Algorithms for Mapping Parallel Processes onto Grid and Torus Architectures

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Abstract. Static mapping is the assignment of parallel processes to the processing elements (PEs) of a parallel system, where the assignment does not change during the application's lifetime. In our scenario we model an application's computations and their dependencies by an application graph. This graph is first partitioned into (nearly) equally sized blocks. These blocks need to communicate at block boundaries. To assign the processes to PEs, our goal is to compute a communication-efficient bijective mapping between the blocks and the PEs.

This approach of partitioning followed by bijective mapping has many degrees of freedom. Thus, users and developers of parallel applications need to know more about which choices work for which application graphs and which parallel architectures. To this end, we not only develop new mapping algorithms (derived from known greedy methods). We also perform extensive experiments involving different classes of application graphs (meshes and complex networks), architectures of parallel computers (grids and tori), as well as different partitioners and mapping algorithms. Surprisingly, the quality of the partitions, unless very poor, has little influence on the quality of the mapping.

More importantly, one of our new mapping algorithms always yields the best results in terms of the quality measure maximum congestion when the application graphs are complex networks. In case of meshes as application graphs, this mapping algorithm always leads in terms of maximum congestion *and* maximum dilation, another common quality measure.

1 Introduction

Symmetric dependencies of computations within a parallel application can be modeled by an undirected graph G_a , called *application graph*, e.g. the mesh of a numerical simulation. Iterative algorithms in such a simulation act upon the vertices of G_a , and for each such vertex require the values of the neighboring vertices from the previous iteration. Thus, a vertex of G_a represents some computation, and an edge of G_a indicates a dependency between computations, i.e. an exchange of data. It is important to note that this modeling is not restricted to simulations at all. In fact, the nodes of G_a could represent arbitrary parallel processes and the edges symmetric communication requirements between the processes.

Typically, running an application on computers with distributed parallelism requires the application graph to be spread over the computer's processing elements. One way to carry out this task, called *static* mapping, is to (i) partition the application graph G_a into blocks of equal size (or of equal weight in case the computational requirements at the nodes are not homogeneous) for load balancing purposes and (ii) map the blocks of G_a onto the processing elements (PEs) of a parallel computer, see Figure 1. Mapping may involve the communication graph G_c , whose vertices represent the blocks of G_a 's partition and whose edges indicate block neighborhood and therefore communication between different PEs.

The parallel computer is often represented as a graph G_p , called *processor graph* (or topology graph), the vertices of which represent the PEs, and the edges of which represent physical communication links between the PEs. We require that G_c has the same number of vertices as G_p and make the assumption that G_p is sparse, which is true for many real architectures today [1]. In this paper we address the problem of finding a bijective mapping Π of G_c 's vertex set onto G_p 's vertex set (processors) that is communication-efficient. We refer to Π as *bijective topology mapping* or simply *mapping*. One can also see the problem as embedding the guest graph G_c into the host graph G_p .



Fig. 1. (a) Application graph G_a with 4-way partition indicated by colors. (b) Communication graph G_c induced by G_a and the partition. G_c expresses the neighborhood relations of G_a 's blocks. Edge weights (shown through width) indicate communication volumes between blocks. (c) Processor graph G_p . Nodes and edges represent the PEs and the communication links, respectively. Communication between the green and the red block in G_c , i.e. via e_c , requires two hops in G_p .

Motivation. Communication costs are crucial for the scalability of many parallel applications. Static mapping, in turn, is crucial when it comes to keeping communication costs under control through (i) providing a partitioning with few edges between blocks and (ii) mapping nearby blocks onto nearby PEs: due to the sparse nature of many large-scale parallel computers, communication costs may vary by several orders of magnitude depending on the distance between the PEs involved [2]. Also, numerous recent applications involve massive *complex networks* such as social networks or web graphs [3]. These networks usually lead to denser communication graphs and make improved mapping strategies even more desirable.

Contribution. We investigate numerous algorithms for static mapping, the scenario being that an application graph is first partitioned into blocks, followed by a bijective mapping of the blocks onto the nodes of a processor graph. The graph partitioners we employ are the state-of-the-art packages METIS [4] and KAHIP [5]. While METIS is widely used for graph partitioning and has been employed for mapping before, it is the first time that the high-quality partitioner KAHIP is used in the mapping context.

To assess and improve the performance of mapping algorithms, we implement several state-of-the-art methods. Moreover and more importantly, we develop and implement two new algorithms as straightforward, yet very effective adaptations of existing greedy algorithms.

The three most striking results of our extensive mapping experiments on meshes and complex networks as application graphs, as well as grids and tori as processor graphs, are: First, the strengths and weaknesses of the mapping algorithms are, to a large extent, independent of the class of application graphs (mesh or complex network) and the processor graphs. Second, the graph partitioner and its partitioning quality is of minor importance for the quality of the mapping. Third, for complex networks as application graphs, one of our new mapping algorithms always yields the best quality in terms maximum congestion. In case of meshes, this mapping algorithm always leads in terms of maximum congestion *and* maximum dilation.

2 Preliminaries

2.1 Problem Description

We represent the communication of a parallel application as a graph $G_c = (V_c, E_c, \omega_c)$, where a weight $\omega_c(\{u, v\}), \{u, v\} \in E_c$, indicates the volume of communication between u and v, i.e. between the corresponding blocks of the application graph.

The parallel computer takes the form of a graph $G_p = (V_p, E_p, \omega_p)$, the processor graph. Here, $\omega_p : E_p \mapsto \mathbb{N}$ indicates the bandwidths of the physical communication links. We require $|V_p| = |V_c|$.

Our aim is to find a bijective topology mapping (short mapping) $\Pi : V_c \mapsto V_p$ that minimizes the overhead due to communication between the processes. A first graph-theoretic definition of the overhead (costs) was given in [6]. In the following we present three aspects of overhead (for more in-depth definitions see [7]).

An edge $e_c = \{u_c, v_c\}$ of G_c gives rise to communication between $\Pi(u_c)$ and $\Pi(v_c)$ on G_p . Sending a unit of information along a path P in G_p with edges $e_1, \ldots e_l$ takes time at least $t(P) = \sum_{i=1}^l (1/\omega_p(e_i))$. Sending all information via an edge $e_c = \{u_c, v_c\} \in E_c$, i.e. from processes in u_c to processes in v_c , then takes time at least

$$d(e_c) = d(e_c, \Pi) = \omega_c(u_c, v_c) \ t(\Pi(u_c), \Pi(v_c)), \text{ where}$$

$$t(u_p, v_p) = \min(t(P) \mid P \text{ connects } u_p \text{ and } v_p)$$
(1)

Thus, maximum and average dilation, defined as

$$mD(\Pi) = \max_{e_c \in E_c} d(e_c) \text{ and}$$
(2)

$$aD(\Pi) = (\sum_{e_c \in E_c} d(e_c))/|E_c|$$
, (3)

respectively, provide lower bounds for the communication time of a parallel application, $mD(\Pi)$ being the tighter lower bound.

When multiple messages are exchanged at the same time, more than one of them may be routed via the same edge. Hence, if c(e) denotes the total volume of communication routed via $e \in E_p$, divided by the bandwidth $\omega(e)$, then the maximum (weighted) congestion

$$mC(\Pi) := \max_{e_c \in E_c} c(e_c) \tag{4}$$

provides another lower bound for the time. Minimizing $mD(\Pi)$, $aD(\Pi)$ and $mC(\Pi)$ is NP-hard, cf. Garey and Johnson [8] and more recent work [7,9]. Due to the problem's complexity, exact mapping methods are only practical in special cases. Leighton's book [10] discusses embeddings between arrays, trees, and hypercubes.

