)\omega) = (J(t)\omega) \times \omega + \tau + d_\tau(t) \tag{3}$$

with the matrix F defined as

$$F(\sigma) := \frac{1}{2}(\frac{1-\sigma^T\sigma}{2}I + \sigma_\times + \sigma\sigma^T). \tag{4}$$

In the above $J(t)$ is the inertia matrix while $d_\tau(t)$ models the additive disturbances acting on the system. We consider that these disturbances are not necessarily vanishing but rather have an unknown bound $||d_\tau||_{[0,t]} := \sup_{s\in[0,t]} d_\tau(s)$.

The class of rigid bodies which are treated in this work satisfy the following working assumption.

Assumption 1 (A.1): The inertia matrix is decomposed into a known, constant matrix J_0 and a unknown, time-varying matrix $J_1(t)$

$$J(t) = J_0 + J_1(t), \tag{5}$$

with $J_1(t)$ being uniformly bounded

$$\underline{J_1} \leq ||J_1(t)|| \leq \overline{J_1} \tag{6}$$

$$||\dot{J}_1(t)|| \leq \overline{\dot{J}_1}, \tag{7}$$

with *known* bounds $\underline{J_1}, \overline{J_1}$ and *unknown* bound $\overline{\dot{J}_1}$.

For later use, we also note the following linear parametrisation identities

$$\dot{J}_1(t)\omega = \psi_1(\omega)d_1(t) \tag{8}$$

$$(J_1(t)\omega) \times \omega = \psi_2(\omega)d_2(t), \tag{9}$$

with known functions ψ_1, ψ_2 of appropriate dimensions and d_1, d_2 vectors containing the elements of $\dot{J}_1(t)$, $J_1(t)$ respectively.

Finally, it follows that for the total inertia matrix we obtain explicit bounds such that

$$J_m \leq ||J(t)|| \leq J_M. \tag{10}$$

Since the problem at hand is to stabilise a desired orientation, represented as σ_d, we naturally need to define an associate error and express the corresponding error kinematics. By defining the MRP error as

$$\delta\sigma := \sigma - \sigma_d, \tag{11}$$

we can derive the error kinematics and obtain the following equations [5]

$$\dot{\delta\sigma} = F(\delta\sigma)\omega \tag{12}$$

$$\frac{d}{dt}(J(t)\omega) = (J\omega) \times \omega + \tau + d_\tau \tag{13}$$

with

$$F(\delta\sigma) = \frac{1}{2}(\frac{1-\delta\sigma^T\delta\sigma}{2}I + \delta\sigma_\times + \delta\sigma\delta\sigma^T). \tag{14}$$

For particular properties of the MRP parametrisation σ and matrix F one can consult for example [5], [3].

C. Prescribed-time stability

Before procedding to the control design and the main result of this work, we will define certain functions that will be used throughout and that have particular properties which will allow to establish the prescribed time stability claim.

We define the following scaling functions:

$$\mu_1(t) := \frac{t_f}{(t_f - t)} \tag{15}$$

$$\mu(t) := \frac{t_f^{1+m}}{(t_f - t)^{1+m}} = \mu_1^{1+m}, \tag{16}$$

for $t \in [0, t_f]$ and some positive integer $m > 0$ to be defined. Properties of these functions have been presented in [13], [14]. In particular, the function μ (similarly for the function μ_1) is monotonically increasing and has the properties

- $\mu(0) = 1$, $\mu(t_f) = +\infty$
- $\mu^{-1}(0) = 1$, $\mu^{-1}(t_f) = 0$.

Finally, we remind the definitions of prescribed-time stability given in [13], [14] which will be used in this work.

