Probabilistic Analysis of Counting Protocols in Large-scale Asynchronous and Anonymous Systems

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***Abstract*—We consider a large system populated by n anonymous nodes that communicate through asynchronous and pairwise interactions. The aim of these interactions is for each node to converge toward a global property of the system, that depends on the initial state of each node. In this paper we focus on both the counting and proportion problems. We show that for any $\delta \in (0,1)$, the number of interactions needed per node to converge is $O(\ln(n/\delta))$ with probability at least $1-\delta$. We also prove that each node can determine, with any high probability, the proportion of nodes that initially started in a given state without knowing the number of nodes in the system. This work provides a precise analysis of the convergence bounds, and shows that using the 4-norm is very effective to derive useful bounds.**

Keywords Large scale anonymous and asynchronous system; Counting problem; Proportion problem; Markov chain; Probabilistic analysis.

I. INTRODUCTION

This paper focuses on the analysis of counting problems in a model in which nodes are identically programmed, with no identity, and they progress in their computation through random pairwise interactions. As motivated by Aspnes [3], the objective of this model is to analyze the conditions under which nodes can converge to a state from which some global property of the system can be locally computed. A considerable amount of work has been done so far to determine which properties can emerge from pairwise interactions between finite-state nodes, together with the derivation of lower bounds on the time and space needed to reach such properties (e.g., [1], [2], [4], [6], [9]). In this work, we are primarily interested in counting problems. Briefly, each node starts independently of each other in one of two input states, say A and B, and the objective for each node is to eventually reach a state from which it can derive with any high probability the exact difference between the number of nodes that started their execution with A, denoted by n_A in the following, and the number of nodes that started their execution with B, denoted by n_B.

The main contribution of this work is an analysis of the time for all the nodes of the system to converge to $n_A - n_B$ with any probability fixed in advance. This analysis improves upon the one obtained by Mocquard et al. [7] thanks to the tools used to derive this convergence time. In [7], the 2-norm of the difference between the vector of states of the nodes and the limiting distribution of these states is analyzed. In the present paper, we use the 4-norm, and we precisely characterize the conditions under which the 4-norm gives tighter bounds with respect to the the 2-norm. To the best of our knowledge, such an analysis has never been achieved before.

In the remaining of the paper we define in Section II both the counting problem and the proportion one, and examine in Section IV the different works that have tackled those problems. In Section III we formally present the model in which this work has been done. We present in Section V the protocols run by the nodes to solve both problems. After the statement of preliminary results in Section VI, we then study in Section VII the moments and the distribution of the difference between the random vector of all agents' values and the limiting distribution of these values. In Section VIII we analyze their bounds and asymptotic behavior when the number n of nodes goes to infinity. The accuracy of our analytic study has been illustrated through numerous simulations whose main results are presented in Section IX.

II. THE ADDRESSED PROBLEMS

We consider a set of n agents, interconnected by a complete graph, that asynchronously start their execution in one of two distinct states A and B. Let n_A (resp. n_B) be the number of agents whose initial state is A (resp. B), and let $\kappa = n_A - n_B$. This paper addresses two related problems, the counting problem and the proportion one, both defined as follows.

a) Counting problem.: A population protocol ran by all the nodes of the system solves the counting problem in τ steps with probability at least $1-\delta$, for any $\delta \in (0,1)$, if for any $t \geq \tau$, it holds that for any node i of the system, i is capable of computing κ with probability at least $1-\delta$.

b) Proportion problem.: A population protocol ran by all the nodes of the system solves the proportion problem in τ steps with probability at least $1-\delta$, for any $\delta \in (0,1)$, if for any $t \geq \tau$, it holds that for any node i of the system, i is capable of computing n_A/n with probability at least $1-\delta$, without having access to the population size n.

III. MODEL AND NOTATIONS

In this work we assume that a collection of nodes are connected by a complete graph and communicate through pairwise and asynchronous interactions. Initially, all the nodes start with an initial state represented by the symbol A or B, and upon interactions, update their local state according to the transaction function f. Interactions between nodes are random:

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at each discrete time, any two agents are randomly selected to interact. The notion of time in population protocols refers to as the successive steps at which interactions occur, while the parallel time refers to as the total number of interactions averaged by n, see Aspnes et al. [3]. Note that nodes do not maintain nor use identifiers, however for ease of presentation, they are numbered $1, 2, \ldots, n$.

We will denote by $C_t^{(i)}$ the state of node i at time t. The configuration of the system at time t is the state of each node at time t and is denoted by $C_t = (C_t^{(1)}, \ldots, C_t^{(n)})$. We denote by X_t the random pair of distinct nodes chosen at time t to interact, and for every $i, j = 1, \ldots, n$, with $i \neq j$, we define $p_{i,j}(t) = \mathbb{P}\{X_t = (i, j)\}$.

We suppose that the sequence $\{X_t,\ t \geq 0\}$ is a sequence of independent and identically distributed random variables. Since C_t is entirely determined by the values of $C_0, X_0, X_1, \ldots, X_{t-1}$, this means in particular that the random variables X_t and C_t are independent and that the stochastic process $C = \{C_t,\ t \geq 0\}$ is a discrete-time homogeneous Markov chain. Note that this Markov chain is very hard to analyze using classical Markov chains methods because its state space is quite complex. Nevertheless, as we will see, the simplicity of the transition function f allows us to get interesting results.

IV. Related Work

The closest problems to the one we address are the computation of the majority (see [4], [6], [2], [9], [1]). In this problem, all the agents start in one of two distinct states and they eventually converge to 1 if $\kappa > 0$ (*i.e.* $n_A > n_B$), and to 0 if $\kappa \leq 0$ (*i.e.* $n_A \leq n_B$). Draief and Vojnovic [4] and Mertzios et al. [6] propose a four-state protocol that solves the majority problem with a convergence parallel time logarithmic in n but only in expectation. Moreover, the expected convergence time is infinite when n_A and n_B are close to each other (that is κ approaches 0). Angluin et al.[2] and Perron et al. [9] propose a three-state protocol that converges with high probability after a convergence parallel time logarithmic in n but only if κ is large enough, *i.e* when $|n_A - n_B| \geq \sqrt{n}\log n$. Alistarh et al. [1] present a nice protocol based on an average-and-conquer method to solve the majority problem. The first type of interaction is close to the one used in this paper while the second one is used to diffuse the result of the computation to the agents that have not decided yet. Actually, to show their convergence time, they need to assume a large number of intermediate states because they need to prove that all the agents with maximum positive values and minimal negative values will have sufficiently enough time to halve their values. Note that in practice, their algorithm does not require more than n state to converge to the majority. Note that the protocol proposed by Jelasity et al. [5] computes the average of the initial values of the nodes, as done in the present paper, however, in their protocol nodes act synchronously. Finally in a previous work [7] we have presented a solution to the majority counting problem, whose originality was a proof of convergence based on tracking the euclidean distance between the vector of all agents' states and the limiting distribution of these states. In a more recent work [8], we have provided an analysis which shows that when nodes can only manipulate integers, then our solution to both the counting problem and the proportion one are optimal both in time and space. In the present paper, we tackle the case where nodes manipulate dyadic rational numbers, and we provide tighter bounds on the time needed for each node to converge to the sought result.