As in previous studies [7], we assume that the routing algorithm sends the messages on uniformly distributed shortest paths in G_p . In particular, the routing algorithm is oblivious to the utilization of the parallel system.

2.2 Graph partitioning

Given a graph G = (V, E) and a number of blocks k > 0, the graph partitioning problem asks for a division of V into k pairwise disjoint subsets V_1, \ldots, V_k (blocks) such that no block is larger than $(1 + \varepsilon) \cdot \left\lceil \frac{|V|}{k} \right\rceil$, where $\varepsilon \ge 0$ is the allowed imbalance. The most widely used objective function is the *edge cut* (whose minimization is \mathcal{NP} -hard [8]), i.e., the total weight of the edges between different blocks. Yet, a more important factor for modeling the communication cost of parallel iterative graph algorithms seems to be the *maximum communication volume* (MCV) [11], which has received growing attention recently, e.g. in the 10th DIMACS Implementation Challenge on graph partitioning. MCV considers the worst communication volume taken over all blocks V_p $(1 \le p \le k)$ and thus penalizes imbalanced communication: $MCV(V_1, \ldots, V_k) :=$ $\max_p \sum_{v \in V_p} |\{V_i \mid \exists \{u, v\} \in E \text{ with } u \in V_i \neq V_p\}|$.

3 Related Work

In this section we give a brief overview of algorithms for static mapping. More on topology mapping can be found in [12, 13] and particularly in Pellegrini's survey [14].

It should be mentioned that partitioning and mapping can be done simultaneously, i.e. communication between PEs is taken into account already during partitioning [15–17]. In this paper, however, we focus on the complementary approach where partitioning and topology mapping form different stages of a software pipeline.

One can apply a wide range of optimization techniques to the topology mapping problem. Hoefler and Snir [7] employ (among others) the Reverse Cuthill-McKee (RCM) algorithm, originally devised for minimizing the bandwidth of a sparse matrix [18]. If both G_c and G_p are sparse, the simultaneous optimization of both graph layouts can lead to good mapping results [19].

A common approach to static mapping, i.e., partitioning and topology mapping combined, is to recursively partition G_a and G_p in the same fashion, i.e. such that the number of blocks and sub-blocks per block is equal on each level [20]. Such a hierarchical approach to mapping may take into account the actual hierarchy of a heterogeneous multi-core cluster [21]. Typically, the number of sub-blocks per block is small. Thus, on the scope of an individual block, an optimal mapping of a block's sub-blocks can be found by evaluating all possibilities. If the number of sub-blocks is two, the method is called *dual recursive bisection*. It has been shown effective in the software SCOTCH [19]. While an optimal mapping of a block's sub-blocks on the scope of an individual block is not an issue in dual recursive bisection, neighboring relations between sub-blocks of different blocks still pose a challenge. In this paper we apply dual recursive bisection to the pair (G_c, G_p) instead of (G_a, G_p) . This (basic) form of dual recursive bisection does not take into account neighboring relations between the sub-blocks of different blocks (as in [7]).

Greedy approaches such as the ones by Hoefler and Snir [7] and Brandfass *et al.* [22] build on the idea of increasing a mapping by successively adding new maps $v_c \rightarrow v_p$ such that (i) v_c has maximal communication volume with one or all of the already mapped vertices of G_c and (ii) v_p has minimal distance to one or all of the already mapped vertices of G_p . For more details see Sections 4.1.

Hoefler and Snir [7] compare RCM, DRB and a greedy approach experimentally on abstractions of three real architectures. While their results do not show a clear winner, they confirm previous studies [14] in that performing mapping at all is worthwhile. It is important to note, however, that Hoefler and Snir perform mapping from reordered matrices, not from partitioned graphs as we do here.

Many metaheuristics have been used to solve the mapping problem. Uçar *et al.* [23] implement a large variety of methods within a clustering approach, among them genetic algorithms, simulated annealing, tabu search, and particle swarm optimization. The authors require, however, that the processor graph is homogeneous, i. e. $t(u_p, v_p)$ depends only on whether $u_p = v_p$ or not. Our approach is more general than theirs in that we allow $t(u_p, v_p)$ to take different values for $u_p \neq v_p$ (see Equation 1).

Bhatele *et al.* [24] discuss topology-aware mappings of different MPI communication patterns on emerging architectures. Better mappings avoid communication hot spots and reduce communication times significantly. Geometric information can also be helpful for finding good mappings on regular architectures such as tori [25].

4 Methods for Topology Mapping

The simplest topology mapping is the identity, i.e. when block i of the application graph (or vertex i of the communication graph G_c) is mapped onto node i of the processor graph G_p , $1 \leq i \leq k$. We refer to this mapping as INITIAL. It depends on how the graph partitioner, in our case METIS or KAHIP, numbers the blocks and on how the nodes of G_p are numbered. In our experiments G_p is a 2D or 3D grid or torus since such topologies are used in real architectures, e.g. tori for BlueGene [26]. The nodes are ordered lexicographically w.r.t. the nodes' canonical integer coordinates. We also carry along a mapping called RANDOM, where the bijection $\Pi : \{1, \ldots, k\} \mapsto \{1, \ldots, k\}$ is random. The latter is done for comparison purposes, keeping in mind that RANDOM is usually a very bad solution.

Four algorithms in our collection, i. e., RCM, DRB, GREEDYALL and GREEDYMIN are from the literature (for RCM and DRB see Section 1 and [18,7]). Algorithms GREEDYALL and GREEDYMIN are described in

Section 4.1 (also see the references therein). There we also specify the last two algorithms, GREEDYALLC and GREEDYMINC, which are variants of GREEDYALL and GREEDYMIN and which, to our knowledge, are new.

4.1 Greedy Algorithms

As a prerequisite for the algorithms described in this section we need to compute $t(\cdot, \cdot)$ once for a given processor graph G_p (see Equation 1). Using Johnson's algorithm [27, 28] we can do so in time $\mathcal{O}(|V_p|^2 \log |V_p|$ $+ |V_p||E_p|$). Since G_p is sparse, this amounts to $\mathcal{O}(|V_p|^2 \log |V_p|) = \mathcal{O}(|V_c|^2 \log |V_c|)$. This running time is not included in the running times for the greedy algorithms in this section, as $t(\cdot, \cdot)$ is computed only once for a given processor graph.

The mapping algorithm GREEDYALL consists of the "construction method" proposed in [22]. Using our terminology, the algorithm starts by picking a node v_c^0 of G_c such that $\sum_{e=\{v_c^0, v_c\}\in E_c} \omega(e)$ is maximal, i. e. v_c^0 is a vertex whose communication with neighboring vertices is heaviest. Then, it computes for each vertex v_p of G_p the term $\sum_{u_p \in V_p} t(u_p, v_p)$. Here, $t(u_p, v_p)$ is the (minimum) time needed to send a unit of information from u_p to v_p (see Section 2.1). A vertex v_p^0 for which this sum is minimal (a most central node in G_p w.r.t. communication time) then becomes the vertex onto which v_c^0 is mapped. The experiments of this paper involve processor graphs which are grids and tori. On the latter all nodes are equally central.

The remaining pairs (v_c^i, v_p^i) , $i \ge 1$, are formed as follows. First, a not yet mapped vertex v_c^i of G_c is found such that $\sum_{j=0}^{i-1} \omega_c(\{v_c^j, v_c^i\})$ is maximal, i.e. v_c^i is a vertex that communicates most heavily with the already mapped vertices. Then, a not yet mapped vertex v_p^i of G_p is found such that $\sum_{j=0}^{i-1} t(v_p^j, v_p^i)$ is minimal, i.e. a vertex that is most central w.r.t. the already mapped vertices of G_p . Note that the choices of v_c^i and v_p^i are independent of each other. Our implementation of GREEDYALL has running time $\mathcal{O}(|V_c|^2)$. This running time is achieved by updating vectors sum_c (sum_p) that, for each vertex v_c (v_p) which has not been mapped yet, stores the sum of the edge weights (distances) to the vertices in G_c (G_p) that have been mapped already. We use the same two vectors in GREEDYALLC, see Algorithm 1.

