

# Optimal Energy Efficiency in Cell-Free Massive MIMO Systems: A Stochastic Geometry Approach

Anastasios Papazafeiropoulos<sup>\*‡</sup>, Hien Quoc Ngo<sup>†</sup>, Pandelis Kourtessis<sup>\*</sup>, Symeon Chatzinotas<sup>‡</sup>, John M. Senior<sup>\*</sup>

<sup>\*</sup>Communications and Intelligent Systems Research Group, University of Hertfordshire, Hatfield, U. K.

<sup>†</sup>School of Electronics, Electrical Engineering, Queen's University Belfast, Belfast BT3 9DT, U.K.

<sup>‡</sup>SnT - securityandtrust.lu, University of Luxembourg, Luxembourg

Email: tapapazaf@gmail.com, hien.ngo@qub.ac.uk, symeon.chatzinotas@uni.lu.

**Abstract**—The increasing demand for green wireless communications and the benefits of the promising cell-free (CF) massive multiple-input-multiple-output (mMIMO) systems towards their optimal energy efficiency (EE) are the focal points of this work. Specifically, despite previous works assuming a uniform placement for the access points (APs), we consider that their locations follow a Poisson point process (PPP) which approaches their opportunistic spatial randomness. Based on stochastic geometry, we derive a lower bound on the average spectral efficiency, and under a realistic power consumption model for CF mMIMO systems, we formulate an EE maximization problem achieving to obtain in closed form the optimal EE per unit area in terms of the pilot reuse factor and the AP density. Note that we have defined the EE per unit area and not just the EE to characterize the energy in systems with multi-point transmission. Thus, we provide important design insights for energy-efficient CF mMIMO systems.

**Index Terms**—Cell-free massive MIMO systems, energy efficiency, stochastic geometry, small cell networks, 5G and beyond MIMO systems.

## I. INTRODUCTION

Recently, cell-free (CF) massive multiple-input-multiple-output (mMIMO) systems has emerged as a promising technology for fifth-generation (5G) networks and beyond [1]. In particular, CF mMIMO systems take advantage of both conventional mMIMO [2] and network MIMO system [3] by assuming that a large number of access points (APs), distributed over a coverage area and coordinated by a central processing unit, serve coherently a smaller number of users in the same time-frequency resources [1]. However, we manage to enjoy the benefits of channel hardening and favorable propagation under certain conditions including multiple antenna APs and low path-losses [4].

Despite the increasing power consumption of CF mMIMO systems as the size of the network increases, only few prior works have studied their energy efficiency (EE) but not in closed-forms but by simulations [5]–[7]. For example, [7] investigated the EE in a user-centric approach with millimeter waves. In parallel, the inevitable irregularity of CF mMIMO systems as they become denser by increasing the number of APs has not

been described by a practical model describing the APs spatial randomness except the work in [4] that focused only on the effects of channel hardening and favorable propagation and not in the derivation of the achievable rate, and the work in [8], which relied on a large number of APs to apply deterministic equivalents and derive the coverage probability.

In this work, we address the EE per unit area of CF mMIMO systems, being of paramount importance for future networks, and contrary to existing works [5], [6], we provide closed-form expressions while we account for the spatial randomness of the APs during the analysis. Specifically, we present a novel analytical model for the downlink performance of CF mMIMO systems with Poisson point process (PPP) distributed multiple-antenna APs. Moreover, we derive a lower bound on the downlink average spectral efficiency (SE) by means of stochastic geometry and under a realistic power consumption CF mMIMO systems, we develop an EE maximization problem. Thus, we obtain the optimal EE per unit area with respect to the pilot reuse factor and AP density while we shed light on the impact of the main system parameters.

The remainder of this paper is structured as follows. Section II provides the system model of a CF massive MIMO system with multiple antennas APs that are PPP distributed. Section III presents the uplink training phase, and Section IV describes the downlink transmission phase. Section V provides the analysis regarding the EE per unit area and Section VI describes the optimization of the EE per unit area in terms of system parameters. The numerical results are given in Section VII, and Section VIII concludes the paper.