V. The Counting and Proportion Protocols

Initially, all the nodes $i, 1 \leq i \leq n$ start with a symbol A or B that provides their initial state $C_0^{(i)}$. Let m be any positive integer. We set

$$C_0^{(i)} = \begin{cases} m & \text{if the initial local state is } A \\ -m & \text{if the initial local state is } B. \end{cases}$$

Interactions between nodes are orchestrated by a random scheduler: at each discrete time $t \geq 0$, any two indices i and j are randomly chosen to interact with probability $p_{i,j}(t)$. Note that the random scheduler is fair, meaning that any possible interaction cannot be avoided forever. Once chosen, the pair of nodes (i, j) interacts and both nodes update their respective state $C_t^{(i)}$ and $C_t^{(j)}$ by applying the following transition function f, leading to state C_{t+1}, given by

$$\begin{aligned}\left(C_{t+1}^{(i)}, C_{t+1}^{(j)}\right) &= f\left(C_t^{(i)}, C_t^{(j)}\right) \\ &= \left(\frac{C_t^{(i)} + C_t^{(j)}}{2}, \frac{C_t^{(i)} + C_t^{(j)}}{2}\right) \qquad (1)\\ \text{and } C_{t+1}^{(h)} &= C_t^{(h)} \text{ for } h \neq i, j.\end{aligned}$$

At any time t, and upon request from the application, any node i of the system can provide its estimation of $\kappa = n_A - n_B$ by returning w_i, defined as

$$w_i = \left\lfloor \frac{C_t^{(i)} n}{m} + \frac{1}{2} \right\rfloor.$$

We show in the following (see Corollary 7) that with any high probability, $w_i = \kappa$ for any node i in the system. Note that the values taken by the variables $C_t^{(i)}$ belong to the set of dyadic numbers (the rational numbers whose denominator is a power of 2) of the interval $[-m, m]$.

The proportion protocol uses the interaction function f defined in Relation (1), and the estimation of the proportion n_A/n is defined as

$$w'_i = \frac{C_t^{(i)} + m}{2m}.$$

Note that node i does not need to know the size n of the population to compute the proportion of nodes that started with symbol A. We show with Corollary 8 (shown in Section VII) that with any high probability and with any high precision, $w'_i \approx n_A/n$ for any node i in the system.

VI. Preliminary Results

The following Lemma states that the sum of the entries of vector C_t is constant.

Lemma *1 ([7]):* For all $t \geq 0$, $\sum_{i=1}^n C_t^{(i)} = \sum_{i=1}^n C_0^{(i)}$.

We denote by ℓ the mean value of the sum of the entries of C_t and by L the row vector of $\mathbb{R}^n$ with all its entries equal to ℓ, that is

$$\ell = \frac{1}{n}\sum_{i=1}^{n} C_t^{(i)} \text{ and } L = (\ell, \ldots, \ell).$$

For every $d \in \mathbb{N} \setminus \{0\}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will use the d-norm and ∞-norm of x denoted by $\|x\|_d$ and $\|x\|_\infty$, defined by

$$\|x\|_d = \left(\sum_{i=1}^{n} |x_i|^d\right)^{1/d} \quad \text{and } \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$

It is well-known that these norms satisfy

$$\|x\|_\infty \le \|x\|_d \le n^{1/d}\|x\|_\infty.$$

This shows in particular that

$$\lim_{d \longrightarrow \infty} \|x\|_d = \|x\|_\infty. \tag{2}$$

For the sake of simplicity we introduce the following notations

$$y_t^{(i)} = C_t^{(i)} - \ell \text{ and } Y_t = (y_t^{(1)}, \ldots, y_t^{(n)}),$$

that is $Y_t = C_t - L$.

The next theorem states that, for every d, the d-norm of vector Y_t is decreasing with t.

Theorem *2:* For all $d \in \{1, 2, \ldots, \infty\}$, the sequence $(\|Y_t\|_d)_{t \ge 0}$ is decreasing.

Proof: Suppose first that d is finite with $d \ge 1$. From Relation (1), we have, for every $t \ge 0$,

$$\|Y_{t+1}\|_d^d = \|Y_t\|_d^d - \sum_{i,j=1}^{n}\left(\left|y_t^{(i)}\right|^d + \left|y_t^{(j)}\right|^d - 2\left|\frac{y_t^{(i)} + y_t^{(j)}}{2}\right|^d\right) 1_{\{X_t=(i,j)\}}. \tag{3}$$

The real function g defined by $g(x) = x^d$ is a convex function on $[0, \infty)$, so for every $a, b \ge 0$, we have

$$a^d + b^d \ge 2\left(\frac{a+b}{2}\right)^d.$$

Taking $a = \left|y_t^{(i)}\right|$, $b = \left|y_t^{(j)}\right|$ and using the fact that $|a| + |b| \ge |a + b|$, we get

$$\left|y_t^{(i)}\right|^d + \left|y_t^{(j)}\right|^d \ge 2\left(\frac{\left|y_t^{(i)}\right| + \left|y_t^{(j)}\right|}{2}\right)^d \ge 2\left|\frac{y_t^{(i)} + y_t^{(j)}}{2}\right|^d,$$

which means that the double sum in (3) is non negative. This proves that $\|Y_{t+1}\|_d^d \le \|Y_t\|_d^d$ i.e. that $\|Y_{t+1}\|_d \le \|Y_t\|_d$. If $d = \infty$, by taking the limit in this inequality, we obtain using (2), $\|Y_{t+1}\|_\infty \le \|Y_t\|_\infty$, which completes the proof. ■

From now on, we suppose as usual in such studies that X_t is uniformly distributed, i.e. that is

$$p_{i,j}(t) = \frac{1}{n(n-1)}.$$

VII. Moments of $\|Y_t\|_2$ and $\|Y_t\|_4$

We study in this section the moments of $\|Y_t\|_2$ and $\|Y_t\|_4$ which will be used to analyze their distributions. To avoid triviality we suppose that $n \ge 3$.

Theorem *3:* For all $t \ge 0$, we have

$$\mathbb{E}\left(\|Y_{t+1}\|_4^4\right) = \left(1 - \frac{7}{4(n-1)}\right)\mathbb{E}\left(\|Y_t\|_4^4\right) + \frac{3}{4n(n-1)}\mathbb{E}\left(\|Y_t\|_2^4\right). \tag{4}$$

Proof: By taking the expectations in Relation (3) and using the fact that X_t and Y_t are independent, we obtain for $d = 4$,

$$\begin{aligned}
&\mathbb{E}\left(\|Y_{t+1}\|_4^4\right) = \mathbb{E}\left(\|Y_t\|_4^4\right) \\
&- \sum_{i,j=1}^{n} \mathbb{E}\left(\left(y_t^{(i)}\right)^4 + \left(y_t^{(j)}\right)^4 - 2\left(\frac{y_t^{(i)} + y_t^{(j)}}{2}\right)^4\right) p_{i,j}(t) \\
&= \mathbb{E}\left(\|Y_t\|_4^4\right) - \frac{1}{n(n-1)} \\
&\times \sum_{i,j=1}^{n} \mathbb{E}\left[\left(y_t^{(i)}\right)^4 + \left(y_t^{(j)}\right)^4 - 2\left(\frac{y_t^{(i)} + y_t^{(j)}}{2}\right)^4\right].
\end{aligned} \tag{5}$$

The double sum in (5) can also be also written as

$$\begin{aligned}
&\sum_{i,j=1}^{n}\left[\left(y_t^{(i)}\right)^4 + \left(y_t^{(j)}\right)^4 - 2\left(\frac{y_t^{(i)} + y_t^{(j)}}{2}\right)^4\right] \\
&= \frac{7}{8}\sum_{i,j=1}^{n}\left(\left(y_t^{(i)}\right)^4 + \left(y_t^{(j)}\right)^4\right) \\
&- \frac{1}{2}\sum_{i,j=1}^{n}\left(y_t^{(i)}\left(y_t^{(j)}\right)^3 + \left(y_t^{(i)}\right)^3 y_t^{(j)}\right) \\
&- \frac{3}{4}\sum_{i,j=1}^{n}\left(y_t^{(i)}\right)^2\left(y_t^{(j)}\right)^2.
\end{aligned}$$