The mapping algorithm GREEDYMIN stems from [7]. Its general idea is the same as that behind GREEDYALL. The only differences are that (i) v_p^0 is picked randomly, (ii) v_c^i ($i \ge 1$) is chosen such that $max_{j=0}^{i-1}\omega_c(\{v_c^i, v_c^j\})$ is maximal, and (iii) v_p^i ($i \ge 1$) is chosen such that $t(v_p^{i-1}, v_p^i)$ is minimal. Again, as in GREEDYALL, the choices of v_c^i and v_p^i are independent of each other. Our implementation of GREEDYMIN (which is less generic than that in [7]) has running time $\mathcal{O}(|V_c|^2)$.

4.2 GreedyAllC and GreedyMinC

Neither GREEDYALL nor GREEDYMIN link the choices of v_i^c and v_i^p . Both algorithms aim at (i) a high communication volume of v_i^c with all or one of the already mapped vertices of G_c and (ii) a high centrality of v_i^p w.r.t. all or one of the already mapped vertices of G_p . The actual increase of communication times caused by the new pair (v_i^c, v_i^p) (increase w.r.t. the partial mapping defined so far) is not considered.

We therefore propose new variants GREEDYALLC and GREEDYMINC. They take this increase of communication time into account. Specifically, the choice of v_p^i depends on the choice of v_c^i (same as in GREEDYALL/ GREEDYMIN). Let v_p be a candidate for being mapped onto by v_c^i . Then, (minimal) times of communication between v_c^i and the vertices of G_c that have been mapped before, i. e. $v_c^0, \ldots v_c^{i-1}$, amount to

$$\sum (\omega_c(\{v_c^i, v_c\} \ t(v_p, \Pi(v_c))) \ | \ \{v_c^i, v_c\} \in E_c \ , v_c \in \{v_c^0, \dots v_c^{i-1}\})$$
(5)

Analogous to GREEDYALLC and GREEDYMINC, we set v_p^i to some v_p such that the expression in Equation 5 is *minimal*. Thus, our objective function for choosing v_p , i. e. Equation 5, is about actual communication times and not just distances on G_p . We have experimented with replacing the sum in Equation 5 by the maximum and found out that this tends to decrease the quality of the mappings. For the pseudocode of GREEDYALLC see Algorithm 1.

Algorithm 1 The algorithm GREEDYALLC. Input: Communication graph $G_c = (V_c, E_c, \omega_c)$ and processor graph $G_p = (V_p, E_p, \omega_p)$ with $|V_c| = |V_p|$. Output: Pairs $(v_c^i, v_p^i), 0 \le i < |V_c|$, such that $\Pi : V_c \mapsto V_p$ defined by $\Pi(v_c^i) = v_p^i$ is a bijective mapping with low values of $mC(\Pi)$, $mD(\Pi)$ and $aD(\Pi)$. 1: Find $v_c^0 \in V_c$ with maximal $\sum_{e \in \{v_c^0, v_c\} \in E_c} \omega(e)$ 2: Find $v_p^0 \in V_p$ with minimal $\sum_{u_p \in V_p} t(u_p, v_p)$ 3: Create vectors sum_c and sum_p of length $|V_c|$ 4: Initialize entries of sum_c to zero and entries of sum_p to one 5: for $i \leftarrow 0, ..., |V_c| - 2$ do $sum_c(v_c^i) \leftarrow -1 /* \text{ Mark } v_c^i \text{ as assigned } */$ 6: $sum_p(v_p^i) \leftarrow \text{INT}_MAX /* \text{ Mark } v_p^i \text{ as assigned }*/$ 7:8: for all $e_c = \{v_c^i, w\} \in E_c$ do 9: if $sum_c[w] \ge 0$ then /* w is not yet assigned */10: $sum_c[w] \leftarrow sum_c[w] + \omega_c(e_c)$ end if 11:end for 12:Pick v_c^{i+1} such that $sum_c(v_c^{i+1})$ is maximal 13:for $j \leftarrow 1, \ldots, |V_p| - 1$ do 14:if $sum_p[j] < INT_MAX$ then 15:16:/* j is not yet assigned */17: $sum_p[j] \leftarrow 0$ for all $e_c = \{v_c^{i+1}, w\} \in E_c$ do 18:19:if $sum_c[w] < 0$ then 20:/* w has already been assigned, i.e. $\Pi(w)$ is defined */ 21: $sum_p[j] \leftarrow sum_p[j] + \omega_c(e_c) * t(j, \Pi(w))$ 22:end if end for 23:24:end if 25:end for Pick v_p^{i+1} such that $sum_p(v_p^{i+1})$ is maximal 26:27: end for

Proposition 1. The running time of GREEDYALLC is $\mathcal{O}(|V_c||E_c|)$.

Proof. The outermost loop from line 5 to line 27 and the inner loop from line 8 to line 12 take amortized time $\mathcal{O}(|E_c|)$. So does the outer loop from line 14 to line 25 and the inner loop from line 18 to line 23. Since the latter two loops are contained in the outermost loop from line 5 to line 27, the running time of Algorithm 1 is indeed $\mathcal{O}(|V_c||E_c|)$. Even a trivial implementation of lines 13 and 26 (with running time $\mathcal{O}(|V_c|)$) does not change the result.

The running time for GREEDYMIN is the same as for GREEDYALLC because the two algorithms differ only at lines 1 to 4, and the running times of both algorithms are not determined by this part.

5 Experiments

In this section we specify our test instances, our experimental setup and the way we evaluate the mapping algorithms.

Test Instances. The application graphs fall into two classes: The class WALSHAWLARGE consists of the eight largest graphs in Walshaw's graph partitioning archive [29], and the class COMPLEXNETS consists of 12 complex networks (see Tables 1 and 2). The latter form a subset of the 15 complex networks used in [30] for partitioning experiments. It turned out, however, that KAHIP [GPMETIS with k-way partitioning, respectively], while respecting the allowed imbalance, occasionally generated empty blocks for the complex network p2p-Gnutella [as-22july06 and loc-gowalla_edges]. Using GPMETIS with recursive bisection instead of k-way partitioning was not an option because GPMETIS then quite often violated the balance constraint and produced blocks heavier than $(1 + \epsilon)$ times the average block size (only on complex networks). For each of the classes WALSHAWLARGE and COMPLEXNETS the benchmarking comprises the following processor graphs.

- 2DGrid(16 \times 16), 2DGrid(32 \times 32), 3DGrid(8 \times 8 \times 8)
- 2DTorus(16 × 16), 2DTorus(32 × 32), 3DTorus(8 × 8 × 8)

Name	#vertices	#edges
fe_tooth	78 136	452 591
fe_rotor	99 617	662 431
598a	110 971	741934
fe_ocean	143 437	409593
144	144 649	1074391
wave	156317	1059331
m14b	214 765	1679018
auto	448 695	3314611

Table 1. Meshes used for benchmarking

Table 2. Complex networks used for benchmarking.

