On Fractional Linear Network Coding Solution of Multiple-Unicast Networks

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Abstract—It is known that there exists a multiple-unicast network which has a rate 1 linear network coding solution if and only if the characteristic of the finite field belongs to a given finite or co-finite set of primes. In this paper, we show that for any non-zero positive rational number $\frac{k}{n}$, there exists a multiple-unicast network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to a given finite or co-finite set of primes.

I. INTRODUCTION

The concept of network coding came into light in the year 2000 by a seminal work of Ahlswede et al.. In [1] the authors showed that the min-cut bound in a multicast network can be achieved if the intermediate nodes are allowed to do operations on incoming messages before forwarding. This kind of operation of intermediate nodes has been termed as network coding. Linear network coding refers to the scheme when all such operations are linear. In subsequent works it has been shown that linear network coding is sufficient to achieve the capacity of multicast networks [2]-[4]. Moreover, a capacity achieving network code can be designed efficiently [3]. As far as a multicast network is concerned, a scalar linear network code over a sufficiently large finite field suffices to achieve the capacity. However, the requirement of field size can be reduced if vector linear network coding is used [5]. In [6], it was shown that binary field is sufficient if the vector length is large enough. These results related to multicast networks were known by the year 2005. After a decade later, there has been a few interesting works related to intricacies involved in linear network coding for multicast network. In one such result it has been shown that a multicast network being linearly solvable over a sufficiently large finite field does not necessarily mean that over all larger fields it is also solvable [7]. It was also shown that existence of a vector linear solution for a certain vector dimension does not necessarily mean that there exists a vector linear solution for all larger vector dimensions [8]. It is important to note that for a multicast network, the characteristic of the finite field does not play an important role in the sense that there does not exist a multicast network which has a scalar/vector linear solution only over a finite field of certain characteristic.

The non-multicast networks have very different properties. For non-multicast networks it has been shown that there exists a network where better throughput (capacity) can be achieved if the nodes are allowed to do non-linear operations [9]. There are non-multicast networks where scalar/vector linear solution not only depends on the size of the field but also on the characteristic of the field [10]-[12]. In particular, it has been shown in [10] that for any system of polynomial equations over integers, there exists a network which has a scalar linear network solution over a finite field if and only if the system of polynomial equation has a root in the same finite field. Therefore, there exist networks which are solvable only over fields of certain characteristics. Interesting examples are the Fano and non-Fano network presented in [9], [13]. The Fano network has a vector linear solution for any vector dimension if and only if the characteristic of the finite field is even. Over a finite field of odd characteristic it has linear coding capacity equal to $\frac{4}{5}$. The non-Fano network has a vector linear solution for any vector dimension if and only if the characteristic of the finite field is odd. If the characteristic is even then its linear coding capacity is equal to $\frac{10}{11}$.

In the references [11] and [12] the authors considered networks where all terminals demand the sum of the symbols generated at the sources. Such networks have been given the name sum-networks. In [11] it was shown that for any finite set of primes there exists a sum-network which has a vector linear solution for any vector dimension if and only if the characteristic of the finite field belongs to the given set. In [12] it was shown that for any sum-network, there exists a multiple-unicast network which has a rate 1 solution if and only if the sum-network has a rate 1 solution. They also show that for any co-finite set of primes there exists a sum-network which has a vector linear solution for any vector dimension if and only if the characteristic of the finite field belong to the given set [12]. Thus, these three results when combined, shows that for any given finite/co-finite set of primes there exists a multiple-unicast network which has a vector linear solution if and only if the characteristic of the finite field belong to the given set.

However, in the works of [10]–[12], the dependency on the characteristic of the field is shown only for either scalar linear network coding [10] or vector linear network coding of any vector length [11], [12]. In this paper, we generalise the previous results to show that for any non-zero positive rational number $\frac{k}{n}$ and for any given finite/co-finite set of prime numbers, there exists a non-multicast network which has a rate $\frac{k}{n}$ fractional linear network code solution if and only of the characteristic of the finite belongs to the given finite/co-finite set of primes.

The organization of the paper is as follows. In Section II we reproduce the standard definitions of fractional linear network coding, vector linear network and scalar linear network coding. In Section III we show that for any non-zero positive rational number $\frac{k}{n}$, and for any finite/co-finite set of primes, there exists a network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. In Section IV we extend the results of Section II to multiple-unicast networks. The paper is concluded in Section V.

II. PRELIMINARIES

A network is represented by a graph G(V, E). The set V is partitioned into three disjoint sets namely, the set of sources S, the set of terminals T, and the rest of the nodes are called the intermediate nodes and their collection is denoted by V'. Without loss of generality the sources are assumed to have no incoming edge and the terminals are assumed to have no outgoing edge. Each source generates an i.i.d random process uniformly distributed over an alphabet \mathcal{A} . The source process at any source is independent of all source processes generated at other sources. Each terminal demands to compute the information generated at a subset of the sources. An edge e originating from node u and ending at node v is denoted by (u, v); where u is denoted by tail(e), and v is denoted by head(e). For an node $v \in V$, the set of edges e for which head(e) = v is denoted by In(v). The information carried by an edge e is denoted by Y_e . Without loss of generality it is assumed that all the edges in the network are unit capacity edges.

In a (k, n) fractional linear network code the alphabet \mathcal{A} is taken as a finite field \mathbb{F}_q . Each source $s_i \in S$ generates a symbol X_i from the finite field \mathbb{F}_q^k . For any edge e, if $tail(e) = s_i$, then $Y_e = A_{\{s_i,e\}}X_i$ where $Y_e \in \mathbb{F}_q^n$, $A_{\{s_i,e\}} \in \mathbb{F}_q^{n \times k}$ and $X_i \in \mathbb{F}_q^k$. If tail(e) is any intermediate node $v \in V'$, then $Y_e = \sum_{\forall e' \in In(v)} A_{\{e',e\}}Y_{e'}$ where $Y_{e}, Y_{e'} \in \mathbb{F}_q^n$, and $A_{\{e',e\}} \in \mathbb{F}_q^{n \times n}$. For any terminal $t \in T$, if t computes symbol X_t , then $X_t = \sum_{\forall e' \in In(t)} A_{\{e',t\}}Y_{e'}$ where $X_t \in \mathbb{F}_q^k, A_{\{e',t\}} \in \mathbb{F}_q^{k \times n}$ and $Y_{e'} \in \mathbb{F}_q^n$. The matrices $A_{\{s_i,e\}}, A_{\{e',e\}}$ and $A_{\{e',t\}}$ shown above are called as the local coding matrices.

If using a (k, n) fractional linear network code all terminals can compute k respective symbols in n uses of the network, then the network is said to have a (k, n) fractional linear network coding solution. The ratio $\frac{k}{n}$ is called the rate. A network is said to have a rate $\frac{k}{n}$ fractional linear network coding solution if it has a (dk, dn) fractional linear network coding solution for any non-zero positive integer d. A (k, k)fractional linear network code is called as a k dimensional vector linear network code and k is called the vector dimension or as the message dimension. If a network has a (k, k)fractional linear network coding solution then it is said that the network has a vector linear solution for k message dimension. If a network has a (1, 1) vector linear network coding solution then the network is said to be scalar linearly solvable.

III. A NETWORK HAVING A RATE $\frac{k}{n}$ FRACTIONAL LINEAR NETWORK CODING SOLUTION IFF THE CHARACTERISTIC BELONGS TO A GIVEN FINITE/CO-FINITE SET OF PRIMES

A. Network having $\frac{k}{n}$ solution iff characteristic belongs to a given finite set of primes.

First we show that for any positive non-zero rational number $\frac{k}{n}$, and for any given finite set of primes, there exists a network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. Towards this end, we consider the network \mathcal{N}_1 presented in Fig. 1. The network shown here in Fig. 1 has both the generalized Fano network shown in [15] and the Fano network shown in [15] as a sub-network. As it can be seen, the network has q+1 sets of sources namely, S_a , S_{b_i} for $1 \leq 1$ $i \leq (q-1)$ and S_c . In the figure, the individual source nodes are indicated by the source message it generates. A source $s_i \in S_a$ generates the message a_i . For $1 \leq i \leq (q-1)$ and $1 \leq j \leq n$ a source $s_j \in S_{b_i}$ generates the message b_{ij} . And a source $s_i \in S_c$ generates the message c_i . There are 2q sets of terminals namely, T_c, T_a, T_{b_i} and T_{c_i} for $1 \le i \le (q-1)$; and each of these sets contains n terminals. Each individual terminal is indicated by the source message it demands.