Consider theses 3 terms separately. For the first one, we have

$$\frac{7}{8}\sum_{i,j=1}^{n}\left(\left(y_t^{(i)}\right)^4 + \left(y_t^{(j)}\right)^4\right) = \frac{7n}{4}\sum_{i=1}^{n}\left(y_t^{(i)}\right)^4 = \frac{7n}{4}\|Y_t\|_4^4.$$

Concerning the second term, since by definition of ℓ, we have

$$\sum_{i=1}^{n} y_t^{(i)} = \sum_{i=1}^{n} C_t^{(i)} - n\ell = 0 \tag{6}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{n}\left(y_t^{(i)}\left(y_t^{(j)}\right)^3 + \left(y_t^{(i)}\right)^3 y_t^{(j)}\right) \\
&= \sum_{j=1}^{n}\left(y_t^{(j)}\right)^3 \sum_{i=1}^{n}\left(y_t^{(i)}\right) = 0.
\end{aligned}$$

Concerning the third term, we have

$$\frac{3}{4}\sum_{i,j=1}^{n}\left(y_t^{(i)}\right)^2\left(y_t^{(j)}\right)^2 = \frac{3}{4}\sum_{i=1}^{n}\left(y_t^{(i)}\right)^2\sum_{j=1}^{n}\left(y_t^{(j)}\right)^2 = \frac{3}{4}\|Y_t\|_2^4.$$

We then obtain

$$\sum_{i,j=1}^{n}\left[\left(y_t^{(i)}\right)^4+\left(y_t^{(j)}\right)^4-2\left(\frac{y_t^{(i)}+y_t^{(j)}}{2}\right)^4\right] = \frac{7n}{4}\|Y_t\|_4^4-\frac{3}{4}\|Y_t\|_2^4.$$

Relation (5) becomes

$$\mathbb{E}\left(\|Y_{t+1}\|_4^4\right) = \mathbb{E}\left(\|Y_t\|_4^4\right)-\frac{7}{4(n-1)}\mathbb{E}\left(\|Y_t\|_4^4\right) + \frac{3}{4n(n-1)}\mathbb{E}\left(\|Y_t\|_2^4\right),$$

that is

$$\mathbb{E}\left(\|Y_{t+1}\|_4^4\right) = \left(1-\frac{7}{4(n-1)}\right)\mathbb{E}\left(\|Y_t\|_4^4\right) + \frac{3}{4n(n-1)}\mathbb{E}\left(\|Y_t\|_2^4\right),$$

which completes the proof. ■

In order for this result to be really interesting, we need to evaluate fourth moment of the 2-norm of Y_t. This is the goal of the next theorem.

Theorem *4:* For all $t \geq 0$, we have

$$\mathbb{E}\left(\|Y_{t+1}\|_2^4\right) = \left(1-\frac{4n-3}{2n(n-1)}\right)\mathbb{E}\left(\|Y_t\|_2^4\right) + \frac{1}{2(n-1)}\mathbb{E}\left(\|Y_t\|_4^4\right). \qquad (7)$$

Proof: Applying Relation (3) with $d=2$ leads to

$$\|Y_{t+1}\|_2^2 = \|Y_t\|_2^2 - \sum_{i,j=1}^{n}\left[\left(y_t^{(i)}\right)^2+\left(y_t^{(j)}\right)^2-2\left(\frac{y_t^{(i)}+y_t^{(j)}}{2}\right)^2\right]1_{\{X_t=(i,j)\}} = \|Y_t\|_2^2-\frac{1}{2}\sum_{i,j=1}^{n}\left(y_t^{(i)}-y_t^{(j)}\right)^2 1_{\{X_t=(i,j)\}}.$$

By taking the conditional expectations with respect to X_t and using the fact that X_t and Y_t are independent, we obtain

$$\begin{aligned}\mathbb{E}&\left(\|Y_{t+1}\|_2^4 \mid X_t=(i,j)\right)\\ &= \mathbb{E}\left(\left(\|Y_{t+1}\|_2^2\right)^2 \mid X_t=(i,j)\right)\\ &= \mathbb{E}\left(\left[\|Y_t\|_2^2-\frac{1}{2}\left(y_t^{(i)}-y_t^{(j)}\right)^2\right]^2\right)\\ &= \mathbb{E}\left(\|Y_t\|_2^4\right)-\mathbb{E}\left(\left(y_t^{(i)}-y_t^{(j)}\right)^2\|Y_t\|_2^2-\frac{1}{4}\left(y_t^{(i)}-y_t^{(j)}\right)^4\right).\end{aligned}$$

Unconditioning, we obtain

$$\mathbb{E}\left(\|Y_{t+1}\|_2^4\right) = \mathbb{E}\left(\|Y_t\|_2^4\right)-\frac{1}{n(n-1)} \times \mathbb{E}\left[\|Y_t\|_2^2\sum_{i,j=1}^{n}\left(y_t^{(i)}-y_t^{(j)}\right)^2-\frac{1}{4}\sum_{i,j=1}^{n}\left(y_t^{(i)}-y_t^{(j)}\right)^4\right].$$

The first double sum can be written, using (6), as

$$\sum_{i,j=1}^{n}\left(y_t^{(i)}-y_t^{(j)}\right)^2 = 2n\|Y_t\|_2^2.$$

In the same way, using (6), the second double sum writes

$$\begin{aligned}\sum_{i,j=1}^{n}&\left(y_t^{(i)}-y_t^{(j)}\right)^4\\ &= \sum_{i,j=1}^{n}\left[\left(y_t^{(i)}\right)^4+\left(y_t^{(j)}\right)^4-4\left(y_t^{(i)}\right)^3\left(y_t^{(j)}\right)\right.\\ &\qquad\left. -4\left(y_t^{(i)}\right)\left(y_t^{(j)}\right)^3+6\left(y_t^{(i)}\right)^2\left(y_t^{(j)}\right)^2\right]\\ &= 2n\|Y_t\|_4^4+6\|Y_t\|_2^4.\end{aligned}$$

Putting together these two results gives

$$\mathbb{E}\left(\|Y_{t+1}\|_2^4\right) = \mathbb{E}\left(\|Y_t\|_2^4\right) - \frac{1}{n(n-1)}\mathbb{E}\left[2n\|Y_t\|_2^4-\frac{n}{2}\|Y_t\|_4^4-\frac{3}{2}\|Y_t\|_2^4\right],$$

that is

$$\mathbb{E}\left(\|Y_{t+1}\|_2^4\right) = \left(1-\frac{4n-3}{2n(n-1)}\right)\mathbb{E}\left(\|Y_t\|_2^4\right) + \frac{1}{2(n-1)}\mathbb{E}\left(\|Y_t\|_4^4\right),$$

which completes the proof. ■

We are now able, using Theorems 3 and 4, to obtain explicit expressions for the moments of $\|Y_t\|_2$ and $\|Y_t\|_4$ in function of those of $\|Y_0\|_2$ and $\|Y_0\|_4$.