Name	#vertices	#edges	Type
PGPgiantcompo	10 680	24316	largest connected component in network of PGP users
email-EuAll	16 805	60 260	network of connections via email
soc-Slashdot0902	28 550	379445	news network
loc-brightkite_edges	56739	212945	location-based friendship network
coAuthorsCiteseer	227 320	814134	citation network
wiki-Talk	232 314	1458806	network of user interactions through edits
citationCiteseer	268 495	1156647	citation network
coAuthorsDBLP	299 067	977676	citation network
web-Google	356 648	2093324	hyperlink network of web pages
coPapersCiteseer	434 102	16036720	citation network
coPapersDBLP	540 486	15245729	citation network
as-skitter	554 930	5797663	network of internet service providers

Experimental Setup. All computations are sequential and done on a workstation with two 4-core Intel(R) Core(TM) i7-2600K processors at 3.40GHz. Our code is written in C++ and compiled with GCC 4.7.1.

Evaluation. The benchmarking of the mapping algorithms described in Section 4 is done separately on the classes WALSHAWLARGE and COMPLEXNETS. First, graphs from both classes are partitioned into 256, 512 and 1024 parts using the graph partitioner KAHIP v. 0.62 (http://algo2.iti.kit.edu/documents/kahip/) [31]. In particular, the meshes and social networks are partitioned with the configuration *eco* and *ecosocial*, respectively. The allowed imbalance is always 1.03, i.e. $\epsilon = 3\%$. To recursively bipartition G_c and G_p during DRB, we also use KAHIP (configurations fast and ecofast, perfect balance).

Since the partitioning process depends on random choices, we run KAHIP with 20 different seeds. For each seed we construct a communication graph G_c from the partition, map G_c onto all processor graphs with the same number of vertices and then compute the minimum, the arithmetic mean and the maximum of the mapping's runtime t, mC (see Equation 4), mD and aD (see Equation 2). Thus we arrive at the values t_{min} , t_{mean} , t_{max} , mC_{min} , etc. (twelve values for each combination of G_c , G_p , and a mapping algorithm).

Next we form the geometric means of the twelve values over all graphs in WALSHAWLARGE and COM-PLEXNETS, respectively. Thus we arrive at twelve values t_{min}^{gm} , ... for any combination of a graph class (WALSHAWLARGE or COMPLEXNETS), a processor graph, and a mapping algorithm. Finally, the last 9 values (all except runtimes) are set into proportion to the corresponding values for INITIAL. This yields the values QmC_{min}^{gm} , QmC_{mean}^{gm} , QmD_{min}^{gm} , QmD_{mean}^{gm} , QmD_{max}^{gm} , QaD_{min}^{gm} , QaD_{mean}^{gm} and QaD_{max}^{gm} . A *Q*-value smaller than one means that the quality is higher than that of INITIAL because we are minimizing.

We also investigate the influence of graph partitioning on the quality of the mapping algorithms. In addition to using KAHIP as described above, we apply two variants of METIS v. 5.1.0 [4]

- 1. We run GPMETIS with the option of k-way partitioning, an allowed imbalance of 1.03 and 20 seeds (imbalance and seed number are as for KAHIP).
- 2. We run NDMETIS with 20 seeds. This results in a fill-reducing ordering of G_a 's adjacency matrix. The ordering is then turned into a partitioning of G_a by going through the vertices in the new order and assigning block numbers such that all blocks have almost equal size (maximal deviation is one vertex). We are aware that using NDMETIS in this way is not a good choice in view of partitioning quality (NDMETIS is made for other purposes). We proceed like this, however, *because* we wish to test our collection of mapping algorithms on partitions with mediocre edge cut and MCV.

We indicate the METIS-based graph partitioning that is underlying a mapping algorithm by using the subscripts G and N when employing GPMETIS and NDMETIS, respectively. As an example, GREEDYALLC_N means that we applied GREEDYALLC to partitions obtained via NDMETIS.

Finally, for each $x \in \{t_{min}^{gm}, \dots, QaD_{max}^{gm}\}$ (this set has 12 values), we form quotients Qx of the form

x from KAHIP partitions	x from KAHIP partitions
\overline{x} from GPMETIS partitions '	\overline{x} from NDMETIS partitions

As an example, $QaD_{max}^{gm} = 3.2098$ for GREEDYALLC_N in Table 5 means that aD_{max}^{gm} is worse by a factor of 3.2098 if NDMETIS is used instead of KAHIP for mapping meshes onto 2DGrid(16 × 16).

6 Results

6.1 Mapping of Meshes onto Grids and Tori

Table 3 shows a comparison of KAHIP partitions with partitions from GPMETIS and NDMETIS. We measure running time, edge cut and MCV. As above, we record the best, mean and worst result over 20 seeds and

Table 3. Performance of GPMETIS and NDMETIS on meshes compared to KAHIP. Values smaller than one indicate that GPMETIS/NDMETIS is faster or that the quality the GPMETIS/NDMETIS-partitions is higher.

	$Time_{min}^{gm}$	$Time_{mean}^{gm}$	$Time_{max}^{gm}$	Cut_{min}^{gm}	Cut_{mean}^{gm}	Cut_{max}^{gm}	MCV_{min}^{gm}	MCV_{mean}^{gm}	MCV^{gm}_{max}
GPMETIS	0.0462	0.0451	0.0440	1.0101	1.0101	1.0121	0.9970	1.0449	1.1601
NDMETIS	0.1026	0.0999	0.0985	2.2075	2.2371	2.2472	6.0976	5.9880	5.7471

calculate the geometric means of these numbers over all meshes in our collection — giving rise to the numbers $Time_{max}^{gm}, \ldots, MCV_{max}^{gm}$ in Table 3. In terms of partition quality, GPMETIS performs significantly poorer than KAHIP only in terms of MCV_{mean}^{gm} and MCV_{max}^{gm} . Here GPMETIS is worse by 4.49% and 16.01%, respectively. The partitions that we derived from NDMETIS (in a deliberately sub-optimal way) fall back drastically both in terms of the edge cut and MCV. In particular, $Cut_{mean}^{gm} = 2.2371$ and $MCV_{mean}^{gm} = 5.988$, which means that the edge cut from NDMETIS is more than double and that MCV increases almost six times if NDMETIS is used instead of KAHIP.

Table 4 shows the quality of the mapping algorithms for (i) partitions based on KAHIP and (ii) mapping onto the 16×16 torus. Tables I - V in the Appendix support the following results.

Table 4. Mapping of meshes onto 2DTorus(16×16). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in milliseconds.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t^{gm}_{max}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD^{gm}_{max}
Random	0.028	0.033	0.044	2.087	2.059	2.030	1.389	1.397	1.432	1.667	1.471	1.249
RCM	0.060	0.070	0.088	1.634	1.640	1.645	1.357	1.454	1.586	1.509	1.389	1.242
DRB	50.75	52.18	54.20	0.862	0.904	0.966	0.821	0.886	0.987	1.039	1.010	0.962
GreedyAll	0.948	0.971	0.997	1.310	1.316	1.324	1.252	1.291	1.369	1.359	1.263	1.164
GreedyMin	0.163	0.168	0.183	1.139	1.162	1.199	1.025	1.080	1.149	0.870	0.776	0.654
GREEDYALLC	0.918	0.953	0.987	0.683	0.707	0.736	0.665	0.706	0.766	0.730	0.780	0.871
GreedyMinC	0.869	0.939	0.100	0.793	0.813	0.844	0.739	0.789	0.849	0.745	0.756	0.780

- 1. The mapping algorithms RANDOM, RCM and GREEDYALL are worse than INITIAL on all accounts. While this was expected for RANDOM, our data show that very simple mapping strategies are not worthwhile if the underlying partition is good.
- 2. The algorithm GREEDYMIN beats INITIAL only in terms of average dilation. The improvement is, however, a major one in some cases, e.g. $QaD_{mean}^{gm} = 0.776$ and $QaD_{max}^{gm} = 0.654$ for the 16 × 16 2D torus (see Table 4). Another strong point of GREEDYMIN is its low running time.
- 3. On all six processor graphs our new mapping algorithm GREEDYALLC yields the best maximum congestion, mC, and the best maximum dilation, mD. This holds not only for the (geometric mean over all meshes of the) average over all seeds, but also if the best or the worst result is taken over all seeds. The quotients are between 0.556 and 0.789. In terms of running time, we are in-between that of GREEDYMIN and DRB.
- 4. DRB yields many major improvements over INITIAL and, discarding average dilation, is worse only once (in terms of QmD_{max}^{gm} on the 3D torus, see Table V in the Appendix). DRB often comes close to GREEDYALLC and sometimes beats it on average dilation.
- 5. GREEDYMINC has its strengths on tori and often beats GREEDYALLC on average dilation (on grids *and* tori). Interestingly, the overall quality of GREEDYALL is much worse than that of GREEDYMIN (both from previous work), while this trend is reversed if we look at the modified versions GREEDYALLC and GREEDYMINC.