Definition 1 (FT-ISS): The system $\dot{x} = f(x, t, d)$ (of arbitrary dimensions of x and d) is said to be fixed-time input-to-state stable in time t_f (FT-ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for all $t \in [0, t_f]$

$$|x(t)| \leq \beta(|x_0|, \frac{t_f}{t_f - t} - 1) + \gamma(||d||_{0,t}). \tag{17}$$

Definition 2 (FT-ISS+C): The system $\dot{x} = f(x, t, d)$ (with x and d of arbitrary dimensions) is said to be fixed-time input-to-state stable in time t_f and convergent to the origin (FT-ISS+C) if there exist class $\mathcal{KL}$ functions β, β_f and a class $\mathcal{K}$ function γ, such that, for all $t \in [0, t_f]$

$$|x(t)| \leq \beta_f(\beta(|x_0|, \frac{t_f}{t_f - t} - 1) + \gamma(||d||_{0,t})). \quad (18)$$

Remark 1: References [13], [14] present a detailed discussion and comparison between the proposed approach and the sliding mode methodology. For the interested reader, technical issues about fixed-time stability, robustness and the high-gain nature of the controllers (near the origin) are discussed therein.

With the above at hand, we can now state the problem that we wish to solve and whose solution will be provided in the following section.

Problem Statement: Design a smooth, time-varying, control law based on the model (12)-(13), with uncertainties characterised by assumption A.1, that provides *prescribed time stabilisation* of the desired attitude σ_d while rejecting the additive disturbances d_τ.

III. Control design

In this section, we present the main result, that is a control law which ensures prescribed-time convergence to zero while attenuating the uncertainties in the inertia matrix and the effect of additive disturbances.

The main result is summarised in the following.

Proposition 1: Consider the system (12)-(13) in closed-loop with the control law

$$\begin{aligned}\tau = &-J_M\Big(k_\omega^1 + \lambda_1||\psi_1(\omega)||^2 + \lambda_2||\psi_2(\omega)||^2\Big)\mu(\omega - \omega_d) \\ &- (J_0\omega) \times \omega. \quad (19)\end{aligned}$$

Then, under assumption A.1, and for k_ω^1, λ_1, λ_2 satisfying the conditions

$$k_0 = 4(k_f + \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_0}{8}), \quad k_f > 0 \quad (20)$$

$$k_1 = k_0 + \frac{4(1+m)}{t_f} \quad (21)$$

$$k_\omega^2 > 0 \quad (22)$$

$$\begin{aligned}k_\omega^1(t, \delta\sigma) = &\, k_\omega^2 + \frac{1+m}{t_f} + \frac{(1+m)^2\mu^2 k_1^2}{2\epsilon_1 t_f^2} + \frac{\epsilon_3}{2J_m^2} \\ &+ \frac{k_1\mu(1+|\delta\sigma|^2)}{4}(\frac{k_1^3\mu}{2\epsilon_2} + 1) + \frac{1+|\delta\sigma|^2}{8\epsilon_0} \quad (23)\end{aligned}$$

$$\lambda_1 \geq \frac{\epsilon_3}{2J_m^2} \quad (24)$$

$$\lambda_2 \geq \frac{\epsilon_3}{2J_m^2}, \quad (25)$$

for any positive scalars $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$, the closed-loop system is fixed-time input-to-state stable in prescribed time t_f and convergent to zero.

Proof: We follow a backstepping design approach [15], [13], [14]. The first step is to consider that ω acts as an input to the subsystem (12) and define an ideal desired input, ω_d, that stabilises $\delta\sigma$ to the origin in prescribed-time. Then, as a second step, we consider the complete system (12)-(13), and design the torque input τ in order for ω to track ω_d in prescibed-time while attenuating the effects of uncertainties and perturbations. A benefit of the backtepping methodology will be that it will naturally lead to the construction of a strict Lyapunov function and thus, simplify the stability and robustness claims.

Let us start by defining the desired ω_d that stabilises the origin of (12). Inspired by the works by [13], [14] we propose the choice

$$\omega_d := -k(t)\delta\sigma \quad (26)$$

with the time-varying gain $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ of the form

$$k(t) = k_1\mu(t) \quad (27)$$

$$k_1 = k_0 + \frac{4(1+m)}{t_f}, \quad k_0 > 0. \quad (28)$$

The resulting $\delta\sigma$ dynamics is given by

$$\dot{\delta\sigma} = -\frac{k_1\mu}{4}(1 + |\delta\sigma|^2)\delta\sigma. \quad (29)$$