Corollary *5:* For every $n \geq 3$ and $t \geq 0$ we have,

$$\mathbb{E}\left(\|Y_t\|_4^4\right) = \frac{6\alpha^t+n\beta^t}{n+6}\mathbb{E}\left(\|Y_0\|_4^4\right)+\frac{3(\beta^t-\alpha^t)}{n+6}\mathbb{E}\left(\|Y_0\|_2^4\right),$$
$$\mathbb{E}\left(\|Y_t\|_2^4\right) = \frac{2n(\beta^t-\alpha^t)}{n+6}\mathbb{E}\left(\|Y_0\|_4^4\right)+\frac{n\alpha^t+6\beta^t}{n+6}\mathbb{E}\left(\|Y_0\|_2^4\right),$$

where

$$\alpha = 1-\frac{2}{n-1} \text{ and } \beta = 1-\frac{7n-6}{4n(n-1)}.$$

Proof: Introducing the column vector $U(t)$ defined by

$$U(t) = \begin{pmatrix}\mathbb{E}\left(\|Y_t\|_4^4\right)\\ \mathbb{E}\left(\|Y_t\|_2^4\right)\end{pmatrix},$$

the Relations (4) and (7) can be written, for $t \geq 1$, as $U(t) = AU(t-1)$, where A is the $(2,2)$ matrix given by

$$A = \begin{pmatrix} 1 - \dfrac{7}{4(n-1)} & \dfrac{3}{4n(n-1)} \\ \\ \dfrac{1}{2(n-1)} & 1 - \dfrac{4n-3}{2n(n-1)} \end{pmatrix}.$$

We easily obtain, for all $t \geq 0$, $U(t) = A^t U(0)$. The eigenvalues of A are

$$\alpha = 1 - \frac{2}{n-1} \text{ and } \beta = 1 - \frac{7n-6}{4n(n-1)}.$$

Note that since $n \geq 3$, we have $0 \leq \alpha < \beta$. The eigenvectors V_α and V_β are

$$V_\alpha = \begin{pmatrix} 1 \\ -n/3 \end{pmatrix} \text{ and } V_\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We then have

$$A^t = \frac{1}{n+6} \begin{pmatrix} 6\alpha^t + n\beta^t & 3(\beta^t - \alpha^t) \\ \\ 2n(\beta^t - \alpha^t) & n\alpha^t + 6\beta^t \end{pmatrix}.$$

This leads to

$$\mathbb{E}\left(\|Y_t\|_4^4\right) = \frac{6\alpha^t + n\beta^t}{n+6} \mathbb{E}\left(\|Y_0\|_4^4\right) + \frac{3(\beta^t - \alpha^t)}{n+6} \mathbb{E}\left(\|Y_0\|_2^4\right)$$

$$\mathbb{E}\left(\|Y_t\|_2^4\right) = \frac{2n(\beta^t - \alpha^t)}{n+6} \mathbb{E}\left(\|Y_0\|_4^4\right) + \frac{n\alpha^t + 6\beta^t}{n+6} \mathbb{E}\left(\|Y_0\|_2^4\right),$$

which completes the proof. ■

It has been shown in [7] that

$$\mathbb{E}\left(\|Y_t\|_2^2\right) = \left(1 - \frac{1}{n-1}\right)^t \mathbb{E}\left(\|Y_0\|_2^2\right). \tag{8}$$

Using this result and the previous theorem, we get the following expression of the variance of $\|Y_t\|_2^2$ which is denoted by $\mathbb{V}\left(\|Y_t\|_2^2\right)$.

$$\begin{aligned} \mathbb{V}\left(\|Y_t\|_2^2\right) &= \mathbb{E}\left(\|Y_t\|_2^4\right) - \mathbb{E}\left(\|Y_t\|_2^2\right)^2 \\ &= \frac{2n(\beta^t - \alpha^t)}{n+6} \mathbb{E}\left(\|Y_0\|_4^4\right) + \frac{n\alpha^t + 6\beta^t}{n+6} \mathbb{E}\left(\|Y_0\|_2^4\right) \\ &\quad - \left(1 - \frac{1}{n-1}\right)^{2t} \mathbb{E}\left(\|Y_0\|_2^2\right)^2. \end{aligned}$$

By definition of $C_0^{(i)}$, using the fact $n = n_A + n_B$ and introducing the notation $p_{n,A} = n_A/n$, we have

$$\ell = \frac{1}{n} \sum_{i=1}^{n} C_0^{(i)} = \frac{1}{n}(n_A - n_B)\, m = (2p_{n,A} - 1)m. \tag{9}$$

This expression leads to

$$\begin{aligned} \|Y_0\|_2^2 &= \sum_{i=1}^{n} \left(y_0^{(i)}\right)^2 = \sum_{i=1}^{n} \left(C_0^{(i)}\right)^2 - n\ell^2 = nm^2 - n\ell^2 \\ &= 4nm^2 p_{n,A}(1 - p_{n,A}). \end{aligned} \tag{10}$$

We then have

$$\|Y_0\|_2^4 = 16n^2 m^4 p_{n,A}^2 (1 - p_{n,A})^2 \tag{11}$$

and in the same way, we obtain using the relation $n_A - n_B = n\ell/m$

$$\begin{aligned} \|Y_0\|_4^4 &= \sum_{i=1}^{n} \left(y_0^{(i)}\right)^4 \\ &= \sum_{i=1}^{n} \left[\left(C_0^{(i)}\right)^4 - 4\left(C_0^{(i)}\right)^4 \ell + 6\left(C_0^{(i)}\right)^2 \ell^2 - 4C_0^{(i)} \ell^3 + \ell^4\right] \\ &= nm^4 - 4n\ell^2 m^2 + 6n\ell^2 m^2 - 4n\ell^4 + n\ell^4 \\ &= n(m - \ell)(m + \ell)(m^2 + 3\ell^2). \end{aligned}$$

Using Relation (9), we obtain

$$\|Y_0\|_4^4 = 16nm^4 p_{n,A}(1 - p_{n,A})(3p_{n,A}^2 - 3p_{n,A} + 1). \tag{12}$$

Note that the maximum of function $x(1-x)$ is reached at $x = 1/2$ and its value is equal to $1/4$. The maximum of function $x(1-x)(3x^2 - 3x + 1)$ is reached at points $x = (3 \pm \sqrt{3})/6$ and its value is equal to $1/12$. Thus

$$\|Y_0\|_2^2 \leq nm^2, \;\; \|Y_0\|_2^4 \leq n^2 m^4 \text{ and } \|Y_0\|_4^4 \leq 4nm^4/3. \tag{13}$$

The equality occurs when $p_{n,A} = 1/2$ for $\|Y_0\|_2^2$ and $\|Y_0\|_2^4$ and when $p_{n,A} = (3 \pm \sqrt{3})/6$ for $\|Y_0\|_4^4$.

Theorem *6:* For every $n \geq 3$, for all $\varepsilon > 0$, $\delta \in (0,1)$ and $t \geq \tau$, we have $\mathbb{P}\{\|Y_t\|_\infty \geq \varepsilon\} \leq \delta$, where

$$\tau = \frac{4n(n-1)}{7n-6} \ln\left(\frac{13m^4 n}{3\varepsilon^4 \delta}\right).$$

Proof: Using the inequalities (13), we get

$$\begin{aligned} \mathbb{E}\left(\|Y_t\|_4^4\right) &\leq \frac{4nm^4(6\alpha^t + n\beta^t)}{3(n+6)} + \frac{3n^2 m^4(\beta^t - \alpha^t)}{n+6} \\ &= \frac{1}{n+6}\left[\frac{13m^4 n^2 \beta^t}{3} - (3n-8)m^4 n\alpha^t\right] \\ &\leq \frac{13m^4 n^2 \beta^t}{3(n+6)} \leq \frac{13m^4 n\beta^t}{3}. \end{aligned}$$

From Theorem 2 and the Markov inequality, we have for all $\varepsilon > 0$ and $t \geq \tau$

$$\begin{aligned} \mathbb{P}\{\|Y_t\|_\infty \geq \varepsilon\} &= \mathbb{P}\{\|Y_t\|_\infty^4 \geq \varepsilon^4\} \leq \mathbb{P}\{\|Y_t\|_4^4 \geq \varepsilon^4\} \\ &\leq \mathbb{P}\{\|Y_\tau\|_4^4 \geq \varepsilon^4\} \leq \frac{\mathbb{E}\left(\|Y_\tau\|_4^4\right)}{\varepsilon^4} \leq \frac{13m^4 n\beta^\tau}{3\varepsilon^4}. \end{aligned}$$

For all $x \in [0,1)$, we have $\ln(1-x) \leq -x$ which is equivalent to $(1-x)^\tau \leq e^{-\tau x}$. By definition of τ, this leads to

$$\beta^\tau = \left(1 - \frac{7n-6}{4n(n-1)}\right)^\tau \leq e^{-\tau(7n-6)/(4n(n-1))} = \frac{3\varepsilon^4 \delta}{13m^4 n}.$$

We then obtain for $t \geq \tau$, $\mathbb{P}\{\|Y_t\|_\infty \geq \varepsilon\} \leq \delta$, which completes the proof. ■

The following result shows that after τ pairwise interactions, each node is able to provide the value of κ with probability at least $1 - \delta$, for all $\delta \in (0,1)$.