We now look at the influence of the partitioning quality on the quality of the mapping algorithms (see Table 5 (Table VI in the Appendix provides more evidence). As for KAHIP vs. GPMETIS, the small lead of KAHIP over GPMETIS w.r.t. MCV translates into an even smaller lead of the corresponding mappings. Moreover, this small lead is only on average, and there are cases where GPMETIS partitions lead to better mapping results. As for KAHIP vs. NDMETIS, poor edge cut and/or MCV seem to have a deteriorating effect on mapping quality.

Table 5. Mapping of meshes onto $2DGrid(16 \times 16)$.

Algo	Qt_{min}^{gm}	$\mathcal{Q}t^{gm}_{mean}$	$\mathcal{Q}t_{max}^{gm}$	QmC_{min}^{gm}	QmC^{gm}_{mean}	QmC_{max}^{gm}	QmD_{min}^{gm}	QmD_{mean}^{gm}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
INITIAL _G	1.0205	1.0152	1.1762	0.9930	0.9963	0.9881	1.0010	1.0002	0.9985	1.0042	1.0071	0.9690
INITIAL _N	1.0346	1.0506	1.1932	1.9068	1.9594	2.0013	2.7783	2.8391	2.8141	3.9285	4.2721	4.2889
$\operatorname{Random}_{G}$	0.9944	0.9749	0.9809	0.9961	0.9983	1.0027	0.9968	0.9847	0.9431	1.0251	1.0307	1.0524
$\operatorname{Random}_{N}$	1.0120	0.9887	0.9510	1.9659	2.0315	2.0960	2.5159	2.5489	2.5157	3.8256	3.9547	4.3704
$\mathrm{RCM}_{\mathrm{G}}$	1.0287	1.0087	1.0103	0.9980	1.0020	0.9955	1.0033	1.0169	1.0310	1.0473	1.0225	1.0019
$\mathrm{RCM}_{\mathrm{N}}$	1.1701	1.1362	1.1554	2.0780	2.1376	2.1856	2.8376	2.8998	3.0228	3.6763	3.8054	4.2111
$\mathrm{DRB}_{\mathrm{G}}$	1.0083	1.0034	0.9956	0.9991	1.0096	1.0258	0.9951	1.0112	1.0131	1.0179	1.0441	0.9811
$\mathrm{DRB}_{\mathrm{N}}$	0.9980	1.0119	1.0194	1.9035	2.0223	1.9517	2.6668	2.8051	2.7540	2.5027	2.9897	3.1595
$\mathrm{GREEDYMIN}_{\mathrm{G}}$	1.0039	1.0021	1.0182	1.0110	0.9953	0.9948	1.0295	0.9959	0.9923	1.0031	1.0088	1.0034
${\rm GreedyMin}_{\scriptscriptstyle N}$	1.0095	1.0177	1.1114	1.5176	1.4785	1.4779	2.5373	2.5238	2.5750	1.5832	1.5819	1.6219
$GREEDYALLC_{G}$	1.0091	1.0074	1.0236	0.9999	1.0135	1.0144	1.0491	1.0200	1.0085	1.0153	0.9919	0.9771
$\mathrm{GREEDYALLC}_{\mathrm{N}}$	1.1313	1.1499	1.1819	1.9840	1.9265	1.8896	2.9915	2.9117	2.9811	2.1990	2.6233	3.2098
$GREEDYMINC_{\rm G}$	1.0002	1.0073	1.0205	1.0121	0.9887	0.9809	1.0166	0.9987	0.9979	1.0018	1.0013	0.9449
${\rm Greedy}{\rm Min}{\rm C}_{\scriptscriptstyle N}$	1.1869	1.1879	1.1993	1.6175	1.5741	1.5145	2.5819	2.7487	2.8079	1.6267	2.2717	3.5661

6.2 Mapping of Complex Networks onto Grids and Tori.

Table 6 shows a comparison of KAHIP partitions with partitions from GPMETIS and NDMETIS. For a description of the table see the explanation of Table 3 in Section 6.1.

Table 6. Performance of GPMETIS and NDMETIS on complex networks compared to KAHIP. Values smaller than one indicate that GPMETIS/NDMETIS is faster or that the quality the GPMETIS/NDMETIS-partitions is higher.

	$Time_{min}^{gm}$	$Time_{mean}^{gm}$	$Time_{max}^{gm}$	Cut_{min}^{gm}	Cut_{mean}^{gm}	Cut_{max}^{gm}	MCV_{min}^{gm}	MCV^{gm}_{mean}	MCV^{gm}_{max}
GPMETIS	0.0083	0.0081	0.0078	1.0634	1.0619	1.0560	1.2531	1.2066	1.1536
NDMETIS	0.0262	0.0268	0.0257	2.0284	2.0202	2.0121	1.8416	1.9157	2.0040

Compared to the picture we saw on meshes, KAHIP now also leads in terms of the edge cut. Moreover, the lead of KAHIP in terms of MCV compared to GPMETIS and NDMETIS is even more pronounced (about 20%).

Regarding topology mapping based on KAHIP partitions, we only comment on results that deviate from those that we have described for meshes (especially running times show the same trends). The main differences are in the maximum and average dilation. Sometimes RCM and even RANDOM yield even lower maximum dilation than GREEDYALLC. Moreover, average dilation behaves quite erratically, as is revealed by a comparison between the aD-values of GREEDYMINC in Table 7 and Tables VII through XI in the Appendix.

Table 7. Mapping of complex networks onto $2DTorus(16 \times 16)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in milliseconds.

Algo	$ t_{min}^{gm} $	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.028	0.032	0.042	1.511	1.509	1.513	0.777	0.780	0.810	3.291	3.241	2.840
RCM	0.104	0.122	0.161	1.366	1.416	1.455	0.822	0.868	0.931	2.672	2.919	2.699
DRB	124.8	138.5	154.0	0.982	1.003	1.021	0.853	0.876	0.926	1.084	1.250	1.476
GreedyAll	5.182	5.344	5.684	1.100	1.119	1.131	1.068	1.011	0.985	1.189	1.315	1.301
GreedyMin	0.248	0.258	0.292	1.057	1.054	1.056	1.109	1.056	1.015	0.858	0.676	0.496
GREEDYALLC	5.647	5.871	6.268	0.841	0.839	0.839	0.863	0.820	0.801	0.531	0.442	0.351
GreedyMinC	5.251	5.550	6.153	0.858	0.856	0.855	0.857	0.829	0.808	0.644	0.575	0.481

Regarding maximum congestion, mC, the picture is the same as we saw for meshes: Our new algorithm GREEDYALLC always yields the best results.