Below we list the set of edges which has a source node as its tail.

- 1) (s, u_1) for $\forall s \in \{S_a, S_{b_1}, S_{b_2}, \dots, S_{b_{q-1}}\}.$
- 2) (s, u_2) for $\forall s \in \{S_{b_1}, S_{b_2}, \dots, S_{b_{q-1}}, S_c\}$.
- 3) (a_i, u_{11}) for $1 \le i \le n$.
- 4) (c_i, u_6) for $1 \le i \le n$.
- 5) $(b_{ij}, tail(e_k))$ for $1 \le i, k \le (q-1), i \ne k$, and $1 \le j \le n$.
- 6) (b_{ij}, v_k) for $1 \le i, k \le (q-1), i \ne k$, and $1 \le j \le n$.
- 7) (b_{ij}, w_i) for $1 \le i \le (q-1)$, and $1 \le j \le n$.

We now list the edges which originates at an intermediate node and ends at a intermediate node.

- 8) (u_i, u_{i+2}) for $1 \le i \le 7, i \ne 4$.
- 9) (u_i, u_{i+1}) for i = 4, 8, 9, 11, 13.
- 10) (u_3, u_6) , (u_7, u_{11}) , and (u_8, u_{13})
- 11) e_i for $1 \le i \le (q-1)$
- 12) $(u_4, tail(e_i))$ for $1 \le i \le (q-1)$
- 13) $(head(e_i), u_{13})$ and $(head(e_i), w_i)$ for $1 \le i \le (q-1)$
- 14) (u_{10}, v_i) and (v_i, v'_i) for $1 \le i \le (q-1)$
- 15) (w_i, w'_i) for $1 \le i \le (q-1)$

For any terminal $t_i \in T_c$ there exists an edge (u_{12}, t_i) and t_i demands the message c_i . For any terminal $t_j \in T_{b_i}$ for $1 \le i \le (q-1), 1 \le j \le n$, there exits an edge (v'_i, t_j) where the terminal t_j demands the message b_{ij} . For any terminal $t_i \in T_a$ there exits an edge (u_{14}, t_i) and t_i demands the message a_i . For $1 \le i \le (q-1)$, a terminal $t_j \in T_{c_i}$ for $1 \le j \le n$ is connected to the node w'_i by the edge (w'_i, t_j) and t_j demands the message c_j . The local coding matrices are shown alongside the edges.

Lemma 1. The network in Fig. 1 has a rate $\frac{1}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field divides q.

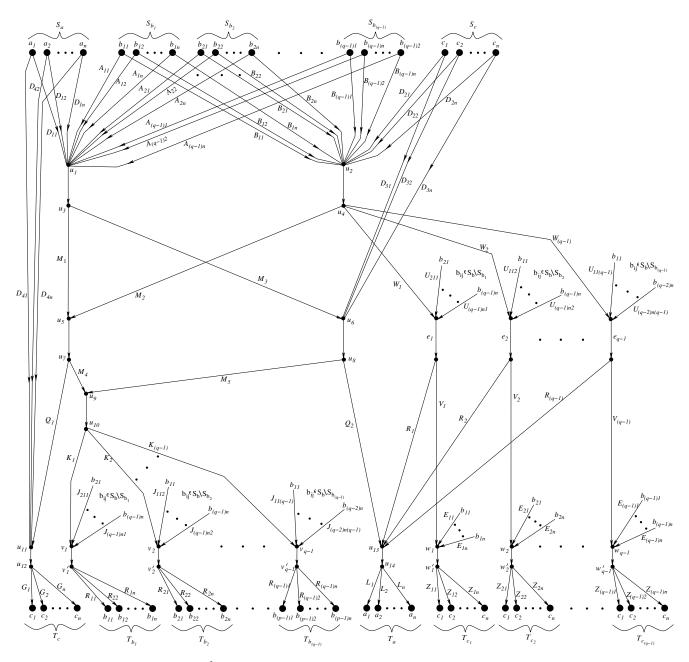


Fig. 1. Network N_1 which has a rate $\frac{1}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field divides q

Proof: Consider a (d, dn) fractional linear network coding solution of the network \mathcal{N}_1 where d is any non-zero positive integer. Then the sizes of the local coding matrices are given in the following. For $1 \leq i \leq n$, the matrices D_{4i} and D_{1i} are of size $dn \times d$ and it left multiplies the information a_i which is a d length vector. Matrices A_{ij} , B_{ij} , U_{ijk} , J_{ijk} and E_{ij} for $1 \leq i, k \leq (q-1), i \neq k$ and $1 \leq j \leq n$ are of size $dn \times d$ and it left multiplies the d length vector b_{ij} . For $1 \leq i \leq n$, the matrices D_{2i} and D_{3i} are of size $dn \times d$ and it left multiplies the information c_i . The following matrices are of size $dn \times dn$: M_i for $1 \leq i \leq 5$, Q_1, Q_2, K_i, R_i, V_i and W_i for $1 \leq i \leq (q-1)$. And the following are the matrices of size $d \times dn$: G_j, R_{ij}, L_j and Z_{ij} for $1 \leq i \leq (q-1)$ and $1 \leq j \leq n$. ALso let I be a $d \times d$ identity matrix. Then,

$$Y_{(u_1,u_3)} = \sum_{i=1}^{n} D_{1i}a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} A_{ij}b_{ij}$$
(1)

$$Y_{(u_2,u_4)} = \sum_{i=1}^{q-1} \sum_{j=1}^{n} B_{ij} b_{ij} + \sum_{i=1}^{n} D_{2i} c_i$$
(2)

$$Y_{(u_5,u_7)} = M_1 Y_{(u_1,u_3)} + M_2 Y_{(u_2,u_4)} = \sum_{i=1}^n M_1 D_{1i} a_i$$

+ $\sum_{i=1}^{q-1} \sum_{j=1}^n (M_1 A_{ij} + M_2 B_{ij}) b_{ij} + \sum_{i=1}^n M_2 D_{2i} c_i$ (3)

$$Y_{(u_{6},u_{8})} = M_{3}Y_{(u_{1},u_{3})} + \sum_{i=1}^{n} D_{3i}c_{i}$$

$$= \sum_{i=1}^{n} M_{3}D_{1i}a_{i} + \sum_{i=1}^{q-1} \sum_{j=1}^{n} M_{3}A_{ij}b_{ij} + \sum_{i=1}^{n} D_{3i}c_{i} (4)$$

$$Y_{(u_{9},u_{10})} = M_{4}Y_{(u_{5},u_{7})} + M_{5}Y_{(u_{6},u_{8})}$$

$$= \sum_{i=1}^{n} (M_{4}M_{1}D_{1i} + M_{5}M_{3}D_{1i})a_{i}$$

$$+ \sum_{i=1}^{q-1} \sum_{j=1}^{n} \{M_{4}(M_{1}A_{ij} + M_{2}B_{ij}) + M_{5}M_{3}A_{ij}\}b_{ij}$$

$$+ \sum_{i=1}^{n} (M_{4}M_{2}D_{2i} + M_{5}D_{3i})c_{i}$$

$$Y_{(u_{11},u_{12})} = \sum_{i=1}^{n} (D_{4i} + Q_{1}M_{1}D_{1i})a_{i}$$
(5)

$$+\sum_{i=1}^{q-1}\sum_{j=1}^{n} (M_1 A_{ij} + M_2 B_{ij}) b_{ij} + \sum_{i=1}^{n} M_2 D_{2i} c_i$$
(6)

for $1 \leq i \leq (q-1)$:

$$Y_{e_i} = W_i Y_{(u_2, u_4)} + \sum_{j=1, j \neq i}^{q-1} \sum_{k=1}^n U_{jki} b_{jk} = \sum_{k=1}^n W_i B_{ik} b_{ik}$$
$$+ \sum_{j=1, j \neq i}^{q-1} \sum_{k=1}^n (W_i B_{jk} + U_{jki}) b_{jk} + \sum_{k=1}^n W_i D_{2k} c_k (7)$$