Corollary *7:* For every $n \geq 3$, for all $\delta \in (0,1)$, for every $t \geq \tau$, we have

$$\mathbb{P}\left\{\left\lfloor \frac{C_t^{(i)} n}{m} + \frac{1}{2} \right\rfloor = \kappa, \text{ for all } i = 1, \ldots, n\right\} \geq 1 - \delta,$$

where

$$\tau = \frac{4n(n-1)}{7n-6}\left(5\ln(n) + \ln(208/3) - \ln(\delta)\right).$$

Proof: By taking $\varepsilon = m/(2n)$ in Theorem 6, we get

$$\tau = \frac{4n(n-1)}{7n-6}\left(5\ln(n) + \ln(208/3) - \ln(\delta)\right)$$

and thus for all $t \geq \tau$ and $\delta \in (0,1)$, we have $\mathbb{P}\left\{\|Y_t\|_\infty \geq m/(2n)\right\} \leq \delta$. Recalling that $Y_t = C_t - L$ and that $\ell = \kappa m/n$, we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\|C_t - L\|_\infty \geq m/(2n)\right\} \leq \delta \\
&\Leftrightarrow \mathbb{P}\left\{\|C_t - L\|_\infty < m/(2n)\right\} \geq 1-\delta \\
&\Leftrightarrow \mathbb{P}\left\{\left|C_t^{(i)} - \frac{\kappa m}{n}\right| < \frac{m}{2n},\ \forall i = 1,\ldots,n\right\} \geq 1-\delta \\
&\Leftrightarrow \mathbb{P}\left\{\left|\frac{C_t^{(i)} n}{m} - \kappa\right| < \frac{1}{2},\ \forall i = 1,\ldots,n\right\} \geq 1-\delta \\
&\Leftrightarrow \mathbb{P}\left\{\kappa < \frac{C_t^{(i)} n}{m} + \frac{1}{2} < \kappa + 1,\ \forall i = 1,\ldots,n\right\} \geq 1-\delta.
\end{aligned}$$

This last inequality implies that

$$\mathbb{P}\left\{\left\lfloor \frac{C_t^{(i)} n}{m} + \frac{1}{2}\right\rfloor = \kappa,\ \text{for all } i = 1,\ldots,n\right\} \geq 1-\delta,$$

which completes the proof. ■

Similarly, the following result shows that after τ pairwise interactions, each node is able to provide the proportion $p_{n,A} = n_A/n$ of nodes having initially the symbol A, with probability at least $1-\delta$, for all $\delta \in (0,1)$.

Corollary *8:* For all $n \geq 3$, $\varepsilon > 0$, $\delta \in (0,1)$ and $t \geq \tau$, we have

$$\mathbb{P}\left\{\left|\frac{C_t^{(i)} + m}{2m} - p_{n,A}\right| < \varepsilon,\ \text{for all } i = 1,\ldots,n\right\} \geq 1-\delta,$$

where

$$\tau = \frac{4n(n-1)}{7n-6}\left(\ln(n) - \ln(48/13) - 4\ln(\varepsilon) - \ln(\delta)\right).$$

Proof: By replacing ε by $2m\varepsilon$ in Theorem 6, we get

$$\tau = \frac{4n(n-1)}{7n-6}\left(\ln(n) - \ln(48/13) - 4\ln(\varepsilon) - \ln(\delta)\right)$$

and thus for all $t \geq \tau$, $\varepsilon > 0$ and $\delta \in (0,1)$, we have $\mathbb{P}\left\{\|Y_t\|_\infty \geq 2m\varepsilon\right\} \leq \delta$. Recalling that $Y_t = C_t - L$ and that $\ell = (2p_{n,A} - 1)m$, we obtain following the same lines used in the proof of Corollary 7

$$\begin{aligned}
&\mathbb{P}\left\{\|C_t - L\|_\infty \geq 2m\varepsilon\right\} \leq \delta \\
&\Leftrightarrow \mathbb{P}\left\{\left|\frac{C_t^{(i)} + m}{2m} - p_{n,A}\right| < \varepsilon, \forall i = 1,\ldots,n\right\} \geq 1-\delta,
\end{aligned}$$

which completes the proof. ■

VIII. The Strength of the 4-Norm over the 2-Norm

In this section, we show how effective the 4-norm is to derive useful bounds on the convergence speed of our solution. More precisely, we compare the results obtained when using the 4-norm with the ones previously got with the 2-norm. In the following, we obtain bounds and the asymptotic behavior of the moments and of the distribution of $\|Y_t\|_2$ and $\|Y_t\|_4$ when the time parameter t is equal to $an\ln(n)$, for a real constant $a > 0$ and when n goes to infinity. This choice of t is meaningful because it is of the same order as the lower bound τ of the number of steps needed for any node i in the system to converge either to $p_{n,A} = n_A/n$ or to κ with any high probability. Finally, without any loss of generality we suppose that $m = 1$ (recall that m and $-m$ are the initial value respectively associated to symbols A and B, see Section V).

Recalling Relations (10) and (12), we introduce the notation

$$\begin{aligned}
D_{n,1} &= 4p_{n,A}(1-p_{n,A}) = \frac{\|Y_0\|_2^2}{n}, \\
D_{n,2} &= 16p_{n,A}(1-p_{n,A})(3p_{n,A}^2 - 3p_{n,A} + 1) = \frac{\|Y_0\|_4^4}{n}.
\end{aligned}$$

Using (13), we have $D_{n,1} \leq 1$ and $D_{n,2} \leq 4/3$, for all $n \geq 1$.

Theorem *9:* For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right) \leq n^{1-a} D_{n,1} \leq n^{1-a}$$

and

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right) \underset{n\longrightarrow\infty}{\sim} n^{1-a} D_{n,1}.$$

Proof: See Appendix. ■

Theorem *10:* For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\begin{aligned}
&\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \\
&\quad\leq D_{n,1}^2 n^{2(1-a)} + 2(D_{n,2} + 3D_{n,1}^2) n^{(4-7a)/4} \\
&\quad\leq n^{2(1-a)} + (26/3) n^{(4-7a)/4}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \\
&\underset{n\longrightarrow\infty}{\sim} \begin{cases} D_{n,1}^2 n^{2(1-a)} & \text{if } a \in (0,4) \\ 2(D_{n,2} + 3D_{n,1}^2) n^{(4-7a)/4} & \text{if } a \in (4,+\infty) \\ (2D_{n,2} + 7D_{n,1}^2) n^{-6} & \text{if } a = 4. \end{cases}
\end{aligned}$$

Proof: See Appendix. ■

Theorem *11:* For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \underset{n\longrightarrow\infty}{\sim} (D_{n,2} + 3D_{n,1}^2) n^{(4-7a)/4},$$

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \leq 48n^{-1-2a} + (13/3) n^{(4-7a)/4}.$$

Proof: See Appendix. ■

Using these results and the Markov inequality, we obtain the following bounds on the ∞-norm of $Y_{\lceil an\ln(n)\rceil}$.