## II. SYSTEM MODEL

We consider a CF massive MIMO system with  $N \geq 1$  antennas per AP. Most importantly, we assume that the AP locations follow a homogeneous PPP  $\Phi_{AP}$  with intensity  $\lambda_{AP}$  [AP/km<sup>2</sup>]. Let  $M$  be the number of APs found in a region  $\mathcal{A}$  of size  $S(\mathcal{A})$  in a random realization of the PPP  $\Phi_{AP}$ . Then,  $M$  is a Poisson random variable with mean value

$$\mathbb{E}[M] = \lambda_{AP} S(\mathcal{A}). \quad (1)$$

Hence,  $\mathcal{W} = MN$ , denoting the total number of antennas in  $\mathcal{A}$ , is a Poisson random variable with mean  $\mathbb{E}[\mathcal{W}] = N\lambda_{\text{AP}}S(\mathcal{A})$ .

Based on the network MIMO concept, all the APs, connected via a perfect fronthaul network to a central processing unit for coding and decoding of the data signals, serve simultaneously all the single-antenna users on the same time-frequency resource. In particular, we assume that in a given  $\Phi_{\text{AP}}$ , the number of users, selected at random from a large set based on some scheduling algorithm, is given by  $K$  while their locations follow an independent stationary point process [9]. Note that the choice of densities should fulfill the condition  $\mathcal{W} \gg K$  [4].

Slivnyak's theorem allows the consideration of a typical user, selected at random among the users to analyze the network performance [10]. Especially, for the sake of exposition, we assume that the typical user, indexed by  $k$ , is located at the origin.

#### A. Channel Model

Let a given realization of the PPP  $\Phi_{\text{AP}}$  with  $M$  APs, the  $N \times 1$  channel vector  $\mathbf{h}_{mk}$  between the  $m$ th AP and the typical user is expressed as

$$\mathbf{h}_{mk} = l_{mk}^{1/2} \mathbf{g}_{mk}, \quad (2)$$

where  $l_{mk} = \min(1, r_{mk}^{-\alpha})$  and  $\mathbf{g}_{mk}$  represent independent path-loss and small-scale fading. The expression of the path-loss describes a non-singular bounded model, where  $\alpha > 0$  is the path-loss exponent and  $r_{mk}$  is the distance between the  $m$ th AP and the  $k$ th user [11]. The selection of this path-loss model is justified based on its suitability to describe short distances as in CF mMIMO systems [4]. Moreover, we assume that the distances between the  $m$ th AP located at  $\mathbf{x}_m$  in  $\mathbb{R}^2$  and the various users in  $\mathcal{A} \setminus \{\mathbf{x}_m \in \mathcal{A}\}$  follow the uniform distribution and are independent. In addition,  $\mathbf{g}_{mk}$  consists of identically distributed (i.i.d.)  $\mathcal{CN}(0, 1)$  random elements, i.e., we assume independent Rayleigh fading [1].

Let a narrowband channel that consists of coherence blocks with duration  $T_c$  in s and bandwidth  $B_c$  in Hz, i.e., we have  $\tau_c = B_c T_c$  samples while independent channel realizations appear in every block [12]. We consider the time-division-duplex (TDD) protocol with an uplink training phase of  $\tau_{\text{tr}}$  samples and two data transmission phases of  $\tau_{\text{d}}$  (downlink) and  $\tau_{\text{up}}$  (uplink) samples, respectively. Thus,  $\tau_c = \tau_{\text{tr}} + \tau_{\text{up}} + \tau_{\text{d}}$ . This work focuses on the downlink data transmission phase with duration that can be expressed by  $\tau_{\text{d}} = \xi(\tau_c - \tau_{\text{tr}})$  with  $\xi \leq 1$  expressing the downlink payload fraction transmission [13].

### III. UPLINK CHANNEL ESTIMATION

We assume  $K \gg \tau_{\text{tr}}$  and we introduce the reuse factor  $\zeta = K/\tau_{\text{tr}}$ , where  $\zeta$  is assumed an optimization variable. The physical meaning is that  $\mathbb{E}[\zeta]$  users share the same pilot sequences on the average.

The received  $N \times \tau_{\text{tr}}$  channel vector by the  $m$ th AP is given by

$$\tilde{\mathbf{y}}_m^{\text{tr}} = \sum_{i=1}^K \sqrt{\tau_{\text{tr}} \rho_{\text{tr}}} l_{mi}^{1/2} \mathbf{g}_{mi} \boldsymbol{\psi}_i^{\text{H}} + \mathbf{n}_m^{\text{tr}}, \quad (3)$$

where  $\rho_{\text{tr}}$  is the average transmit power and  $\boldsymbol{\psi}_k \in \mathbb{C}^{\tau_{\text{tr}} \times 1}$  with  $\|\boldsymbol{\psi}_k\|^2 = 1$  is the normalized pilot sequence transmitted by user  $k$  while  $\mathbf{n}_m^{\text{tr}}$  is the  $N \times \tau_{\text{tr}}$  additive noise vector at the  $m$ th AP consisted of i.i.d.  $\mathcal{CN}(0, 1)$  random variables.