Using the scaling function $\mu(t)$[1] and the transformation

$$e_\sigma := \mu(t)\delta\sigma, \quad (30)$$

we re-express the (desired) error kinematics as

$$\begin{aligned}\dot{e}_\sigma &= \frac{1+m}{t_f}\mu^{\frac{1}{1+m}}e_\sigma - \frac{k_1\mu}{4}(1 + |\mu^{-1}e_\sigma|^2)e_\sigma \\ &= -\mu\Big(\frac{k_1}{4}(1 + |\mu^{-1}e_\sigma|^2) - \frac{1+m}{t_f}\mu^{-\frac{m}{1+m}}\Big)e_\sigma. \quad (31)\end{aligned}$$

Following a similar analysis as in [13], [14], and using in particular the property $\mu^{-\frac{m}{1+m}} \leq 1$, one can establish prescribed-time stability using the Lyapunov function

$$V_\sigma(e_\sigma) = \frac{1}{2}|e_\sigma|^2. \quad (32)$$

Since it will be used later on, we write the time derivative of V_σ along trajectories of (31) and obtain the bound

$$\dot{V}_\sigma \leq -\frac{\mu}{4}(k_0 + k_1|\mu^{-1}e_\sigma|^2)|e_\sigma|^2. \quad (33)$$

We look now at the real error kinematics that are given by

$$\begin{aligned}\dot{e}_\sigma = &-\mu\Big(\frac{k_1}{4}(1 + |\mu^{-1}e_\sigma|^2) - \frac{1+m}{t_f}\mu^{-\frac{m}{1+m}}\Big)e_\sigma \\ &+ \mu F(\mu^{-1}e_\sigma)(\omega - \omega_d). \quad (34)\end{aligned}$$

Calculating now V_σ's time derivative results in

$$\begin{aligned}\dot{V}_\sigma \leq &-\frac{\mu}{4}(k_0 + k_1|\mu^{-1}e_\sigma|^2)|e_\sigma|^2 \\ &+ \mu e_\sigma^T F(\mu^{-1}e_\sigma)(\omega - \omega_d). \quad (35)\end{aligned}$$

We proceed now with the second step and consider both the attitude kinematics and dynamics. We first define the error between the angular velocity ω and the desired angular velocity ω_d defined in (26)-(28) as

$$\delta\omega := \omega - \omega_d. \quad (36)$$

[1]To enhance readability, we will drop the explicit dependence on time and simply write μ when refering to $\mu(t)$.

Then, the angular velocity error dynamics takes the form

$$\begin{aligned}\dot{\delta\omega} &= J^{-1}(t)\Big((J\omega)\times\omega - \dot{J}(t)\omega + \tau + d_\tau\Big) - \dot{\omega}_d\\ &= J^{-1}(t)\Big((J_0\omega)\times\omega + (J_1(t)\omega)\times\omega - \dot{J}_1(t)\omega + \tau + d_\tau\Big)\\ &- \dot{\omega}_d, \end{aligned} \tag{37}$$

where to obtain the second equality we used the splitting of the inertia matrix as per assumption A.1. Noticing that the second and third terms of (37) can be factored as per (8), (9), we can express (37) as

$$\begin{aligned}\dot{\delta\omega} &= J^{-1}\Big((J_0\omega)\times\omega - \psi_1(\omega)d_1 + \psi_2(\omega)d_2 + \tau + d_\tau\Big)\\ &- \dot{\omega}_d \end{aligned} \tag{38}$$

Now, as with the scaled MRP error e_σ, we define the scaled angular velocity error

$$e_\omega := \mu(t)\delta\omega, \tag{39}$$

and obtain its dynamics

$$\begin{aligned}\dot{e}_\omega &= J^{-1}\Big((J_0\omega)\times\omega - \psi_1(\omega)d_1 + \psi_2(\omega)d_2 + \tau + d_\tau\Big)\\ &- \dot{\omega}_d + \frac{1+m}{t_f}\mu^{\frac{1}{1+m}}e_\omega. \end{aligned} \tag{40}$$