Theorem *12:* For all $a \in (0, +\infty)$, $n \geq 3$ and $\varepsilon > 0$, we have

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{n^{1-a}}{\varepsilon^2} \quad (14)$$

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{n^{2(1-a)} + (26/3)n^{(4-7a)/4}}{\varepsilon^4} \quad (15)$$

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{48n^{-1-2a} + (13/3)n^{(4-7a)/4}}{\varepsilon^4}. \quad (16)$$

Proof: See Appendix. ∎

Note that the bound of (14) has been obtained from the results of [7] and both bounds of (15) and (16) have been obtained from our new results on on the use of the 4-norm. It is quite immediate to check that for all $a \in (0, +\infty)$, $n \geq 3$ and $\varepsilon > 0$, bound of (16) is less than or equal to bound of (15). In order to compare the bounds of (14) and (16), we suppose that $\varepsilon = n^{-b}$, with $b > 0$. With these values of ε, we denote respectively by $f_n(a, b)$ and $g_n(a, b)$ the bounds of (14) and (16), that is

$$f_n(a, b) = n^{1-a+2b},$$
$$g_n(a, b) = 48n^{-1-2a+4b} + \frac{13}{3}n^{(4-7a+16b)/4}.$$

The comparison between $f_n(a, b)$ and $g_n(a, b)$ consists in determining the domains D_2 and D_4 defined by

$$D_2 = \{(a, b) \mid f_n(a, b) \leq g_n(a, b) \text{ and } \lim_{n\longrightarrow\infty} f_n(a, b) = 0\},$$

$$D_4 = \{(a, b) \mid g_n(a, b) \leq f_n(a, b) \text{ and } \lim_{n\longrightarrow\infty} g_n(a, b) = 0\}.$$

In order to be complete, we denote by $\overline{D}$ the domain in which the bounds (14) and (16) are useless, i.e. the domain defined by

$$\overline{D} = \{(a, b) \mid \lim_{n\longrightarrow\infty} f_n(a, b) \neq 0 \text{ and } \lim_{n\longrightarrow\infty} g_n(a, b) \neq 0\}.$$

Note that the condition $\lim_{n\longrightarrow\infty} f_n(a, b) \neq 0$ (resp. $\lim_{n\longrightarrow\infty} g_n(a, b) \neq 0$) is equivalent to $\lim_{n\longrightarrow\infty} f_n(a, b) = \infty$ (resp. $\lim_{n\longrightarrow\infty} g_n(a, b) = \infty$).

Domain D_4 (resp. D_2) represents the region in which the 4-norm (resp. 2-norm) gives tighter bounds with respect to the 2-norm (resp. 4-norm).

Theorem *13:* For all $a \in (0, +\infty)$ and $n \geq 3$, we have

$$D_2 = \{(a, b) \mid 2b + 1 < a \leq 8b/3\},$$
$$D_4 = \{(a, b) \mid a > 8b/3 \text{ and } a \geq 16b/7 + 4/7\},$$
$$\overline{D} = \{(a, b) \mid a \leq 2b + 1 \text{ and } a \leq 16b/7 + 4/7\}.$$

Proof: See Appendix. ∎

Figure 1 shows these different domains and thus the benefit obtained by using either the 4-norm or the 2-norm.

It is easily checked in Figure 1 that the percentage of points (a, b) belonging to domain D_4, which corresponds to the use of the 4-norm, is much more greater than the percentage of points (a, b) belonging to domain D_2, which corresponds to the use of the 2-norm.

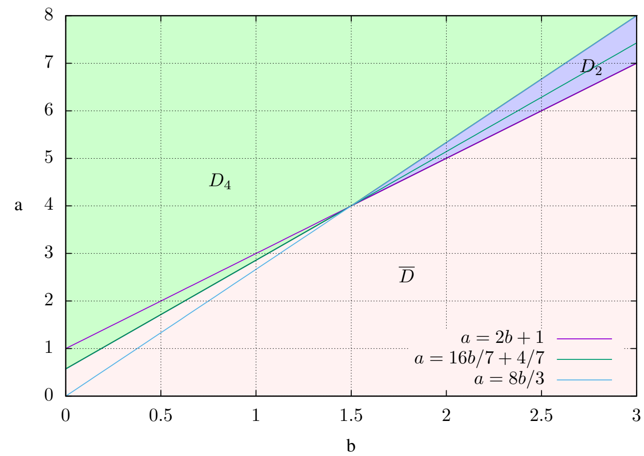

Figure 1. Domains D_2, D_4 and $\overline{D}$ for $a \in (0, 8]$ and $b \in (0, 3]$.

In order to illustrate this figure with numerical values, we have for $a = 2$ and $b = 1/4$,

$$f_n(2, 1/4) = \frac{1}{\sqrt{n}} \text{ and } g_n(2, 1/4) = \frac{13}{3n\sqrt{n}} + \frac{48}{n^4}$$

and Figure 1 shows, since $(2, 1/4) \in D_4$, that

$$\mathbb{P}\left\{\|Y_{\lceil 2n\ln(n)\rceil}\|_\infty \geq \frac{1}{n^{1/4}}\right\} \leq \frac{13}{3n\sqrt{n}} + \frac{48}{n^4} \leq \frac{1}{\sqrt{n}}.$$

For $a = 7.5$ and $b = 3$, we have

$$f_n(7.5, 3) = \frac{1}{n^{1/2}} \text{ and } g_n(7.5, 3) = \frac{13}{3n^{1/8}} + \frac{48}{n^4}$$

and Figure 1 shows, since $(7.5, 3) \in D_2$, that

$$\mathbb{P}\left\{\|Y_{\lceil 7.5n\ln(n)\rceil}\|_\infty \geq \frac{1}{n^3}\right\} \leq \frac{1}{n^{1/2}} \leq \frac{13}{3n^{1/8}} + \frac{48}{n^4}.$$

For $a = 2$ and $b = 1$, we have

$$f_n(2, 1) = n \text{ and } g_n(2, 1) = \frac{13n^{13/4}}{3} + \frac{48}{n}$$

and Figure 1 shows, since $(2, 1) \in \overline{D}$, that those values are useless to bound the quantity $\mathbb{P}\left\{\|Y_{\lceil 7.5n\ln(n)\rceil}\|_\infty \geq 1/n^3\right\}$.

IX. Experimental Evaluation of the Counting Problems

This section shows how tight our bounds are, by comparing the relation of Corollary 7 to the results obtained via extensive simulations in two cases, for the first $\kappa = 0$ and for the second $\kappa = n/2$. We also compare these bounds to the ones obtained by [7] in which the analysis is based on the 2-norm. The counting problem is equivalent to the proportion problem with $\varepsilon = 1/(2n)$, per example the counting problem with $n = 1000$ (see Figure 3) is like the proportion problem with $\varepsilon = 0.0005$, the advantage of this problem is that it could be compared with the results obtained in [7].