for
$$1 \le i \le (q-1)$$
:

$$Y_{(v_i,v'_i)} = K_i Y_{(u_9,u_{10})} + \sum_{j=1,j\neq i}^{q-1} \sum_{k=1}^n J_{jki} b_{jk}$$

$$= \sum_{k=1}^n K_i (M_4 M_1 D_{1k} + M_5 M_3 D_{1k}) a_k$$

$$+ \sum_{j=1}^n K_i \{M_4 (M_1 A_{ij} + M_2 B_{ij}) + M_5 M_3 A_{ij}\} b_{ij}$$

$$+ \sum_{k=1,k\neq i}^{q-1} \sum_{j=1}^n \{J_{kji} + K_i (M_4 (M_1 A_{kj} + M_2 B_{kj}) + M_5 M_3 A_{kj})\} b_{kj}$$

$$+ M_5 M_3 A_{kj}) \} b_{kj}$$

$$+\sum_{j=1}^{\infty} K_i (M_4 M_2 D_{2j} + M_5 D_{3j}) c_j \tag{8}$$

for
$$1 \le i \le (q-1)$$
:
 $Y_{(w_i,w'_i)} = V_i Y_{e_i} + \sum_{j=1}^n E_{ij} b_{ij} = \sum_{k=1}^n (V_i W_i B_{ik} + E_{ik}) b_{ik}$
 $+ \sum_{j=1, j \ne i}^{q-1} \sum_{k=1}^n \{V_i (W_i B_{jk} + U_{jki})\} b_{jk}$
 $+ \sum_{k=1}^n V_i W_i D_{2k} c_k$
(9)

$$Y_{(u_{13},u_{14})} = Q_2 Y_{(u_6,u_8)} + \sum_{i=1}^{q-1} R_i Y_{e_i} = \sum_{i=1}^n Q_2 M_3 D_{1i} a_i$$

+
$$\sum_{i=1}^{q-1} \sum_{j=1}^n \{Q_2 M_3 A_{ij} + R_i W_i B_{ij} + \sum_{k=1,k\neq i}^{q-1} R_k (W_k B_{ij} + U_{ijk})\} b_{ij}$$

+
$$\sum_{i=1}^n \{Q_2 D_{3i} + (\sum_{k=1}^{q-1} R_k W_k) D_{2i}\} c_i$$
(10)

Since a terminal $t_i \in T_c$ computes c_i , we have the following inequalities. For $1 \leq i, j \leq n$ since the component of a_i is zero at all $t_j \in T_c$,

$$G_j(Q_1M_1D_{1i} + D_{4i}) = 0 (11)$$

As the components of for b_{ij} is also zero at all $t_k \in T_c$, for $1 \le i \le (q-1)$ and $1 \le j, k \le n$ we have,

$$G_k\{Q_1(M_1A_{ij} + M_2B_{ij})\} = 0$$
(12)

Now, since the terminal $t_i \in T_c$ retrieves c_i for $1 \le i, j \le n$ and $j \ne i$ we have,

$$G_i(Q_1 M_2 D_{2i}) = I (13)$$

$$G_i(Q_1 M_2 D_{2j}) = 0 (14)$$

Now consider the *n* terminals in the set T_{b_i} for $1 \le i \le (q-1)$. Since the component of a_k for $1 \le k \le n$ at $t_j \in T_{b_i}$ for $1 \le j \le n$ is zero, we have, for $1 \le i \le (q-1)$ and $1 \le j, k \le n$:

$$R_{ij}K_i(M_4M_1D_{1k} + M_5M_3D_{1k}) = 0$$
(15)

Since the terminal $t_j \in T_{b_i}$ computes the information b_{ij} , we have, for $1 \leq i, k \leq (q-1), i \neq k, 1 \leq j, m, l \leq n$ and $m \neq j$:

$$R_{ij}K_i\{M_4(M_1A_{ij} + M_2B_{ij}) + M_5M_3A_{ij}\} = I$$
(16)

$$R_{ij}K_i\{M_4(M_1A_{im} + M_2B_{im}) + M_5M_3A_{im}\} = 0$$
(17)

$$R_{ij}\{K_i(M_4(M_1A_{kl} + M_2B_{kl}) + M_5M_3A_{kl}) + J_{kli}\} = 0$$
(18)

Since the component of c_k for $1 \le k \le n$ is zero at $t_j \in T_{b_i}$, we have for $1 \le i \le (q-1)$ and $1 \le j, k \le n$,

$$R_{ij}K_i(M_4M_2D_{2k} + M_5D_{3k}) = 0 (19)$$

Let us consider the terminals in the set T_a . Since $t_i \in T_a$ computes the message a_i , for $1 \leq i, j \leq n$ and $j \neq i$, we have:

$$L_i Q_2 M_3 D_{1i} = I \tag{20}$$

$$L_i Q_2 M_3 D_{1j} = 0 (21)$$

At any $t_l \in T_a$ for $1 \le i \le (q-1)$ and $1 \le l, j \le n$ the component of b_{ij} is zero. So we have,

$$L_{l}\{Q_{2}M_{3}A_{ij} + R_{i}W_{i}B_{ij} + \sum_{k=1,k\neq i}^{q-1} R_{k}(W_{k}B_{ij} + U_{ijk})\}$$
(22)

At a terminal $t_i \in T_a$, since the component of c_i is zero, for newly defined matrices are square matrices of size $dn \times dn$. $1 \leq i, j \leq n$:

$$L_{j}\{Q_{2}D_{3i} + \left(\sum_{k=1}^{q-1} R_{k}W_{k}\right)D_{2i}\} = 0$$
 (23)

Now consider the terminals in the set T_{c_i} for $1 \le i \le (q-1)$. Since at $t_l \in T_{c_i}$ the component of b_{ij} for $1 \le i \le (q-1)$ and $1 \leq l, j \leq n$ is zero, we have,

$$Z_{il}(V_i W_i B_{ij} + E_{ij}) = 0 (24)$$

For $1 \le k \le (q-1), k \ne i$ the component of b_{kj} for $1 \le j \le n$ is zero as well,

$$Z_{il}\{V_i(W_iB_{kj} + U_{kji})\} = 0$$
(25)

Since $t_l \in T_{c_i}$ computes c_l , for $1 \leq l, m \leq n, l \neq m$, we have,

$$Z_{il}V_iW_iD_{2l} = I (26)$$

$$Z_{il}V_iW_iD_{2m} = 0 (27)$$

The rest of the proof requires a lemma and a corollary which are presented next.

Let us consider $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}^T$ and B = $\begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$ where A_i and B_i for $1 \leq i \leq n$ are matrices of size $d \times dn$ and $dn \times d$ respectively. Let I_{dn} denote an identity matrix of size $dn \times dn$.

Lemma 2. If $A_iB_i = I$ but $A_iB_j = 0$ for $1 \le i, j \le n, i \ne j$, then $AB = I_{dn}$; and both A and B has an unique inverse.

Proof:

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$$
$$= \begin{bmatrix} A_1 B_1 & A_1 B_2 & \cdots & A_1 B_n \\ A_2 B_1 & A_2 B_2 & \cdots & A_2 B_n \\ \vdots & \vdots & \vdots & \vdots \\ A_n B_1 & A_n B_2 & \cdots & A_n B_n \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = I_{dn}$$

Now since both A and B are matrices of size $dn \times dn$, for $AB = I_{dn}$ to hold, both of A and B has to be full rank matrices. This completes the proof.

Corollary 3. For $1 \le i, j \le n$, if $A_i B_j = 0$, then AB = 0.