Regarding the influence of partitioning quality on the quality of the mappings we see that the higher partitioning quality of KAHIP compared to GPMETIS (in terms of the edge cut and MCV) does not translate into considerably better mappings, see Table 8 (for additional evidence see Table XII in the Appendix). As in the case of meshes, the partitions that we derived from NDMETIS (in a deliberately sub-optimal way) lead to poor mappings.

Table 8. Mapping of complex networks onto $2DGrid(16 \times 16)$.

Algo	Qt_{min}^{gm}	$\mathcal{Q}t^{gm}_{mean}$	$\mathcal{Q}t_{max}^{gm}$	QmC_{min}^{gm}	QmC^{gm}_{mean}	$\mathcal{Q}mC^{gm}_{max}$	QmD_{min}^{gm}	QmD_{mean}^{gm}	$\mathcal{Q}mD_{max}^{gm}$	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
INITIAL _G	0.9723	0.9838	0.9812	1.0243	1.0407	1.0623	0.9914	0.9955	1.0209	1.2791	1.4033	1.4395
INITIAL _N	0.9684	0.9963	0.9832	10.004	10.146	10.195	2.9752	2.8494	2.8637	3.9055	3.7684	3.5497
Random _g	1.0292	1.0234	1.0285	0.9832	0.9910	0.9983	1.0866	1.1270	1.1680	1.0194	1.0167	1.1733
$\operatorname{Random}_{N}$	1.0127	1.0272	1.0027	7.5795	7.7739	7.9801	2.7521	2.8475	2.9019	1.8046	1.5884	1.6585
$\mathrm{RCM}_{\mathrm{G}}$	1.0970	1.0851	0.9894	1.0087	0.9939	0.9991	1.1129	1.1367	1.1333	0.9571	0.9581	1.0146
RCM _N	0.9611	0.9963	0.9008	7.5698	7.8690	8.2057	3.2024	3.2317	2.9598	1.6997	1.6083	1.4822
$\mathrm{DRB}_{\mathrm{G}}$	1.0486	1.0492	1.0620	1.0076	1.0147	1.0027	1.0624	1.0745	1.0913	0.9775	1.0832	1.0793
$\mathrm{DRB}_{\mathrm{N}}$	0.4242	0.4094	0.3885	7.6865	8.1313	8.5734	3.4637	3.7612	3.9759	2.0400	2.1615	1.7987
$GREEDYMIN_{\rm G}$	1.0375	1.0341	1.0102	1.0284	1.0267	1.0278	1.0060	1.0440	1.0791	1.0846	1.1068	1.0990
$GREEDYMIN_{\rm N}$	0.6850	0.6860	0.7321	9.0648	9.1580	9.4715	4.4637	4.3656	4.4046	2.5795	2.5664	2.5504
$GREEDYALLC_{G}$	1.0637	1.0637	1.0608	1.0450	1.0392	1.0392	1.0527	1.0582	1.0594	1.3140	1.0434	0.9078
${\rm GREEDYALLC}_{\rm N}$	0.3690	0.3700	0.3559	7.9916	8.0772	8.1791	3.5868	3.5635	3.6554	4.0989	2.5944	1.6957
$GREEDYMINC_{G}$	1.0847	1.0731	1.0698	1.0454	1.0411	1.0347	1.1240	1.1579	1.2136	1.0790	1.0542	0.9809
$GREEDYMINC_{\scriptscriptstyle N}$	0.3688	0.3721	0.3769	9.2443	9.3823	9.8506	5.2981	5.1511	5.1905	2.4001	1.7601	1.3172

7 Conclusions and Future Work

We performed extensive static mapping experiments, our scenario being a consecutive pipeline of graph partitioning and bijective topology mapping. These experiments involved two classes of application graphs (8 meshes, 12 complex networks), three ways to partition the application graphs (one by KAHIP, two by METIS), six processor graphs (3 grids, 3 tori) and 8 mapping algorithms.

Our results indicate that the strengths and weaknesses of the mapping algorithms are, to a large extent, independent of the class of application graphs (mesh or complex network) and the processor graphs. The main differences are in the maximum and average dilation. Especially the latter behaves erratically in the case of complex networks.

Second, the quality of the partitions, both in terms of edge cut and MCV, has little influence on the quality of the mapping, except in cases where MCV is very poor. Thus, even MCV is not a good indicator of how well a partition can be mapped onto a processor graph — at least within the realm of our experiments.

Third, our variant of a greedy mapping algorithm by Brandfass *et al.*, i.e. GREEDYALLC, clearly dominates all state-of-the art algorithms we considered in terms of maximum congestion. The running time of our algorithm is $\mathcal{O}(|V_c||E_c|)$, where V_c and E_c is the vertex and the edge set of the communication graph, respectively (and therefore usually fairly small).

If the weak influence of partition quality on mapping quality is affirmed for more classes of application graphs and more parallel architectures, improvements of static mapping are likely to come only out of new combinations of partitioning and mapping. In the future we will investigate how to minimize the communication volume specified in Equation 5 by such a coupled approach.

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Appendix

Table 9. Mapping of meshes onto $2DGrid(16 \times 16)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in milliseconds.

Algo	t_{min}^{gm}	t_{mean}^{gm}	t_{max}^{gm}	QmC_{min}^{gm}	QmC_{mean}^{gm}	QmC_{max}^{gm}	QmD_{min}^{gm}	QmD_{mean}^{gm}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
Random	0.029	0.033	0.040	2.295	2.264	2.243	1.941	1.926	1.861	1.744	1.550	1.423
RCM	0.057	0.068	0.079	1.564	1.563	1.572	1.288	1.297	1.304	1.670	1.530	1.393
DRB	49.40	50.82	52.68	0.756	0.818	0.881	0.697	0.750	0.773	0.913	0.974	1.012
GREEDYALL	0.951	0.973	0.992	1.679	1.680	1.678	1.359	1.361	1.292	1.782	1.720	1.579
GreedyMin	0.160	0.164	0.177	1.120	1.192	1.274	1.092	1.136	1.175	0.871	0.803	0.714
GREEDYALLC	0.906	0.942	0.995	0.665	0.722	0.789	0.626	0.665	0.736	0.882	1.027	1.253
GreedyMinC	0.828	0.875	0.919	0.817	0.890	0.957	0.750	0.785	0.817	0.787	0.792	0.901

Table 10. Mapping of meshes onto 2DGrid(32×32). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	QmC^{gm}_{mean}	QmC^{gm}_{max}	$ QmD_{min}^{gm} $	QmD_{mean}^{gm}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
Random	0.093	0.110	0.135	2.933	2.871	2.806	2.268	2.169	2.084	1.769	1.637	1.440
RCM	0.205	0.239	0.273	1.776	1.782	1.781	1.441	1.433	1.418	1.784	1.683	1.481
DRB	216.5	221.6	227.7	0.715	0.757	0.806	0.655	0.690	0.756	0.952	1.060	1.125
GREEDYALL	17.55	18.53	18.97	2.116	2.084	2.068	1.609	1.565	1.522	1.956	1.838	1.632
GreedyMin	3.229	3.867	3.947	1.294	1.371	1.442	1.289	1.320	1.408	0.936	0.858	0.732
GREEDYALLC	17.20	18.05	18.89	0.569	0.626	0.681	0.556	0.615	0.689	0.876	1.137	1.381
GREEDYMINC	15.98	16.71	17.70	0.887	0.948	1.040	0.770	0.837	0.916	0.869	0.892	0.966