We continue now with the definition of a candidate Lyapunov function for the e_ω-subsystem given by

$$V_\omega(e_\omega) = \frac{1}{2}|e_\omega|^2. \tag{41}$$

Before proceeding, notice that $\dot{\omega}_d$ equals

$$\begin{aligned}&\dot{\omega}_d = -k_1\Big(\frac{1+m}{t_f}\mu^{\frac{2+m}{1+m}}\delta\sigma + \mu F(\delta\sigma)(\omega_d - \delta\omega)\Big) =\\ &-k_1\ \Big(\ \frac{1+m}{t_f}\mu^{\frac{1}{1+m}}e_\sigma + \mu F(\mu^{-1}e_\sigma)\omega_d - \mu F(\mu^{-1}e_\sigma)\mu^{-1}e_\omega\Big)\end{aligned}$$

and, in addition, consider the following form of the control law

$$\begin{aligned}\tau &= -J_M\Big(k_\omega^1(t,e_\sigma) + \lambda_1||\psi_1(\omega)||^2 + \lambda_2||\psi_2(\omega)||^2\Big)e_\omega\\ &- (J_0\omega)\times\omega,\end{aligned}$$

with k_ω^1, λ_1, λ_2 positive scalars to be properly defined.

The time derivative of V_ω along trajectories of (40) yields

$$\begin{aligned}\dot{V}_\omega &= \frac{1+m}{t_f}\mu^{\frac{-m}{1+m}}\mu|e_\omega|^2 + \frac{1+m}{t_f}\mu^{\frac{-m}{1+m}}\mu^2 k_1 e_\omega^T e_\sigma\\ &+ k_1\mu^2 e_\omega^T F(\mu^{-1}e_\sigma)(\omega_d + e_\omega) + \mu e_\omega^T J^{-1}\tau\\ &+ \mu e_\omega^T J^{-1}\psi_1(\omega)d_1 + \mu e_\omega^T J^{-1}\psi_2(\omega)d_2 + \mu e_\omega^T J^{-1}d_\tau\\ &\le \frac{1+m}{t_f}\mu|e_\omega|^2 + \frac{(1+m)^2\mu^3k_1^2}{2t_f^2\epsilon_1}\mu|e_\omega|^2 + \frac{\epsilon_1\mu}{2}|e_\sigma|^2\\ &+ \frac{k_1^4\mu^3}{8\epsilon_2}(1+|\mu^{-1}e_\sigma|^2)|e_\omega|^2 + \frac{\epsilon_2}{2}(1+|\mu^{-1}e_\sigma|^2)|e_\sigma|^2\\ &+ \frac{k_1\mu^2}{4}(1+|\mu^{-1}e_\sigma|^2)|e_\omega|^2 + \frac{\mu\epsilon_3}{2J_m^2}||\psi_1||^2|e_\omega|^2 + \frac{\mu}{2\epsilon_3}|d_1|^2\\ &+ \frac{\mu\epsilon_3}{2J_m^2}||\psi_2||^2|e_\omega|^2 + \frac{\mu}{2\epsilon_3}|d_2|^2 + \frac{\mu\epsilon_3}{2J_m^2}|e_\omega|^2 + \frac{\mu}{2\epsilon_3}|d_\tau|^2\\ &- \Big(k_\omega^1 + \lambda_1||\psi_1(\omega)||^2 + \lambda_2||\psi_2(\omega)||^2\Big)\mu|e_\omega|^2\end{aligned}$$

where we applied Young's inequality in the appropriate cross-terms with positive constants $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. With these bounds, and the expressions in (20)-(25), we can finally obtain

$$\begin{aligned}\dot{V}_\omega &\le -k_\omega^2\mu|e_\omega|^2 + \frac{\epsilon_1+\epsilon_2}{2}(1+|\mu^{-1}e_\sigma|^2)\mu|e_\sigma|^2\\ &+ \frac{\mu}{2\epsilon_3}\Bigg|\begin{bmatrix} d_1\\ d_2\\ d_\tau\end{bmatrix}\Bigg|^2. \end{aligned} \tag{42}$$