A simulation consists in the following steps: first, all the n nodes are initialized to m or $-m$ (without loss of generality, we set $m = 1$). In the following, two scenario are discussed:

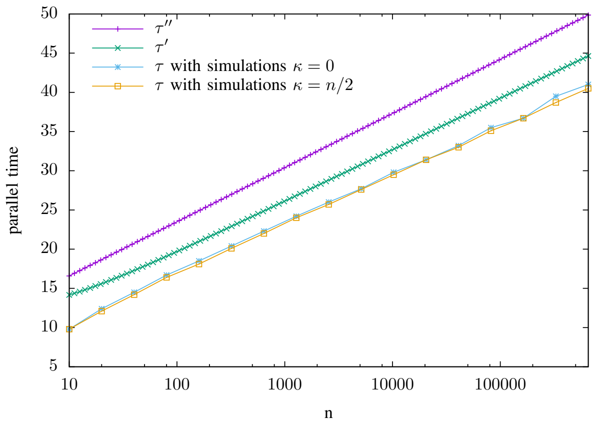

Figure 2. Parallel convergence time as a function of n when $\delta = 10^{-3}$ and $N = 10^5$ for $n < 20000$, $N = 10^4$ for $n > 20000$.

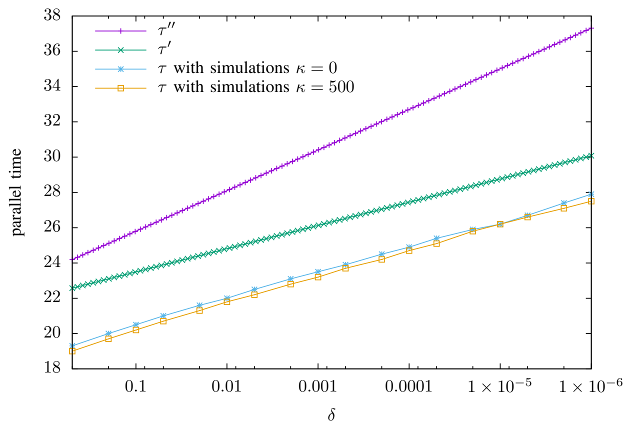

Figure 3. Parallel convergence time as a function of δ when $n = 10^3$ and $N = 10^6$.

in the first one, 50% of the nodes are initialized to m and the other ones are initialized to $-m$ (thus $\kappa = 0$), and in the second one, 75% of the nodes are initialized to m and the other ones are initialized to $-m$ (thus $\kappa = n/2$). Then, at each step of the simulation, two nodes are randomly chosen to interact and update their state. The simulation stops when all the nodes output κ. We have run N independent simulations and have logged the N number of performed interactions $\tau_1 \leq \ldots \leq \tau_N$. The convergence time is then $\tau_{\lceil N(1-\delta) \rceil}$, with $\delta \in (0, 1)$. Figures 2 and 3 depict the convergence parallel time $\tau_{\lceil N(1-\delta) \rceil}/n$ (recall that the convergence parallel time is equal to the convergence time divided by n) for both scenario, the bound τ' obtained from Relation (7), and the bound τ'' obtained from Theorem 4 of [7], i.e. $\tau'' = 4 \ln 2 + 3 \ln n - \ln \delta$. Figure 2 clearly shows that the bound τ' given by Corollary 7 is very close to the parallel time obtained in the simulations, and greatly improves upon our previous results, i.e. τ''. Figure 3 confirms the refinement of our theoretical evaluation. It shows that our present analysis is tighter than our previous one whatever the required precision of the computation.

X. Conclusion

In this paper we have presented a very precise analysis of the time required for each node to solve both the counting and the proportion problems. Our work relies on the use of the 4-norm, and a comparison of bounds derived with both the 4-norm and the 2-norm shows the conditions under which it is beneficial to use the 4-norm or the 2-norm. We might sense that the use of the d-norm, for $d > 4$, would give more refined results but it would give rise to a much more intricate analysis.

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APPENDIX

This appendix is dedicated to the proofs of Theorems 9, 10, 11, 12 and 13.

Recall that we have defined $D_{n,1}$ and $D_{n,2}$ as

$$D_{n,1} = 4p_{n,A}(1-p_{n,A}) = \frac{\|Y_0\|_2^2}{n} \text{ and } D_{n,2} = 16p_{n,A}(1-p_{n,A})(3p_{n,A}^2 - 3p_{n,A} + 1) = \frac{\|Y_0\|_4^4}{n}.$$

Recall also that, using (13), we have $D_{n,1} \leq 1$ and $D_{n,2} \leq 4/3$, for all $n \geq 1$. Moreover, we will frequently use the inequality

$$\text{for all } x \in [0,1) \text{ and } \nu \geq 0,\ (1-x)^\nu \leq e^{-\nu x}.$$

Theorem 9 For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right) \leq n^{1-a} D_{n,1} \leq n^{1-a} \text{ and } \mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right) \underset{n\longrightarrow\infty}{\sim} n^{1-a} D_{n,1}.$$

Proof: When $t = \lceil an\ln(n)\rceil$, we have from Realtions (8) and (10),

$$\begin{aligned}\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right) &= \left(1 - \frac{1}{n-1}\right)^{\lceil an\ln(n)\rceil} nD_{n,1} \leq e^{-\lceil an\ln(n)\rceil/(n-1)} nD_{n,1}\\ &\leq e^{-an\ln(n)/(n-1)} nD_{n,1} \leq e^{-a\ln(n)} nD_{n,1} = n^{1-a} D_{n,1} \leq n^{1-a}.\end{aligned}$$

Following the same lines we also have $\mathbb{E}\left(\|Y_{an\ln(n)}\|_2^2\right) \underset{n\longrightarrow\infty}{\sim} n^{1-a} D_{n,1}$, which completes the proof. ■

In order to prove Theorems 10 and 11, we need the following lemma. We first introduce the notation $\gamma = 1 - \dfrac{1}{n-1}$ and we recall that $\alpha = 1 - \dfrac{2}{n-1}$ and $\beta = 1 - \dfrac{7n-6}{4n(n-1)}$.

Lemma *14:* For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\begin{aligned}&\alpha^{\lceil an\ln(n)\rceil} \leq n^{-2a} \text{ and } \alpha^{\lceil an\ln(n)\rceil} \underset{n\longrightarrow\infty}{\sim} n^{-2a},\\ &\beta^{\lceil an\ln(n)\rceil} \leq n^{-7a/4} \text{ and } \beta^{\lceil an\ln(n)\rceil} \underset{n\longrightarrow\infty}{\sim} n^{-7a/4},\\ &\gamma^{\lceil an\ln(n)\rceil} \leq n^{-a} \text{ and } \gamma^{\lceil an\ln(n)\rceil} \underset{n\longrightarrow\infty}{\sim} n^{-a}.\end{aligned}$$

Proof: Setting $t = \lceil an\ln(n)\rceil$, we easily obtain

$$\begin{aligned}\beta^{\lceil an\ln(n)\rceil} &= \left(1 - \frac{7n-6}{4n(n-1)}\right)^{\lceil an\ln(n)\rceil} \leq e^{-(7n-6)\lceil an\ln(n)\rceil/(4n(n-1))}\\ &\leq e^{-a(7n-6)\ln(n)/(4(n-1))} \leq e^{-7a\ln(n)/4} = n^{-7a/4}.\end{aligned}$$

The same lines lead to $\beta^{\lceil an\ln(n)\rceil} \underset{n\longrightarrow\infty}{\sim} n^{-7a/4}$. In the same way we have

$$\alpha^{\lceil an\ln(n)\rceil} = \left(1 - \frac{2}{n-1}\right)^{\lceil an\ln(n)\rceil} \leq \left(1 - \frac{2}{n-1}\right)^{an\ln(n)} \leq e^{-2an\ln(n)/(n-1)} \leq e^{-2a\ln(n)} = n^{-2a}$$

and also $\alpha^{\lceil an\ln(n)\rceil} \underset{n\longrightarrow\infty}{\sim} n^{-2a}$. The rest of the proof concerning γ is identical. ■