According to [14] and given the distance statistics, the minimum mean-squared error (MMSE) estimate of the channel is given by  $\hat{\mathbf{h}}_{mk} = \mathbb{E}[\mathbf{h}_{mk} \tilde{\mathbf{y}}_{mk}^{\text{H}}] \mathbb{E}^{-1}[\tilde{\mathbf{y}}_{mk} \tilde{\mathbf{y}}_{mk}^{\text{H}}] \tilde{\mathbf{y}}_{mk}$  which results in

$$\hat{\mathbf{h}}_{mk} = \frac{l_{mk}}{\sum_{i=1}^K |\boldsymbol{\psi}_i \boldsymbol{\psi}_i^{\text{H}}|^2 l_{mi} + \frac{1}{\tau_{\text{tr}} \rho_{\text{tr}}}} \tilde{\mathbf{y}}_{mk}, \quad (4)$$

where  $\tilde{\mathbf{y}}_{mk}$  is obtained by projecting  $\tilde{\mathbf{y}}_m^{\text{tr}}$  onto  $\frac{1}{\sqrt{\tau_{\text{tr}} \rho_{\text{tr}}}} \boldsymbol{\psi}_k$ .

The estimation error vector  $\tilde{\mathbf{e}}_{mk} = \mathbf{h}_{mk} - \hat{\mathbf{h}}_{mk}$  is independent of  $\hat{\mathbf{h}}_{mk}$ . Moreover, it follows that  $\mathbf{h}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, l_{mk} \mathbf{I}_N)$ ,  $\hat{\mathbf{h}}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, \sigma_{mk}^2 \mathbf{I}_N)$  and  $\tilde{\mathbf{e}}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, \tilde{\sigma}_{mk}^2 \mathbf{I}_N)$ , where  $\sigma_{mk}^2 = \frac{l_{mk}^2}{d_m}$  and  $\tilde{\sigma}_{mk}^2 = l_{mk} \left(1 - \frac{l_{mk}}{d_m}\right)$  with  $d_m = \left(\sum_{i=1}^K |\boldsymbol{\psi}_i \boldsymbol{\psi}_i^{\text{H}}|^2 l_{mi} + \frac{1}{\tau_{\text{tr}} \rho_{\text{tr}}}\right)$ . To summarize, we have  $\mathbf{h}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, l_{mk} \mathbf{I}_N)$ ,  $\hat{\mathbf{h}}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, \sigma_{mk}^2 \mathbf{I}_N)$  and  $\tilde{\mathbf{e}}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(\mathbf{0}, \tilde{\sigma}_{mk}^2 \mathbf{I}_N)$ .

### IV. DOWNLINK TRANSMISSION

The received signal by the typical user is described equivalently by the following two expressions

$$y_k^{\text{d}} = \sqrt{\rho_{\text{d}}} \sum_{i \in \Phi_{\text{AP}}} \tilde{\mathbf{h}}_i^{\text{H}} \mathbf{s}_i + z_k^{\text{d}} \quad (5)$$

$$= \sqrt{\rho_{\text{d}}} \sum_{m=1}^M \mathbf{h}_{mk}^{\text{H}} \mathbf{s}_m + z_k^{\text{d}}, \quad (6)$$

where, in the first expression, the vector  $\tilde{\mathbf{h}}_i$  expresses the channel between the  $i$ th AP located at  $\mathbf{x}_i \in \mathbb{R}^2$  and the typical user,  $\rho_{\text{d}} > 0$  denotes the corresponding transmit power, and  $\mathbf{s}_i$  is the transmit signal from the  $i$ th AP while  $z_k^{\text{d}} \sim \mathcal{CN}(0, 1)$  is the additive white Gaussian noise at the  $k$ th user. Since a realization of the system includes  $M$  APs, the signal model described by (5) can be written as in (6). The second expression includes the vector  $\mathbf{h}_{mk}$  expressing the channel between the  $m$ th AP and the typical user while  $\mathbf{s}_m$  is the transmit signal from the  $m$ th AP, which is given as