We now define the following matrices. Note that all of these

$$G = \begin{bmatrix} G_1 & G_2 & \cdots & G_n \end{bmatrix}^T$$

$$\begin{bmatrix} Q_1(M_1A_{i1} + M_2B_{i1}) \end{bmatrix}^T$$
(28)

$$Q_1(M_1A_i + M_2B_i) = \begin{vmatrix} Q_1(M_1A_{i2} + M_2B_{i2}) \\ \vdots \\ Q_1(M_1A_i + M_2B_{i2}) \end{vmatrix}$$
(29)

$$\begin{bmatrix} Q_1(M_1A_{in} + M_2B_{in}) \end{bmatrix}$$

$$Q_1M_2D_2 = \begin{bmatrix} Q_1M_2D_{21} & Q_1M_2D_{22} & \cdots & Q_1M_2D_{2n} \end{bmatrix} (30)$$

$$R_iK_i = \begin{bmatrix} R_{i1}K_i & R_{i2}K_i & \cdots & R_{in}K_i \end{bmatrix}^T$$
(31)

$$M_4 M_1 D_1 + M_5 M_3 D_1 = \begin{bmatrix} M_4 M_1 D_{11} + M_5 M_3 D_{11} \\ M_4 M_1 D_{12} + M_5 M_3 D_{12} \\ \vdots \\ M_4 M_1 D_{1n} + M_5 M_3 D_{1n} \end{bmatrix}^T (32)$$

$$M_{4}(M_{1}A_{i} + M_{2}B_{i}) + M_{5}M_{3}A_{i}$$

$$= \begin{bmatrix} M_{4}(M_{1}A_{i1} + M_{2}B_{i1}) + M_{5}M_{3}A_{i1} \\ M_{4}(M_{1}A_{i2} + M_{2}B_{i2}) + M_{5}M_{3}A_{i2} \\ \vdots \\ M_{4}(M_{1}A_{in} + M_{2}B_{in}) + M_{5}M_{3}A_{in} \end{bmatrix}^{T} (33)$$

$$\begin{bmatrix} M_{4}M_{2}D_{21} + M_{5}D_{31} \\ M_{4}M_{2}D_{22} + M_{5}D_{32} \end{bmatrix}^{T}$$

$$M_4 M_2 D_2 + M_5 D_3 = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ \vdots \\ M_4 M_2 D_{2n} + M_5 D_{3n} \end{bmatrix}$$
(34)

$$L = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^T$$
(35)
$$Q_2 M_3 D_1 = \begin{bmatrix} Q_2 M_3 D_{11} & Q_2 M_3 D_{12} & \cdots & Q_2 M_3 D_{1n} \end{bmatrix}$$
(36)

$$Q_{2}M_{3}A_{i} + R_{i}W_{i}B_{i} + \sum_{k=1,k\neq i}^{q-1} R_{k}(W_{k}B_{i}+U_{ik})$$

$$= \begin{bmatrix} Q_{2}M_{3}A_{i1} + R_{i}W_{i}B_{i1} + \sum_{k=1,k\neq i}^{q-1} R_{k}(W_{k}B_{i1}+U_{i1k}) \\ Q_{2}M_{3}A_{i2} + R_{i}W_{i}B_{i2} + \sum_{k=1,k\neq i}^{q-1} R_{k}(W_{k}B_{i2}+U_{i2k}) \\ \vdots \\ Q_{2}M_{3}A_{in} + R_{i}W_{i}B_{in} + \sum_{k=1,k\neq i}^{q-1} R_{k}(W_{k}B_{in}+U_{ink}) \end{bmatrix}^{T}$$
(37)
$$Q_{2}D_{3} + \left(\sum_{k=1}^{q-1} R_{k}W_{k}\right)D_{2}$$

$$= \begin{bmatrix} Q_{2}D_{31} + \left(\sum_{k=1}^{q-1} R_{k}W_{k}\right)D_{21} \\ Q_{2}D_{32} + \left(\sum_{k=1}^{q-1} R_{k}W_{k}\right)D_{22} \\ \vdots \\ Q_{2}D_{3n} + \left(\sum_{k=1}^{q-1} R_{k}W_{k}\right)D_{2n} \end{bmatrix}^{T}$$
(38)
$$Z_{i}V_{i} = \begin{bmatrix} Z_{i1}V_{i} \quad Z_{i2}V_{i} \quad \cdots \quad Z_{in}V_{i} \end{bmatrix}^{T}$$
(39)

$$\begin{bmatrix} V_i(W_iB_{kn} + U_{kni}) \end{bmatrix}$$

$$W_iD_2 = \begin{bmatrix} W_iD_{21} & W_iD_{22} & \cdots & W_iD_{2n} \end{bmatrix}$$
(41)

For the rest of this section we redefine I as an identity matrix of size $dn \times dn$. Using the Corollary 3, from equations (12), (28) and (29), we have, for $1 \le i \le (q-1)$:

$$G\{Q_1(M_1A_i + M_2B_i)\} = 0 \tag{42}$$

Using lemma 2, from equations (13), (14), (28) and (30) we get,

$$G(Q_1 M_2 D_2) = I \tag{43}$$

Using Corollary 3 and equations (15), (31) and (32) we have, for $1 \le i \le (q-1)$:

$$R_i K_i (M_4 M_1 D_1 + M_5 M_3 D_1) = 0 (44)$$

Using Lemma 2 and equations (16), (17), (31) and (33) we have, for $1 \le i \le (q-1)$:

$$R_i K_i \{ M_4 (M_1 A_i + M_2 B_i) + M_5 M_3 A_i \} = I$$
 (45)

Using Corollary 3 on equations (19), (31) and (34) we have, for $1 \le i \le (q-1)$:

$$R_i K_i (M_4 M_2 D_2 + M_5 D_3) = 0 (46)$$

Using Lemma 2 on equations (20), (21), (35) and (36) for we get:

$$LQ_2M_3D_1 = I \tag{47}$$

Using Corollary 3 and equations (22), (35) and (37) we get, for $1 \le i \le (q-1)$:

$$L(Q_2M_3A_i + R_iW_iB_i + \sum_{k=1,k\neq i}^{q-1} R_k(W_kB_i + U_{ik})) = 0 \quad (48)$$

Using Lemma 2, and equations (23), (35) and (38) we have:

$$L(Q_2D_3 + \left(\sum_{k=1}^{q-1} R_k W_k\right) D_2) = 0$$
(49)

Using Corollary 3 and equations (25), (39) and (40) we get, for $1 \le i, k \le (q-1), k \ne i$,

$$Z_i V_i (W_i B_k + U_{ki}) = 0 (50)$$

Using Lemma 2, and equations (26), (27), (39) and (41) we have:

$$Z_i V_i W_i D_2 = I \tag{51}$$

Now, from equation (43) the matrices G, D_2 , M_2 and Q_1 are invertible. From equation (45), for $1 \le i \le (q-1)$, K_i and R_i are invertible. D_1 , L and Q_2 is invertible from equation (47). From equation (51) V_i and Z_i are invertible. From (42) since G and Q_1 both are invertible, we have:

$$M_1 A_i + M_2 B_i = 0 (52)$$

Since bot R_i and K_i are invertible, we have from equation (44):

$$M_4 M_1 D_1 + M_5 M_3 D_1 = 0 (53)$$

And from (46) we have:

$$M_4 M_2 D_2 + M_5 D_3 = 0 \tag{54}$$

Since D_1 is also invertible, from equation (53) we must have,

$$M_4 M_1 + M_5 M_3 = 0 (55)$$

Substituting (52) in (45) we have

$$R_i K_i M_5 M_3 A_i = I \tag{56}$$

Since Z_i and V_i are invertible, from equation (50) we must have, for $1 \le i, k \le (q-1), i \le k$:

$$W_i B_k + U_{ki} = 0 \tag{57}$$

Substituting equation (57) in equation (48), and noting that L is invertible, we have for $1 \le i \le (q-1)$:

$$Q_2 M_3 A_i + R_i W_i B_i = 0 (58)$$

 M_3 and A_i are invertible from (56) for $1 \le i \le (q-1)$, and since Q_2 is invertible, $Q_2M_3A_i$ is invertible, which leads that $R_iW_iB_i$ is invertible from equation (58), and hence B_i is invertible. So from (58), for $1 \le i \le (q-1)$,

$$Q_2 M_3 A_i B_i^{-1} + R_i W_i = 0 (59)$$

Since L is invertible, we have from (49)

$$Q_2 D_3 + \sum_{k=1}^{q-1} R_k W_k D_2 = 0$$
(60)

Since D_2 is invertible, we have:

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} R_k W_k = 0$$
(61)

Substituting $R_k W_k$ for $1 \le k \le (q-1)$, from (59) we have,

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} -Q_2 M_3 A_k B_k^{-1} = 0$$
 (62)

Since M_2 , A_i and B_i are invertible for $1 \le i \le (q-1)$, from (52) M_1 is also invertible, and hence we have, for $1 \le i \le (q-1)$:

$$A_i B_i^{-1} = -M_1^{-1} M_2 \tag{63}$$

Substituting equation (63) in equation (62) we have:

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} -Q_2 M_3 (-M_1^{-1} M_2) = 0$$

Since all constituent matrices are square,

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} Q_2 M_3 M_1^{-1} M_2 = 0$$
 (64)

Since M_5M_3 is invertible from equation (56), and M_1 is invertible as shown above, from equation (55) we have:

$$M_3 M_1^{-1} = -M_5^{-1} M_4 \tag{65}$$

Substituting equation (65) in equation (64) we have:

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} Q_2 (-M_5^{-1} M_4) M_2 = 0$$
 (66)

Now, from equation (54) we have:

$$D_3 D_2^{-1} = -M_5^{-1} M_4 M_2 \tag{67}$$

Substituting this in (66) we have:

$$Q_2 D_3 D_2^{-1} + \sum_{k=1}^{q-1} Q_2 D_3 D_2^{-1} = 0$$

(q) Q_2 D_3 D_2^{-1} = 0

Since Q_2, D_3 and D_2 are all invertible matrices, it must be that q = 0. Now the fact that an element is equal to zero in a finite field if and only if the characteristic divides the element proves that the network has a rate $\frac{1}{n}$ fractional linear network coding solution only if the characteristic of the finite field divides q.

We now show that the network \mathcal{N}_1 has a (1, n) fractional linear network coding solution if q = 0. For this section, let \bar{a}_i denote an *n*-length vector whose i^{th} component is a_i and all other component is zero. Let \bar{c}_i to denote an *n*-length vector whose i^{th} component is c_i and all other component is zero. Also let \bar{b}_{ij} denotes an *n*-length vector which has zero in all of its components but the j^{th} one, which is equal to b_{ij} . Note that a_i, c_i for $1 \le i \le n$ and b_{ij} for $1 \le i \le (q-1), 1 \le j \le n$ are the source processes. Now, it can be seen that by choosing the appropriate local coding matrices, the messages shown below can be transmitted by the corresponding edges.

$$Y_{(u_1,u_3)} = \sum_{i=1}^{n} \bar{a}_i + \sum_{i=1}^{p-1} \sum_{j=1}^{n} \bar{b}_{ij}$$

$$Y_{(u_2,u_4)} = \sum_{i=1}^{q-1} \sum_{j=1}^{n} \bar{b}_{ij} + \sum_{i=1}^{n} \bar{c}_i$$

$$Y_{(u_5,u_7)} = Y_{(u_1,u_3)} - Y_{(u_2,u_4)} = \sum_{i=1}^{n} \bar{a}_i - \sum_{i=1}^{n} \bar{c}_i$$

$$Y_{(u_6,u_8)} = Y_{(u_1,u_3)} - \sum_{i=1}^{n} \bar{c}_i = \sum_{i=1}^{n} \bar{a}_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} \bar{b}_{ij} - \sum_{i=1}^{n} \bar{c}_i$$
for $1 \le i \le q-1$: $Y_{e_i} = \sum_{j=1}^{n} \bar{b}_{ij} + \sum_{i=1}^{n} \bar{c}_i$

$$Y_{(u_9,u_{10})} = Y_{(u_6,u_8)} - Y_{(u_5,u_7)} = \sum_{i=1}^{q-1} \sum_{j=1}^{n} \bar{b}_{ij}$$

$$Y_{(u_{13},u_{14})} = Y_{(u_6,u_8)} - \sum_{i=1}^{q-1} Y_{e_i} = \sum_{i=1}^{n} \bar{a}_i - \sum_{i=1}^{n} \bar{c}_i - \sum_{i=1}^{q-1} \sum_{i=1}^{n} \bar{c}_i$$

$$= \sum_{i=1}^{n} \bar{a}_i - \sum_{i=1}^{q} \sum_{i=1}^{n} \bar{c}_i = \sum_{i=1}^{n} \bar{a}_i - \sum_{i=1}^{n} \bar{a}_i$$

$$Y_{(u_{11},u_{12})} = \sum_{i=1}^{n} \bar{a}_i - Y_{(u_5,u_7)} = \sum_{i=1}^{n} \bar{c}_i$$
for $1 \le i \le q-1$:
$$Y_{(v_i,v_i')} = Y_{(u_9,u_{10})} - \sum_{k=1,k\neq i}^{q-1} \sum_{j=1}^{n} \bar{b}_{ij} = \sum_{j=1}^{n} \bar{b}_{ij}$$

for
$$1 \le i \le q-1$$
: $Y_{(w_i,w_i')} = Y_{e_i} - \sum_{j=1}^n \bar{b}_{ij} = \sum_{i=1}^n \bar{c}_i$

Let $\check{u}(i)$ be a unit row vector of length n which has i^{th} component equal to one and all other component equal to zero. Then from the vector $\sum_{i=1}^{n} \bar{a}_i$, a_i for any $1 \le i \le n$ can be determined by the dot product $\check{u}(i) \cdot (\sum_{i=1}^{n} \bar{a}_i)$. Similarly for any $1 \le i \le (q-1)$, $b_{ij} = \check{u}(j) \cdot (\sum_{i=1}^{n} \bar{b}_{ij})$. For $1 \le i \le n$, c_i can be determined similarly from $\sum_{i=1}^{n} \bar{c}_i$.

Theorem 4. For any non-zero positive rational number $\frac{k}{n}$ and for any finite set of prime numbers $\{p_1, p_2, \ldots, p_l\}$ there exists a network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belong to the given set of primes.

Proof: Let us consider the union of k copies of the network \mathcal{N}_1 shown in Fig. 1 each for $q = p_1 \times p_2 \times \cdots \times p_l$. Denote the i^{th} copy as \mathcal{N}_{1i} . Note that each source and each terminal has k copies in the union. Join all copies of any source or terminal into a single source or terminal respectively. Name this new network as \mathcal{N}'_1 . We show below that \mathcal{N}'_1 has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belong to the set $\{p_1, p_2, \ldots, p_l\}$. Before we proceed further, consider the following property of \mathcal{N}_1 and \mathcal{N}'_1 .

Lemma 5. If \mathcal{N}'_1 has a (dk, dn) fractional linear network coding solution for any non-zero positive integer d, then \mathcal{N}_1 has a (dk, dkn) fractional linear network coding solution.

Proof: This is true since the information that can be sent using the network \mathcal{N}'_1 for x times, can be sent using the network \mathcal{N}_1 kx times. This is because \mathcal{N}'_1 has k copies of \mathcal{N}_1 .

First consider the only if part. Say \mathcal{N}'_1 has a rate $\frac{k}{n}$ fractional linear network coding solution even if the characteristic does not belong to the set $\{p_1, p_2, \ldots, p_l\}$. Then from Lemma 5, the network \mathcal{N}_1 has a rate $\frac{1}{n}$ fractional linear network coding solution even if the characteristic does not belong to the given set of primes. However, as shown in Lemma 1, the \mathcal{N}_1 has a rate $\frac{1}{n}$ fractional linear network coding solution if and only if q = 0 over the finite field. But, as $q = p_1 \times p_2 \times \cdots \times p_l$, q = 0 if and only if one of the prime number from the set $\{p_1, p_2, \ldots, p_l\}$ is zero over the finite field. The latter is the case if and only if the characteristic of the finite field is one of the primes in the set. Hence this is a contradiction to the case that \mathcal{N}_1 has a rate $\frac{1}{n}$ fractional linear network coding solution even if the characteristic does not belong to the given set of primes.

Now consider the if part. Since \mathcal{N}_{1i} for $1 \leq i \leq k$ has a (1, n) fractional linear network coding solution, a (k, n) fractional linear network coding solution can be constructed by keeping the same local coding matrices in all of the copies and sending the *i*th component of each source through \mathcal{N}_{1i} .