Theorem 10 For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \leq D_{n,1}^2 n^{2(1-a)} + 2(D_{n,2} + 3D_{n,1}^2) n^{(4-7a)/4} \leq n^{2(1-a)} + (26/3) n^{(4-7a)/4}$$

and

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \underset{n\longrightarrow\infty}{\sim} \begin{cases} D_{n,1}^2 n^{2(1-a)} & \text{if } a \in (0,4)\\ 2(D_{n,2} + 3D_{n,1}^2) n^{(4-7a)/4} & \text{if } a \in (4,+\infty)\\ (2D_{n,2} + 7D_{n,1}^2) n^{-6} & \text{if } a = 4.\end{cases}$$

Proof: From Corollary 5 and from Relation (11), we obtain

$$\mathbb{E}\left(\|Y_t\|_2^4\right) = \frac{n}{n+6}\left[2(D_{n,2} + 3D_{n,1}^2) n\beta^t + D_{n,1}^2 n^2 \alpha^t - 2D_{n,2} n\alpha^t\right].$$

Setting $t = \lceil an\ln(n)\rceil$, using Lemma 14 and Relation (13) we get the inequalities and

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \underset{n\longrightarrow\infty}{\sim} 2(D_{n,2}+3D_{n,1}^2)n^{(4-7a)/4} + D_{n,1}^2 n^{2(1-a)} - 2D_{n,2}n^{1-2a}.$$

For all $a > 0$, we have $1-2a < 2(1-a)$ and $1-2a < (4-7a)/4$. Furthermore, $2(1-a) \geq (4-7a)/4 \Longleftrightarrow a \leq 4$. Thus

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right) \underset{n\longrightarrow\infty}{\sim} \mathbb{1}_{\{a\geq 4\}} 2(D_{n,2}+3D_{n,1}^2)n^{(4-7a)/4} + \mathbb{1}_{\{a\leq 4\}} D_{n,1}^2 n^{2(1-a)},$$

which completes the proof. ∎

Theorem 11 For all $a \in (0,+\infty)$ and $n \geq 3$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \underset{n\longrightarrow\infty}{\sim} (D_{n,2}+3D_{n,1}^2)n^{(4-7a)/4},$$

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \leq 48n^{-1-2a} + (13/3)n^{(4-7a)/4}.$$

Proof: From Corollary 5 and from Relation (12), we have

$$\mathbb{E}\left(\|Y_t\|_4^4\right) = \frac{n}{n+6}\left[(D_{n,2}+3D_{n,1}^2)n\beta^t + 6D_{n,2}\alpha^t - 3D_{n,1}^2 n\alpha^t\right].$$

Setting $t = \lceil an\ln(n)\rceil$ and using Lemma 14 we obtain

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \underset{n\longrightarrow\infty}{\sim} (D_{n,2}+3D_{n,1}^2)n^{(4-7a)/4} + 6D_{n,2}n^{-2a} - 3D_{n,1}^2 n^{-2a+1}$$

Since for $a > 0$, $(4-7a)/4 > -2a+1 > -2a$, we have

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \underset{n\longrightarrow\infty}{\sim} (D_{n,2}+3D_{n,1}^2)n^{(4-7a)/4}.$$

Now it is easy to check that $6D_{n,2} - 3D_{n,1}^2 n \leq 48/n$. We thus obtain

$$\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right) \leq 48n^{-1-2a} + (13/3)n^{(4-7a)/4}.$$

which completes the proof. ∎

We then have the following results for the distribution of $\|Y_t\|_\infty$, with $t = \lceil an\ln(n)\rceil$.

Theorem 12 For all $a \in (0,+\infty)$, $n \geq 3$ and $\epsilon > 0$, we have

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{n^{1-a}}{\varepsilon^2},$$

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{n^{2(1-a)} + (26/3)n^{(4-7a)/4}}{\varepsilon^4},$$

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} \leq \frac{48n^{-1-2a} + (13/3)n^{(4-7a)/4}}{\varepsilon^4}.$$

Proof: It suffices to use the inequality $\|x\|_\infty \leq \|x\|_d$ and the Markov inequality. Indeed, using Theorem 9, we have

$$\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} = \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty^2 \geq \varepsilon^2\} \leq \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_2^2 \geq \varepsilon^2\} \leq \frac{\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^2\right)}{\varepsilon^2} \leq \frac{n^{1-a}}{\varepsilon^2}.$$

In the same way, using Theorem 10, we get

$$\begin{aligned}\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} &= \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty^4 \geq \varepsilon^4\} \leq \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_2^4 \geq \varepsilon^4\} \\ &\leq \frac{\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_2^4\right)}{\varepsilon^4} \leq \frac{n^{2(1-a)} + (26/3)n^{(4-7a)/4}}{\varepsilon^4}.\end{aligned}$$

Finally, using Theorem 11, we obtain

$$\begin{aligned}\mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty \geq \varepsilon\} &= \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_\infty^4 \geq \varepsilon^4\} \leq \mathbb{P}\{\|Y_{\lceil an\ln(n)\rceil}\|_4^4 \geq \varepsilon^4\} \\ &\leq \frac{\mathbb{E}\left(\|Y_{\lceil an\ln(n)\rceil}\|_4^4\right)}{\varepsilon^4} \leq \frac{48n^{-1-2a} + (13/3)n^{(4-7a)/4}}{\varepsilon^4},\end{aligned}$$

which completes the proof. ∎

Recall that the domains D_2, D_4 and $\overline{D}$ have been respectively defined by

$$D_2 = \{(a,b) \mid f_n(a,b) \leq g_n(a,b) \text{ and } \lim_{n\longrightarrow\infty} f_n(a,b) = 0\},$$

$$D_4 = \{(a,b) \mid g_n(a,b) \le f_n(a,b) \text{ and } \lim_{n\longrightarrow\infty} g_n(a,b) = 0\},$$
$$\overline{D} = \{(a,b) \mid \lim_{n\longrightarrow\infty} f_n(a,b) \neq 0 \text{ and } \lim_{n\longrightarrow\infty} g_n(a,b) \neq 0\},$$

where the bounds $f_n(a,b)$ and $g_n(a,b)$ are given by

$$f_n(a,b) = n^{1-a+2b},\ g_n(a,b) = 48n^{-1-2a+4b} + \frac{13}{3}n^{(4-7a+16b)/4}.$$

Theorem 13 For all $a \in (0,+\infty)$ and $n \ge 3$, we have

$$\begin{aligned} D_2 &= \{(a,b) \mid 2b+1 < a \le 8b/3\}, \\ D_4 &= \{(a,b) \mid a > 8b/3 \text{ and } a \ge 16b/7 + 4/7\}, \\ \overline{D} &= \{(a,b) \mid a \le 2b+1 \text{ and } a \le 16b/7 + 4/7\}. \end{aligned}$$

Proof: Consider first domain D_2. The condition $\lim_{n\longrightarrow\infty} f_n(a,b) = 0$ is equivalent to $1-a+2b<0$, that is $a > 2b+1$. Moreover we have

$$f_n(a,b) \le g_n(a,b) \Longleftrightarrow 1 \le 48n^{2b-a-2} + \frac{13}{3}n^{2b-3a/4}.$$

Thus, if $a \le 8b/3$ then we have $f_n(a,b) \le g_n(a,b)$. Consider now domain $\overline{D}$. The condition $\lim_{n\longrightarrow\infty} g_n(a,b) = 0$ is equivalent to $-1-2a+4b<0$ and $1-7a/4+4b<0$, which is equivalent to $a > 16b/7 + 4/7$. We thus have

$$\overline{D} = \{(a,b) \mid a \le 2b+1 \text{ and } a \le 16b/7 + 4/7\}.$$

The rest of the plane is domain D_4. ∎