$$\mathbf{s}_m = \sum_{k=1}^K \sqrt{\eta_{mk}} \mathbf{f}_{mk} q_k, \quad (7)$$

with  $q_k \in \mathbb{C}$  being the normalized transmit data symbol for user  $k$  satisfying  $\mathbb{E}[|q_k|^2] = 1$  and the vector  $\mathbf{f}_{mk} \in \mathbb{C}^N$  describing the linear precoder. Especially, we assume conjugate beamforming given by  $\mathbf{f}_{mk} = \hat{\mathbf{h}}_{mk}$ . We denote  $\eta_{mk} = \mu \sigma_{mk}^{-4}$ , where the parameter  $\mu$  is obtained by means of the constraint of

the transmit power  $\mathbb{E} \left[ \frac{\rho_d}{K} \mathbf{s}_m \mathbf{s}_m^H \right] = \rho_d$ . This selection regarding  $\eta_{mk}$ , corresponds to a statistical channel inversion power-control policy, which aims at easing the following algebraic manipulations [9]. It allows each AP to allocate more power to the most distant users and less power to the closest ones. Notably, the scaling does not result in any loss in the performance since the parameter  $\mu$  is changed accordingly.

Henceforth, for the sake of algebraic manipulations, we denote  $\mathbf{h}_k = [\mathbf{h}_{1k}^T \cdots \mathbf{h}_{Mk}^T] \sim \mathcal{CN}(\mathbf{0}, \mathbf{L}_k)$ ,  $\hat{\mathbf{h}}_k = [\hat{\mathbf{h}}_{1k}^T \cdots \hat{\mathbf{h}}_{Mk}^T] \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Phi}_k)$  and  $\tilde{\mathbf{e}}_k \in \mathbb{C}^{\mathcal{W} \times 1} \sim \mathcal{CN}(\mathbf{0}, \mathbf{L}_k - \mathbf{\Phi}_k)$ . The matrices  $\mathbf{L}_k \in \mathbb{C}^{\mathcal{W} \times \mathcal{W}}$ ,  $\mathbf{\Phi}_k = \mathbf{L}_k^2 \mathbf{D}^{-1} \in \mathbb{C}^{\mathcal{W} \times \mathcal{W}}$ , and  $\mathbf{D} \in \mathbb{C}^{\mathcal{W} \times \mathcal{W}}$  are block diagonal matrices with elements given by the matrices  $[\mathbf{L}_k]_{ww} = l_{mk} \mathbf{I}_N$ ,  $[\mathbf{\Phi}_k]_{ww} = \sigma_{mk}^2 \mathbf{I}_N$ ,  $[\mathbf{D}]_{ww} = d_m \mathbf{I}_N$ , and  $[\mathbf{D}]_{ww} = d_m \mathbf{I}_N$ , respectively, for  $w = 1, \dots, \mathcal{W}$  and  $\mathcal{W} = MN$ . We also define  $\mathbf{C}_k = \mathbf{\Phi}_k^{-1}$  with  $[\mathbf{C}_k]_{ww} = c_{mk} \mathbf{I}_N$ , where  $c_{mk} = \sigma_{mk}^{-2}$ . The received signal by the typical user is given by means of (6) and (7) as

$$y_k^d = \sqrt{\rho_d} \left( \mathbb{E} \left[ \sum_{m=1}^M \eta_{mk}^{1/2} \mathbf{h}_{mk}^H \hat{\mathbf{h}}_{mk} \right] q_k + \sum_{m=1}^M \eta_{mk}^{1/2} \mathbf{h}_{mk}^H \hat{\mathbf{h}}_{mk} q_k - \mathbb{E} \left[ \sum_{m=1}^M \eta_{mk}^{1/2} \mathbf{h}_{mk}^H \hat{\mathbf{h}}_{mk} \right] q_k + \sum_{i \neq k} \sum_{m=1}^M \eta_{mi}^{1/2} \mathbf{h}_{mi}^H \hat{\mathbf{h}}_{mi} q_i \right) + z_k^d, \quad (8)$$

where we have also followed the approach in [15] to derive below the SINR<sup>1</sup>. By treating the unknown terms as uncorrelated additive noise, we derive the effective SINR of the downlink transmission from all the multi-antenna APs to the typical user, conditioned on the number of APs and their distances from the users as

$$\gamma_k = \frac{\left| \mathbb{E} \left[ \mathbf{h}_k^H \mathbf{C}_k \hat{\mathbf{h}}_k \right] \right|^2}{\sum_{i=1}^K \mathbb{E} \left[ \left| \mathbf{h}_k^H \mathbf{C}_i \hat{\mathbf{h}}_i \right|^2 \right] - \left| \mathbb{E} \left[ \mathbf{h}_k^H \mathbf{C}_k \hat{\mathbf{h}}_k \right] \right|^2 + \frac{1}{\mu p_d}}. \quad (9)$$

*Proposition 1:* Given a realization of the network with  $M$  APs and  $K$  users, the effective SINR of the downlink transmission from the PPP distributed  $N$  antennas APs to the typical user in a CF massive MIMO system, accounting for pilot contamination and conjugate beamforming, is given by (10) at the top of the next page.