The previous bounds on $\dot{V}_\omega$ in (42) along with the bounds on $\dot{V}_\sigma$ in (35), that in the transformed errors become (after some bounding with $\epsilon_0 > 0$)

$$\begin{aligned}\dot{V}_\sigma &\le -\frac{\mu}{4}(k_0 + k_1|\mu^{-1}e_\sigma|^2)|e_\sigma|^2 + \mu e_\sigma^T F(\mu^{-1}e_\sigma)\mu^{-1}e_\omega\\ &\le -\frac{\mu}{4}(k_0 + k_1|\mu^{-1}e_\sigma|^2)|e_\sigma|^2 + \frac{\epsilon_0}{8}(1+|\mu^{-1}e_\sigma|^2)\mu|e_\sigma|^2\\ &+ \frac{1}{8\epsilon_0}(1+|\mu^{-1}e_\sigma|^2)\mu|e_\omega|^2, \end{aligned} \tag{43}$$

can be combined to establish that

$$\begin{aligned}&\frac{d}{dt}(V_\sigma + V_\omega) \le -k_f\mu|e_\sigma|^2 - k_\omega^2\mu|e_\omega|^2 + \Bigg|\begin{bmatrix} d_1\\ d_2\\ d_\tau\end{bmatrix}\Bigg|^2\\ &\le -2\mathtt{min}(k_\omega^2, k_f)\ \mu\ (V_\sigma + V_\omega) + \frac{\mu}{2\epsilon_0}\Bigg|\begin{bmatrix} d_1\\ d_2\\ d_\tau\end{bmatrix}\Bigg|^2, \end{aligned} \tag{44}$$

with the gains satisfying

$$\begin{aligned}k_0 &= 4(k_f + \frac{\epsilon_1+\epsilon_2}{2} + \frac{\epsilon_0}{8}), \quad k_f > 0\\ k_1 &= k_0 + \frac{4(1+m)}{t_f}\\ k_\omega^2 &> 0\\ k_\omega^1(t,\delta\sigma) &= k_\omega^2 + \frac{1+m}{t_f} + \frac{(1+m)^2\mu^2k_1^2}{2\epsilon_1 t_f^2} + \frac{\epsilon_3}{2J_m^2}\\ &+ \frac{k_1\mu(1+|\delta\sigma|^2)}{4}(\frac{k_1^3\mu}{2\epsilon_2} + 1) + \frac{1+|\delta\sigma|^2}{8\epsilon_0}\\ \lambda_1 &\ge \frac{\epsilon_3}{2J_m^2}\\ \lambda_2 &\ge \frac{\epsilon_3}{2J_m^2}.\end{aligned}$$

Following the definition 2, we can thus establish the claim of fixed-time input-to-state stability in time t_f and convergence to the origin. ■

IV. Simulations

The smooth behavior of the closed-loop system is now illustrated in a simple simulation scenario. Simulations were run for the model of a CubeSat used in [16], [17]. This CubeSat is a rectangular parallelepiped of dimensions 20 (cm)×10 (cm)×10 (cm) and mass 2 (kg) assumed to nominally be homogeneously distributed. As such the nominal inertia matrix is given as $J_0 = \mathtt{diag}(J_0^1, J_0^2, J_0^3) = \mathtt{diag}(87, 83, 37)$ (kg cm^2). The uncertainty on the inertia matrix is modeled as $J_1 = \mathtt{diag}(J_0^1 \cdot 0.1 \cdot \sin(2t), J_0^2 \cdot 0.1 \cdot \sin(t+\pi/4), J_0^3 \cdot \cos(0.5t + 3\pi/4))$ so that the first two inertias vary around 10 percent of the nominal value while the third can vary 100 percent. The elements of both the nominal and the uncertain inertias are depicted in Fig. 1.