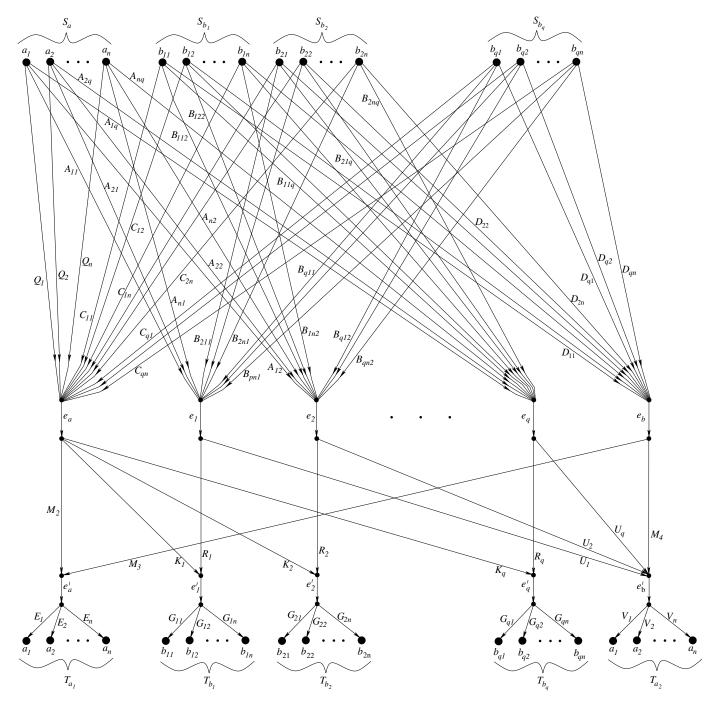


Fig. 2. A network N_2 which has a rate 1/n fractional linear network coding solution if and only if the characteristic of the finite field does not divide q.

B. Network having $\frac{k}{n}$ solution iff characteristic belongs to a given co-finite set of primes.

The outline of the contents in this section is similar to that of the last sub-section. Consider the network \mathcal{N}_2 shown in Fig. 2. The sources are partitioned into q+1 sets: S_a and S_{b_i} for $1 \le i \le q$. Each of these sets has n sources. In Fig. 2 the sources are indicated by the message symbol it generates. A source $s_i \in S_a$ generates the message a_i . Similarly a source $s_j \in S_{b_i}$ generates the message b_{ij} for $1 \le i \le q, 1 \le j \le n$. The set of terminals are partitioned into q + 2 disjoint sets

namely, T_{a_1}, T_{a_2} and T_{b_i} for $1 \le i \le q$. The sets T_{a_1} and T_{a_2} has n terminals in each. Each individual terminal is indicated by the source message it demands. We have the following edges in the network.

- 1) e_a, e_b, e'_a and e'_b 2) e_i and e'_i for $1 \le i \le q$
- 3) $(s, tail(e_a))$ for $\forall s \in S_a \cup \{\bigcup_{i=1}^q S_{b_i}\}$ 4) $(s, tail(e_i))$ for $1 \le i \le q$ and $\forall s \in \{S_a, \bigcup_{j=1, j \ne i}^q S_{b_i}\}$ 5) $(s, tail(e_b))$ for $\forall s \in \bigcup_{i=1}^q S_{b_i}$ 6) $(head(e_a), tail(e'_a))$ and $(head(e_b), tail(e'_a))$

- 7) $(head(e_b), tail(e'_b))$
- 8) $(head(e_i), tail(e'_i))$ for $1 \le i \le q$
- 9) $(head(e_a), tail(e'_i))$ for $1 \le i \le q$
- 10) $(head(e_i), tail(e'_b))$ for $1 \le i \le q$

From each of the nodes $head(e'_a)$, $head(e'_i)$ for $1 \le i \le q$ and $head(e_b)'$, n outgoing edge emanates and the *head* node of all such edges is a terminal. The set of n terminals which have a path from node $head(e'_a)$ are denoted by T_{a_1} . Similarly, the set of n terminals which have a path from node $head(e'_b)$ are denoted by T_{a_2} . And the n terminals in the set T_{b_i} for $1 \le i \le q$ are connected from the node $head(e'_i)$ by an edge.

Before we show that for any non-zero positive rational number $\frac{k}{n}$ and for a given co-finite set of primes there exist a network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belong to the given set, consider the following lemma.

Lemma 6. The network shown in Fig. 2 has a rate $\frac{1}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field does not divides q.

Proof: Consider a (d, dn) fractional linear network coding solution of the network. The local coding matrices are shown next to the edges. The matrices Q_i for $1 \le i \le n$ and A_{ij} for $1 \le i \le n, 1 \le j \le q$ are of size $dn \times d$ and left multiplies the massage a_i . The matrices C_{ij} for $1 \le i \le q, 1 \le j \le n$, B_{ijk} for $1 \le i, k \le q, i \ne k, 1 \le j \le n$, and D_{ij} for $1 \le i \le q$, $1 \le j \le n$ left multiplies b_{ij} and are of size $dn \times d$. The matrices M_2, M_3, M_4 and K_i, R_i and U_i for $1 \le i \le q$ are of sizes $dn \times dn$. And the matrices of size $d \times dn$ are E_j, G_{ij} and V_j for $1 \le i \le q$ and $1 \le j \le n$. Also let I be a $d \times d$ identity matrix. The following is the list of messages carried by some of the edges of the network.

$$Y_{e_a} = \sum_{i=1}^{n} Q_i a_i + \sum_{i=1}^{q} \sum_{j=1}^{n} C_{ij} b_{ij}$$
(68)

for
$$1 \leq i \leq q$$

:

$$Y_{e_i} = \sum_{j=1}^{n} A_{ji} a_j + \sum_{j=1, j \neq i}^{q} \sum_{k=1}^{n} B_{jki} b_{jk}$$
(69)

$$Y_{e_b} = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} b_{ij}$$
(70)

$$Y_{e'_a} = M_2 Y_{e_a} + M_3 Y_{e_b} = \sum_{i=1}^n M_2 Q_i a_i + \sum_{i=1}^q \sum_{j=1}^n (M_2 C_{ij} + M_3 D_{ij}) b_{ij}$$
(71)
for $1 \le i \le q$:

$$Y_{e'_{i}} = K_{i}Y_{e_{a}} + R_{i}Y_{e_{i}} = \sum_{j=1}^{n} (K_{i}Q_{j} + R_{i}A_{ji})a_{j}$$
$$+ \sum_{k=1}^{n} K_{i}C_{ik}b_{ik} + \sum_{j=1, j\neq i}^{q} \sum_{k=1}^{n} (K_{i}C_{jk} + R_{i}B_{jki})b_{jk}$$
(72)

$$Y_{e'_{b}} = \sum_{i=1}^{q} U_{i}Y_{e_{i}} + M_{4}Y_{e_{b}} = \sum_{i=1}^{q} \sum_{j=1}^{n} U_{i}A_{ji}a_{j}$$
$$+ \sum_{j=1}^{q} \sum_{k=1}^{n} \left(\sum_{i=1, i \neq j}^{q} U_{i}B_{jki} + M_{4}D_{jk} \right) b_{jk}$$
(73)

Because of the demands of the terminals the following inequalities must be satisfied. Since any terminal $t_i \in T_{a_1}$ computes a_i , we have, for $1 \le i, j \le n, j \ne i$:

$$E_i M_2 Q_i = I \tag{74}$$

$$E_i M_2 Q_j = 0 \tag{75}$$

At $t_k \in T_a$ the component of b_{ij} is zero for $1 \le i \le q, 1 \le j, k \le n$ we have:

$$E_k(M_2C_{ij} + M_3D_{ij}) = 0 (76)$$

Now consider the terminals in the set T_{b_i} for $1 \le i \le q$. Since at any terminal $t_j \in T_{b_i}$ for $1 \le j \le n$ the component of a_k for $1 \le k \le n$ is zero, we have,

$$G_{ij}(K_iQ_k + R_iA_{ki}) = 0 \tag{77}$$

Because $t_j \in T_{b_i}$ computes b_{ij} for $1 \le i \le q, 1 \le j, k \le n, k \ne j$ we have:

$$G_{ij}(K_i C_{ij}) = I \tag{78}$$

$$G_{ij}(K_i C_{ik}) = 0 (79)$$

As the component of any b_{kr} at $t_j \in T_{b_i}$ is zero if $k \neq i$, we have for $1 \leq i, k \leq q, i \neq k, 1 \leq j, r \leq n$ we have:

$$G_{ij}(K_iC_{kr} + R_iB_{kri}) = 0 \tag{80}$$

We now consider the set T_{a_2} . Since the terminal $t_i \in T_{a_2}$ computes a_i we have, for $1 \le i, j \le n, j \ne i$

$$V_i(\sum_{k=1}^q U_k A_{ik}) = I \tag{81}$$

$$V_i(\sum_{k=1}^{q} U_k A_{jk}) = 0$$
(82)