Table 11. Mapping of meshes onto $3DGrid(8 \times 8 \times 8)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t^{gm}_{max}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.047	0.056	0.070	2.169	2.131	2.094	1.749	1.739	1.690	1.664	1.508	1.329
RCM	0.113	0.129	0.148	1.631	1.622	1.616	1.289	1.314	1.305	1.476	1.372	1.216
DRB	112.6	115.6	119.8	0.814	0.856	0.920	0.719	0.769	0.819	0.935	0.995	0.946
GREEDYALL	4.006	4.050	4.101	1.723	1.704	1.688	1.348	1.326	1.267	1.670	1.618	1.470
GreedyMin	0.687	0.708	0.722	1.214	1.232	1.255	1.051	1.061	1.082	0.976	0.887	0.780
GREEDYALLC	3.875	3.996	4.104	0.683	0.713	0.736	0.617	0.633	0.638	0.876	1.055	1.151
GREEDYMINC	3.683	3.871	4.067	0.827	0.859	0.891	0.695	0.718	0.757	1.268	1.339	1.388

Table 12. Mapping of meshes onto 2DTorus(32×32). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	QmC^{gm}_{mean}	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD_{mean}^{gm}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
Random	0.090	0.111	0.139	2.656	2.609	2.566	1.487	1.4728	1.444	1.661	1.491	1.369
RCM	0.213	0.249	0.295	1.942	1.942	1.932	1.480	1.5121	1.552	1.536	1.400	1.312
DRB	212.9	217.0	221.6	0.794	0.843	0.896	0.782	0.8642	0.938	1.059	1.068	1.111
GREEDYALL	17.79	18.07	18.42	1.529	1.526	1.533	1.638	1.7147	1.765	1.407	1.319	1.215
GREEDYMIN	4.031	4.068	4.110	1.303	1.336	1.360	1.166	1.2162	1.259	0.892	0.787	0.683
GREEDYALLC	17.53	18.08	18.94	0.569	0.611	0.647	0.609	0.6843	0.752	0.726	0.815	0.899
GreedyMinC	16.93	17.68	18.30	0.778	0.820	0.859	0.752	0.8148	0.886	0.892	0.973	1.028

Table 13. Mapping of meshes onto $3\text{DTorus}(8 \times 8 \times 8)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t^{gm}_{max}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
Random	0.049	0.058	0.075	2.052	2.013	2.000	1.307	1.331	1.337	1.571	1.447	1.392
RCM	0.115	0.131	0.154	1.638	1.637	1.641	1.317	1.395	1.502	1.485	1.378	1.360
DRB	115.8	118.5	121.2	0.931	0.972	1.049	0.859	0.943	1.039	1.057	1.028	1.045
GREEDYALL	3.848	3.910	4.028	1.303	1.230	1.312	1.161	1.214	1.221	1.262	1.196	1.173
GREEDYMIN	0.656	0.670	0.684	1.209	1.217	1.240	1.002	1.049	1.073	0.903	0.797	0.726
GREEDYALLC	3.882	4.050	4.178	0.751	0.757	0.767	0.683	0.719	0.711	0.755	0.806	0.913
GREEDYMINC	3.897	4.125	4.463	0.840	0.842	0.858	0.727	0.761	0.790	0.739	0.743	0.783

Algo	$\mathcal{Q}t_{min}^{gm}$	$\mathcal{Q}t^{gm}_{mean}$	$\mathcal{Q}t_{max}^{gm}$	QmC_{min}^{gm}	$\mathcal{Q}mC^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	$\mathcal{Q}mD^{gm}_{mean}$	$\mathcal{Q}mD_{max}^{gm}$	QaD_{min}^{gm}	$\mathcal{Q}aD^{gm}_{mean}$	QaD_{max}^{gm}
INITIAL _G	1.0361	0.9916	0.8434	0.9987	0.9964	0.9968	1.0042	1.0032	0.9917	0.9844	1.0209	1.0812
INITIAL _N	0.9799	0.9805	0.8702	1.6731	1.6923	1.7122	2.5803	2.7067	2.8812	1.5955	1.5739	1.5854
$\operatorname{Random}_{G}$	1.0016	1.0165	0.9776	1.0019	1.0014	1.0027	1.0179	1.0257	1.0847	1.0188	1.0256	1.0516
$\operatorname{Random}_{N}$	1.0269	1.0453	1.0364	1.9834	2.0339	2.0892	2.6819	2.7297	2.9002	3.8130	4.0997	4.3485
$\mathrm{RCM}_{\mathrm{G}}$	1.0212	1.0093	1.1226	1.0140	1.0005	1.0014	1.0219	1.0256	0.9761	1.0133	1.0188	1.0131
$\mathrm{RCM}_{\mathrm{N}}$	1.1623	1.1217	1.1519	2.0952	2.1371	2.1744	2.7086	2.7540	2.7208	3.7498	4.0711	4.3358
DRB_{G}	1.0058	0.9997	0.9919	1.0062	1.0070	1.0161	1.0041	1.0061	1.0642	1.0238	1.0044	1.0358
$\mathrm{DRB}_{\mathrm{N}}$	0.9996	1.0072	1.0058	1.9550	2.0459	1.9538	2.8690	2.9701	3.1304	2.5652	3.1615	3.8322
$\mathrm{GREEDYMIN}_{\mathrm{G}}$	1.0065	1.0013	1.0038	1.0162	0.9976	1.0011	0.9978	0.9865	0.9606	1.0100	1.0115	1.0202
$\mathrm{GREEDYMIN}_{\mathrm{N}}$	1.0185	1.0110	1.0252	1.5790	1.5512	1.5467	2.4673	2.5399	2.5795	1.6415	1.6700	1.7920
$GREEDYALLC_{G}$	1.0063	1.0052	1.0014	0.9941	0.9996	0.9941	1.0085	0.9930	0.9828	1.0682	1.0244	0.9929
$GREEDYALLC_N$	1.1169	1.1379	1.1481	2.0367	1.9924	1.9810	2.8488	3.0034	3.0069	2.0127	2.3230	2.9728
$GREEDYMINC_{\rm G}$	1.0059	1.0101	1.0793	0.9915	0.9928	0.9991	0.9775	0.9935	1.0491	1.0182	1.0197	1.0345
$GREEDYMINC_{\scriptscriptstyle N}$	1.1779	1.1475	1.1270	1.7459	1.7276	1.6917	2.4912	2.6535	2.7047	1.7459	2.4074	3.3671

Table 14. Mapping of meshes onto $2DTorus(16 \times 16)$.

Table 15. Mapping of complex networks onto 2DGrid(16×16). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in milliseconds.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.028	0.032	0.042	1.593	1.597	1.597	0.987	0.925	0.943	3.737	3.831	3.361
RCM	0.101	0.120	0.155	1.362	1.419	1.457	0.968	0.956	1.032	2.783	3.267	3.411
DRB	124.8	138.4	154.8	0.937	0.957	0.989	0.762	0.750	0.778	1.012	1.154	1.562
GreedyAll	5.176	5.337	5.610	1.078	1.098	1.104	1.096	0.975	0.925	1.365	1.625	1.630
GreedyMin	0.245	0.256	0.230	1.043	1.045	1.038	1.003	0.929	0.897	0.774	0.627	0.453
GREEDYALLC	5.622	5.837	6.173	0.799	0.813	0.827	0.798	0.738	0.730	0.847	1.197	1.461
GREEDYMINC	5.243	5.488	5.836	0.849	0.854	0.856	0.711	0.674	0.669	0.554	0.548	0.557