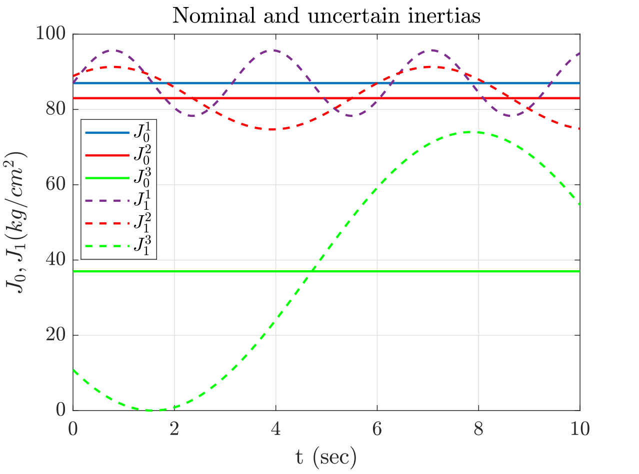

Fig. 1. Nominal J_0 (solid lines) and uncertain inertias $J_0 + J_1(t)$ (dashed lines)

The initial conditions are taken for the angular velocity as $\omega_0 = (-0.1, 0.05, 0)^T$ (rad/sec) and for the MRP parameters such that the initial error be $\delta\sigma_0 = (0.9, -0.7, 0)$. The disturbances are modeled as $d_\tau(t) = (\sqrt{(t)}; -1; -0.1 \cdot \sin(10t + \pi/4))^T$, which is a time-varying vector with both uniformly bounded and strictly increasing components. These are illustrated in Fig.2.

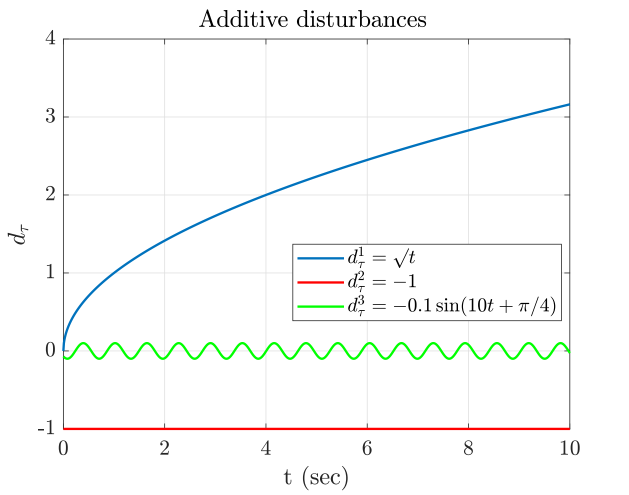

Fig. 2. Additive disturbances.

The gains were obtained from the conditions of Proposition 1 and by setting $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $m = 2$, $k_f = 0.1$, $k_\omega^2 = 0.6$ and $\lambda_1 = \lambda_2 = 0.01 + \frac{\epsilon_3}{2J_m^2}$.

We performed two sets of simulations: one for a desired fixed-time $t_f = 10$ (sec) and one for $t_f = 5$ (sec). For the first simulation, the behaviour of the MRP errors, $\delta\sigma$, is depicted in Fig.3 while the behavior of the angular velocities is shown in Fig.4. In both cases, we notice the very smooth behaviour despite the presence of uncertainties and disturbances as well as the convergence in the predefined time t_f.

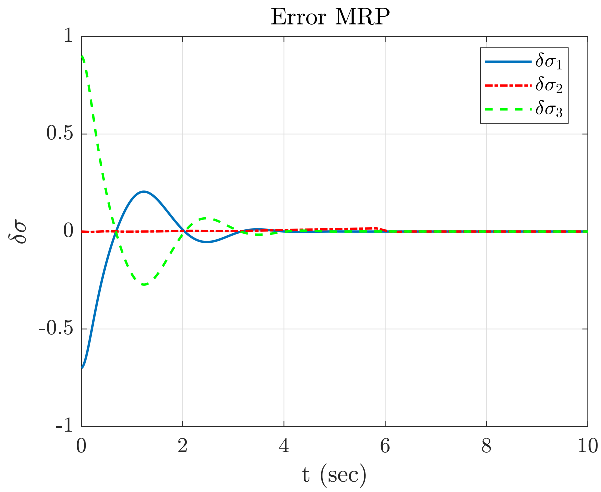

Fig. 3. MRP errors - Fixed time $t_f = 10$ (sec).