The component of b_{jk} is zero at $t_i \in T_{a_2}$ for $1 \le j \le q, 1 \le i, k \le n$, and hence we have,

$$V_i \left(\left(\sum_{r=1, r \neq j}^{q} U_r B_{jkr} \right) + M_4 D_{jk} \right) = 0$$
 (83)

We now define the following matrices.

$$E = \begin{bmatrix} E_1 & E_2 & \cdots & E_n \end{bmatrix}^T$$

$$M_2 Q = \begin{bmatrix} M_2 Q_1 & M_2 Q_2 & \cdots & M_2 Q_n \end{bmatrix}$$
(84)
(85)

$$M_{2}C_{i} + M_{3}D_{i} = \begin{bmatrix} M_{2}C_{i1} + M_{3}D_{i1} \\ M_{2}C_{i2} + M_{3}D_{i2} \\ \vdots \\ M_{2}C_{i2} + M_{2}D_{i2} \end{bmatrix}^{T}$$
(86)

$$G_i = \begin{bmatrix} G_{i1} & G_{i2} & \cdots & G_{in} \end{bmatrix}^T$$

$$(87)$$

$$K_{i}Q + R_{i}A_{i} = \begin{vmatrix} K_{i}Q_{1} + R_{i}A_{1i} \\ K_{i}Q_{2} + R_{i}A_{2i} \\ \vdots \\ K_{i}Q_{n} + R_{i}A_{ni} \end{vmatrix}^{-1}$$
(88)

$$K_i C_i = \begin{bmatrix} K_i C_{i1} & K_i C_{i2} & \cdots & K_i C_{in} \end{bmatrix}$$

$$K_i C_{k1} + R_i B_{k1i} \begin{bmatrix} K_i C_{k1} + R_i B_{k1i} \\ K_i C_{k2} + R_i B_{k2i} \end{bmatrix}^T$$

$$(89)$$

$$K_i C_k + R_i B_{ki} = \begin{bmatrix} \vdots \\ K_i C_{kn} + R_i B_{kni} \end{bmatrix}$$
(90)

$$V = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}^T$$

$$\begin{bmatrix} \sum_{i=1}^{q} & U_i & A_i \end{bmatrix}^T$$
(91)

$$\sum_{k=1}^{q} U_k A_k = \begin{bmatrix} \sum_{k=1}^{q} U_k A_{1k} \\ \sum_{k=1}^{q} U_k A_{2k} \\ \vdots \\ \sum_{k=1}^{q} U_k A_{nk} \end{bmatrix}^{q}$$
(92)

$$(\sum_{r=1,r\neq j}^{q} U_{r}B_{jr}) + M_{4}D_{j} = \begin{bmatrix} (\sum_{r=1,r\neq j}^{q} U_{r}B_{j1r}) + M_{4}D_{j1} \\ (\sum_{r=1,r\neq j}^{q} U_{r}B_{j2r}) + M_{4}D_{j2} \\ \vdots \\ (\sum_{r=1,r\neq j}^{q} U_{r}B_{jnr}) + M_{4}D_{jn} \end{bmatrix}^{r}$$
(93)

Note that all of these newly defined matrices are square and are of size $dn \times dn$. For the rest of the paper let *I* denote an identity matrix of size $dn \times dn$. Applying Lemma 2 on equations (74) and (75) and using the newly defined matrices in equation (84) and (85) we get:

$$EM_2Q = I \tag{94}$$

From Corollary 3 and equations (76), (84) and (86) we get, for $1 \le i \le q$:

$$E(M_2C_i + M_3D_i) = 0 (95)$$

Similarly using Corollary 3 and equations (77), (87) and (88) we get, for $1 \le i \le q$:

$$G_i(K_iQ + R_iA_i) = 0 (96)$$

Using Lemma 2 and equations (78), (79), (87) and (89) we get, for $1 \le i \le q$:

$$G_i(K_iC_i) = I \tag{97}$$

From Corollary 3 and equations (80), (87) and (90) we have, for $1 \le i, k \le n, i \ne k$:

$$G_i(K_iC_k + R_iB_{ki}) = 0 \tag{98}$$

Employing Lemma 2 and equations (81), (82), (91) and (92) we have: a

$$V(\sum_{k=1}^{q} U_k A_k) = I \tag{99}$$

From Corollary 3 and equations (83), (91) and (93) we have, for $1 \le j \le q$:

$$V\left(\left(\sum_{r=1, r\neq j}^{q} U_r B_{jr}\right) + M_4 D_j\right) = 0$$
 (100)

The matrices E, M_2 and Q are invertible from equation (94). Matrix G_i, K_i and C_i for $1 \le i \le q$ is invertible from equation (97). And V is invertible from equation (99). Since E is invertible we have from equation (95):

$$M_2 C_i + M_3 D_i = 0 \tag{101}$$

As both M_2 and C_i are invertible matrices, their product is a full rank matrix, and hence from equation (101), M_3 is an invertible matrix. This comes from the fact that a matrix of rank equal to a certain value cannot multiply with any other matrices and result in a matrix of rank equal to a greater value. Since G_i is invertible for $1 \le i \le q$, we have from equation (96):

$$K_i Q + R_i A_i = 0 \tag{102}$$

Since both K_i and Q are invertible matrices, their product is a full rank matrix, and hence R_i is an invertible matrix for $1 \le i \le q$. Also from equation (98) we have, for $1 \le i, k \le q, i \ne k$:

$$K_i C_k + R_i B_{ki} = 0 \tag{103}$$

And since V is invertible we have from equation (100), for $1 \le i \le q$:

$$\left(\sum_{r=1, r\neq i}^{P} U_r B_{ir}\right) + M_4 D_i = 0 \tag{104}$$

Substituting D_i from equation (101) in equation (104) we get, for $1 \le i \le q$:

$$\left(\sum_{r=1,r\neq i}^{q} U_r B_{ir}\right) - M_4 M_3^{-1} M_2 C_i = 0$$

Substituting B_{ir} from equation (103) we get :

$$-\left(\sum_{r=1,r\neq i}^{q} U_r R_r^{-1} K_r C_i\right) - M_4 M_3^{-1} M_2 C_i = 0$$

Substituting $R_r^{-1}K_r$ from equation (102)weget :

$$\left(\sum_{r=1,r\neq i}^{q} U_r A_r Q^{-1} C_i\right) - M_4 M_3^{-1} M_2 C_i = 0$$

Substituting Q^{-1} from equation (94) we get:

$$\left(\sum_{r=1,r\neq i}^{i} U_r A_r E M_2 C_i\right) - M_4 M_3^{-1} M_2 C_i = 0$$

or,
$$\left(\left(\sum_{r=1,r\neq i}^{q} U_r A_r E\right) - M_4 M_3^{-1}\right) M_2 C_i = 0$$

Since M_2 and C_i both are invertible :

$$\left(\sum_{r=1, r\neq i}^{q} U_r A_r E\right) - M_4 M_3^{-1} = 0$$

Substituting $\sum_{r=1,r\neq i}^{q} U_r A_r$ from equation (99) we get : $(V^{-1} - U_i A_i) E - M_4 M_3^{-1} = 0$

or,
$$V^{-1} - U_i A_i = M_4 M_3^{-1} E^{-1}$$

or, $U_i A_i = V^{-1} - M_4 M_3^{-1} E^{-1}$ (105)

Now, substituting equation (105) in equation (99) we get

$$V(\sum_{i=1}^{q} V^{-1} - M_4 M_3^{-1} E^{-1}) = I$$

$$\sum_{i=1}^{q} V(V^{-1} - M_4 M_3^{-1} E^{-1}) = I$$

$$\sum_{i=1}^{q} I - V M_4 M_3^{-1} E^{-1} = I$$

$$(q-1)I = qIV M_4 M_3^{-1} E^{-1}$$
(106)

In equation (106), if q = 0, then the equation becomes -I = 0. Hence $q \neq 0$ is a necessary condition for the network \mathcal{N}_2 to have a rate $\frac{1}{n}$ fractional linear network coding solution. Then from the fact that an element in a finite field is equal to zero if and only if the characteristic of the finite field divides that element, it can be concluded that $q \neq 0$ if and only if the characteristic of the finite field and only if the characteristic of the finite field divides that