Table 16. Mapping of complex networks onto 2DGrid(32×32). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	QmC^{gm}_{mean}	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
Random	0.094	0.112	0.145	1.849	1.836	1.817	0.820	0.812	0.858	5.086	5.603	5.046
RCM	0.412	0.504	0.641	1.510	1.575	1.623	0.824	0.871	0.951	3.840	4.773	4.528
DRB	415.0	445.1	484.8	0.881	0.911	0.948	0.617	0.636	0.685	0.951	1.207	1.477
GREEDYALL	72.81	75.82	79.82	1.152	1.150	1.143	0.914	0.875	0.872	1.992	2.448	2.514
GreedyMin	3.307	4.141	4.316	1.047	1.050	1.046	0.867	0.849	0.858	0.681	0.584	0.437
GREEDYALLC	96.07	99.59	103.2	0.720	0.728	0.730	0.650	0.631	0.642	0.784	0.804	0.830
GreedyMinC	85.73	88.87	92.26	0.802	0.843	0.873	0.642	0.668	0.715	2.900	3.304	2.860

Table 17. Mapping of complex networks onto $3DGrid(8 \times 8 \times 8)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.049	0.057	0.074	1.539	1.527	1.519	0.878	0.850	0.884	3.459	3.872	3.479
RCM	0.209	0.258	0.335	1.381	1.410	1.437	0.889	0.880	0.900	2.686	3.307	3.177
DRB	254.0	275.0	230.0	0.935	0.952	0.967	0.751	0.757	0.751	0.911	1.197	1.504
GreedyAll	20,78	21.49	22.47	1.108	1.104	1.112	1.134	1.083	1.035	1.412	1.538	1.548
GreedyMin	0.882	0.905	0.927	1.097	1.100	1.100	0.930	0.883	0.879	0.937	0.856	0.658
GREEDYALLC	25.42	26.63	28.10	0.791	0.793	0.802	0.849	0.804	0.794	0.988	1.008	0.978
GreedyMinC	23.38	24.43	25.65	0.857	0.886	0.907	0.811	0.791	0.804	3.522	4.509	4.087

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.093	0.110	0.138	1.748	1.733	1.720	0.627	0.644	0.691	4.906	5.019	4.675
RCM	0.420	0.513	0.650	1.524	1.577	1.618	0.723	0.762	0.824	3.820	4.422	4.475
DRB	412.5	442.1	480.1	0.925	0.946	0.958	0.697	0.764	0.826	0.942	1.318	1.971
GreedyAll	74.24	75.93	78.76	1.171	1.181	1.200	0.989	0.973	0.993	1.750	2.176	2.493
GreedyMin	4.428	4.475	4.532	1.115	1.116	1.107	0.941	0.958	0.993	0.785	0.710	0.557
GREEDYALLC	96.29	99.57	103.6	0.770	0.769	0.764	0.749	0.755	0.771	0.578	0.522	0.439
GreedyMinC	86.33	89.39	93.14	0.837	0.838	0.839	0.735	0.760	0.797	2.732	3.256	3.034

Table 18. Mapping of complex networks onto 2DTorus(32×32). Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in milliseconds.

Table 19. Mapping of complex networks onto $3DTorus(8 \times 8 \times 8)$. Times t_{min}^{gm} , t_{mean}^{gm} and t_{max}^{gm} are in **milliseconds**.

Algo	t_{min}^{gm}	t^{gm}_{mean}	t_{max}^{gm}	QmC_{min}^{gm}	$Qm C^{gm}_{mean}$	QmC^{gm}_{max}	QmD_{min}^{gm}	QmD^{gm}_{mean}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD^{gm}_{mean}	QaD_{max}^{gm}
Random	0.050	0.058	0.072	1.479	1.474	1.469	0.762	0.753	0.773	3.029	3.498	3.251
RCM	0.211	259.9	342.2	1.383	1.404	1.419	0.813	0.831	0.910	2.558	3.255	3.130
DRB	257.3	279.1	302.5	0.995	1.014	1.028	0.893	0.925	0.994	1.003	1.418	1.804
GREEDYALL	20.72	21.36	22.59	1.114	1.118	1.127	1.187	1.139	1.089	1.184	1.380	1.477
GreedyMin	0.841	0.866	0.886	1.100	1.099	1.095	0.956	0.945	0.945	0.893	0.816	0.634
GREEDYALLC	25.48	26.71	28.15	0.852	0.847	0.839	0.974	0.958	0.943	0.660	0.620	0.519
GREEDYMINC	23.57	24.56	25.74	0.875	0.873	0.870	0.971	0.968	0.988	1.215	1.618	1.582

Table 20. Mapping of complex networks onto $2DTorus(16 \times 16)$.

Algo	Qt_{min}^{gm}	$\mathcal{Q}t^{gm}_{mean}$	$\mathcal{Q}t_{max}^{gm}$	$ \mathcal{Q}mC_{min}^{gm} $	QmC^{gm}_{mean}	$\mathcal{Q}mC^{gm}_{max}$	QmD_{min}^{gm}	QmD_{mean}^{gm}	QmD_{max}^{gm}	QaD_{min}^{gm}	QaD_{mean}^{gm}	QaD_{max}^{gm}
INITIAL _G	0.9714	1.0045	1.0365	1.0369	1.0357	1.0463	0.9555	0.9859	1.0147	1.2189	1.2921	1.3684
INITIAL _N	1.0000	0.9888	1.0098	8.7294	8.9164	9.0664	3.2390	3.0742	2.9958	3.2217	2.9771	2.5263
$\operatorname{Random}_{G}$	1.0197	1.0232	1.0457	0.9888	0.9887	0.9873	1.0984	1.1191	1.1202	0.9969	1.0419	1.1427
$\operatorname{Random}_{N}$	1.0153	1.0125	1.1086	7.6895	7.7816	7.8792	2.8851	2.9030	2.8530	1.8989	1.7575	1.5831
$\mathrm{RCM}_{\mathrm{G}}$	1.1057	1.0787	0.9371	0.9843	0.9860	0.9910	1.1042	1.1155	1.1473	0.9775	1.0073	1.1084
$\mathrm{RCM}_{\mathrm{N}}$	0.9755	0.9936	0.8855	7.2693	7.3256	7.5483	3.1131	3.3055	3.2221	1.5371	1.5224	1.6418
$\mathrm{DRB}_{\mathrm{G}}$	1.0430	1.0454	1.0633	1.0021	1.0092	1.0118	1.0575	1.0732	1.1035	0.9560	1.0777	1.1059
$\mathrm{DRB}_{\mathrm{N}}$	0.4275	0.4097	0.3889	8.0650	8.1325	8.4128	3.2878	3.5476	3.5615	2.1035	2.0679	1.6679
${\rm GREEDYMIN}_{\rm G}$	1.0334	1.0333	1.0187	1.0192	1.0181	1.0180	1.0293	1.0652	1.1093	1.1002	1.1122	1.1105
${\rm GreedyMin}_{\scriptscriptstyle N}$	0.6853	0.6817	0.6872	8.3516	8.4982	8.7077	3.4357	3.4170	3.3840	2.5083	2.4816	2.3758
$GREEDYALLC_{\rm G}$	1.0608	1.0673	1.0932	1.0340	1.0304	1.0287	0.9963	1.0285	1.0338	1.1039	1.1171	1.1198
$GREEDYALLC_N$	0.3670	0.3733	0.3912	8.1861	8.3093	8.4513	4.1153	4.1375	4.1129	2.4514	2.3375	2.3238
$GREEDYMINC_{\rm G}$	1.0827	1.0731	0.9870	1.0304	1.0283	1.0335	1.0148	1.0280	1.0331	1.0229	1.0711	1.1837
$GREEDYMINC_{\scriptscriptstyle N}$	0.3723	0.3708	0.3484	8.1740	8.3004	8.4825	3.7402	3.7732	3.7904	2.0304	1.7733	1.6501