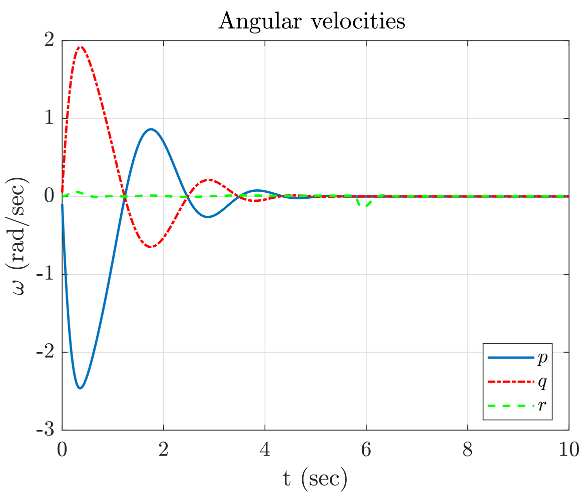

Fig. 4. Angular velocities Fixed time $t_f = 10$ (sec).

Similar results are shown for the second set of simulations. For this case, the attitude errors are shown in Fig.5 while the angular velocities is shown in Fig.6.

V. Conclusions–Perspectives

We proposed a solution to the problem of stabilisation in predefined time of the attitude of a rigid body. The proposed control law is smooth and thus, does not inherit the deficiencies of designs based on (dis)continuous designs. The proposed design allows the stabilisation to the desired attitude while attenuating the effects of additive disturbances and (time-varying) uncertainties in the inertia matrix. The performance of the resulting closed-loop is illustrated on a CubeSat example through numerical simulations in MATLAB. Furthermore, smooth attenuation of extreme cases of time-variations of the inertia matrix and the disturbances was also demonstrated.

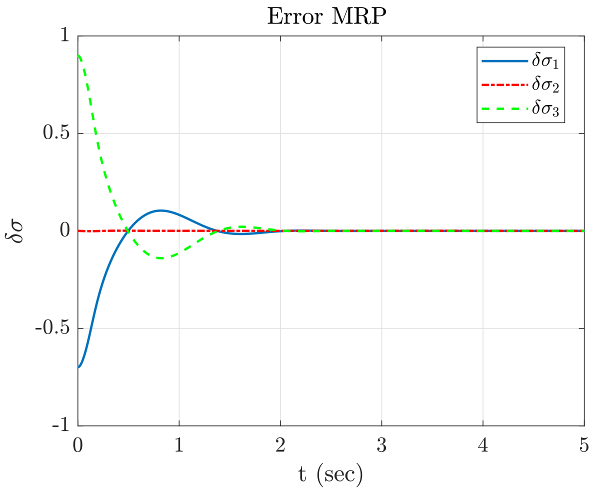

Fig. 5. MRP errors - Fixed time $t_f = 5$ (sec).

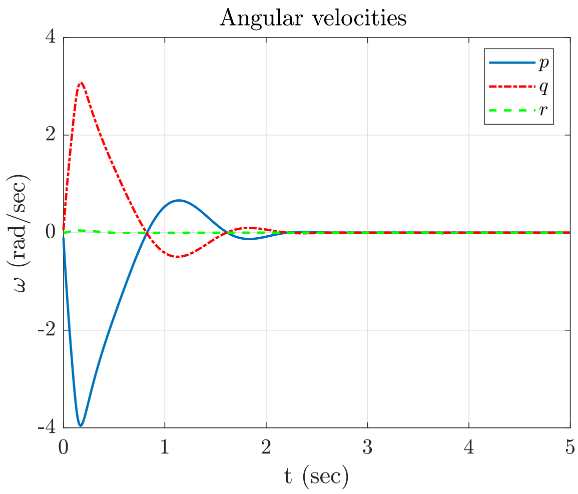

Fig. 6. Angular velocities Fixed time $t_f = 5$ (sec).

Future works will focus on extending this setting to treat saturation and state constraints following the developments, based on barrier-Lyapunov functions, in [18]. Realistic scenarios where the available onboard sensors (IMU, star trackers) have to be considered and as such the control design requires the incorporation of an attitude/velocity observer is another interesting direction.

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