We now show that \mathcal{N}_2 has a (1, n) fractional linear network coding solution if the q has an inverse in the finite field. Note that, as discussed above, q has an inverse if and only if the characteristic of the finite field does not divides q. Let an n-length vector whose i^{th} component is a_i and all other components are zero be denoted by \bar{a}_i . Also let n-length vector whose j^{th} component is b_{ij} and all other components are zero be denoted by the notation \bar{b}_{ij} . Note that a_j and b_{ij} for $1 \le i \le q, 1 \le j \le n$ are the source processes. Then it can be seen that for proper local coding matrices the following information can be transmitted by the corresponding edges.

$$\begin{split} Y_{e_a} &= \sum_{j=1}^n \bar{a}_j + \sum_{i=1}^q \sum_{j=1}^n \bar{b}_{ij} \\ \text{for } 1 \leq i \leq q : \quad Y_{e_i} = \sum_{j=1}^n \bar{a}_j + \sum_{k=1, k \neq i}^q \sum_{j=1}^n \bar{b}_{kj} \\ Y_{e_b} &= \sum_{i=1}^q \sum_{j=1}^n \bar{b}_{ij} \\ Y_{e'_a} &= Y_{e_a} - Y_{e_b} = \sum_{j=1}^n \bar{a}_j \\ \text{for } 1 \leq i \leq q : \quad Y_{e'_i} = Y_{e_a} - Y_{e_i} = \sum_{j=1}^n \bar{b}_{ij} \\ Y_{e'_b} &= q^{-1} \{ \sum_{i=1}^q Y_{e_i} - (q-1)Y_{e_b} \} = q^{-1} \{ q(\sum_{j=1}^n \bar{a}_j) \} \end{split}$$

$$+ (q-1)\left(\sum_{i=1}^{q}\sum_{j=1}^{n}\bar{b}_{ij}\right) - (q-1)\left(\sum_{i=1}^{q}\sum_{j=1}^{n}\bar{b}_{ij}\right)\right\} = \sum_{j=1}^{n}\bar{a}_{j}$$

Let $\check{u}(j)$ be a unit row vector of length n which has j^{th} component equal to one and all other component equal to zero. Then from the dot product of $\check{u}(j)$ and $\sum_{j=1}^{n} \bar{a}_{j}$, message a_{j} can be retrieved. Similarly from the dot product of $\check{u}(j)$ and $\sum_{j=1}^{n} \bar{b}_{ij}$, b_{ij} can be determined.

Theorem 7. For any non-zero positive rational number $\frac{k}{n}$ and for any finite set of prime numbers $\{p_1, p_2, \ldots, p_l\}$ there exists a network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field does not belong to the given set of primes.

Proof: Let q be equal to $p_1.p_2....p_l$ in \mathcal{N}_2 . Let us construct \mathcal{N}'_2 by joining n copies of \mathcal{N}_2 at the corresponding sources and the terminals, in a similar way \mathcal{N}'_1 was constructed from \mathcal{N}_1 . It can be also seen that Lemma 5 holds true when \mathcal{N}_1 and \mathcal{N}'_1 are replaced by \mathcal{N}_2 and \mathcal{N}'_2 respectively. So if \mathcal{N}'_2 has a (k, n) fractional linear network coding solution then \mathcal{N}_2 has a (k, kn) fractional linear network coding solution.

Now say \mathcal{N}'_2 has a rate $\frac{k}{n}$ fractional linear network coding solution even if the characteristic of the finite belongs to the set $\{p_1, p_2, \ldots, p_l\}$. Since $q = p_1.p_2.\ldots.p_l$, note that q = 0 over such a field. Then, \mathcal{N}_2 has a rate $\frac{k}{kn} = \frac{1}{n}$ fractional linear network coding solution over a finite field in which q = 0. However, this is in contradiction with Lemma 6.

If however, the characteristic does not belong to the given set of primes, then, since there are *n* copies of \mathcal{N}_2 in \mathcal{N}'_2 , and each copy has a (1, n) fractional linear network coding solution, a (k, n) fractional linear network coding solution can easily be constructed for \mathcal{N}'_2 .

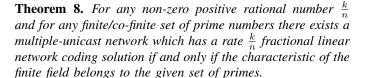
IV. A multiple-unicast network having a rate $\frac{k}{n}$ fractional linear network coding solution iff the characteristic belongs to a given finite/co-finite set of primes

In this section we show that for any non-zero positive rational number $\frac{k}{n}$ and for any finite/co-finite set of primes, there exits a multiple-unicast network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. To prove this result, we first show that for each of the networks \mathcal{N}_1 and \mathcal{N}_2 presented in Section III, there exists a multiple-unicast network which has a (1, n) fractional linear network coding solution if and only if the the corresponding network \mathcal{N}_1 or \mathcal{N}_2 has a (1, n) fractional linear network coding solution.

In a multiple-unicast network, by definition, each source process is generated at only one source node and is demanded by only one terminal. Additionally, each source node generates only one source process, and each terminal demands only one source process. In both the networks \mathcal{N}_1 and \mathcal{N}_2 there exists no source processes which is generated by more than one source node, or no source node generates more than one source process. Moreover, there does not exist any terminal which demands more than one source process. However, there exists more than one terminal which demands the same source process. This is fixed in the following way.

In [14] it was shown that for any network there exists a solvably equivalent multiple-unicast network. To resolve the case of more than one terminals demanding the same source message, the authors considered two such terminals at a time and added a gadget to the two terminals. The same procedure is followed here, only the gadget has been modified. This modified gadget is shown in Fig. 3. It is assumed that the nodes n_1 and n_2 both demanded the same message b in the original network (network before attaching the gadget). After adding the gadget, the modified network has n more source nodes $x_1, s_1, \ldots, s_{n-1}$, and n+1 new terminal nodes $x_4, x_5, t_1, \ldots, t_{n-1}$. Nodes n_1 and n_2 becomes intermediate nodes in the modified construction. This process has to be repeated iteratively for every two terminals in the original network that demand the same source process. In the same way as shown in *Theorem II.1* of [14], it can be shown that after the completion of this process, the resulting network has a (1, n) fractional linear network coding solution if and only if the original network has a (1, n) fractional linear network coding solution.

Hence, as shown above, corresponding to each of the networks \mathcal{N}_1 and \mathcal{N}_2 , there exist multiple-unicast networks \mathcal{N}_1^m and \mathcal{N}_2^m which have a (1,n) fractional linear network coding solution if and only if \mathcal{N}_1 and \mathcal{N}_2 have a (1,n) fractional linear network coding solution respectively. Now by connecting k copies of \mathcal{N}_1^m and \mathcal{N}_2^m in the same way as \mathcal{N}_1' and \mathcal{N}_2' was constructed from \mathcal{N}_1 and \mathcal{N}_2 respectively, the following theorem can be proved in a similar way to Theorem 4 and Theorem 7.



V. CONCLUSION

In this paper we have shown that for any non-zero positive rational number $\frac{k}{n}$ and any finite/co-finite set of prime numbers there exists a multiple unicast network which has a rate $\frac{k}{n}$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. To prove the existence, we have explicitly presented networks having desired properties. The generalized Fano and generalized non-Fano networks presented in [15] are special cases of the networks presented in this paper.

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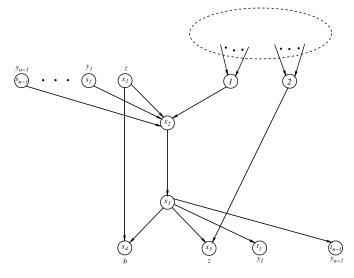


Fig. 3. Gadget which attaches to the two terminals of any arbitrary network indicated by the dotted lines. Nodes s_1, \ldots, s_{n-1} are sources and source s_i generates the messages y_i for $1 \le i \le (n-1)$. The nodes t_1, \ldots, t_{n-1} are terminals and t_i demands y_i for $1 \le i \le (n-1)$. The nodes n_1 and n_2 were terminals in the original network (network indicated by the dotted line) and both of these nodes demanded the message b. The rest of the naming convention has been kept the same as it was in [14].

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