*Proof:* Herein, we shall omit the proof of Proposition 1, which is provided in [16] due to limited space. ■

We observe that the factor  $N$  in the numerator results due to the array gain from the coherent transmission of the  $N$  antennas per AP. Also, the increase in the number of users  $K$  decreases the SINR since  $K$  appears in the denominator in terms of the summations.

<sup>1</sup>This approach exploits channel hardening, which in general does not hold in CF massive MIMO systems with single-antenna APs. However, we assume that certain conditions are met that guarantee channel hardening [4].

## V. EE ANALYSIS

Reasonably, the increasing number of APs in CF massive MIMO systems is expected to increase power consumption. Hence, it is necessary to study the corresponding EE per unit area. Notably, the following definition is novel and also necessary to model CF massive MIMO systems, and in general, architectures with CoMP.

*Definition 1:* The EE per unit area expresses the amount of reliably transmitted information per unit of energy and area, which is defined mathematically as

$$\text{EE} \left[ \text{bit/Joule/km}^2 \right] = \frac{\text{Throughput} \left[ \text{bit/s} \right]}{\text{Area power consumption} \left[ \text{W/km}^2 \right]} = \frac{B_w \left[ \text{Hz} \right] \cdot \text{TSE} \left[ \text{bit/s/Hz} \right]}{\text{APC} \left[ \text{W/km}^2 \right]}, \quad (11)$$

where  $B_w$ , TSE, and APC describe the transmission bandwidth, the total SE (TSE), and the area power consumption (APC), respectively.

### A. Total Spectral Efficiency

The TSE, describing the total SE, is expressed as

$$\text{TSE} = KR \quad \left[ \text{bit/s/Hz} \right], \quad (12)$$

where  $R = R_k$  is the average SE per user over the channel realizations and AP locations since our analysis relies on user  $k$ , being statistical equivalent with any other user in the network. Note that  $K$  corresponds to the sum SE of all users.

Given that the downlink capacity for this network is not known, we present the following tractable lower bound as in [2], [17], [18], which holds for any given realization of  $\Phi_{\text{AP}}$ .

*Lemma 1* ([19]): A lower bound on the downlink ergodic channel capacity of the typical user  $k$  in a CF massive MIMO system with conjugate beamforming and PPP distributed APs for any given realization of  $\Phi_{\text{AP}}$  is provided by

$$R_k = \left( 1 - \frac{K}{\zeta \tau_c} \right) \log_2 (1 + \gamma_k) \quad \text{b/s/Hz}, \quad (13)$$

where  $K$  is the number of users,  $\zeta$  is the pilot reuse factor, and  $\tau_c$  is the channel coherence interval in number of samples while  $\gamma_k$  is given by (10).

The average SE per user is derived below by applying the expectation at (13) over the APs locations.

*Theorem 1:* A lower bound on the downlink average SE per user with conjugate beamforming precoding in a CF massive MIMO system with multi-antenna APs is obtained by

$$\check{R}_k = \left( 1 - \frac{K}{\zeta \tau_c} \right) \log_2 (1 + \bar{\gamma}_k), \quad (14)$$

where  $\bar{\gamma}_k = 1/\check{\gamma}_k$  with  $\check{\gamma}_k$  given by

$$\check{\gamma}_k = \sum_{j=1}^K |\psi_j \psi_k^H|^2 \left( \frac{\alpha - 2}{\alpha \pi N p_d} + K - 1 \right) + \frac{\zeta}{\alpha \pi K \rho_{\text{tr}}} \left( (K-1)(\alpha-2) + \frac{(\alpha-1)}{N p_d} \right) + \lambda_{\text{AP}} (K-1). \quad (15)$$

$$\gamma_k = \frac{M^2 N}{\sum_{i=1}^K \text{tr} \left( \mathbf{C}_i \left( N \mathbf{L}_k + \frac{1}{K p_d} \mathbf{I}_M \right) \right) + N \sum_{i \neq k}^K \text{tr}^2 \left( \mathbf{L}_k \mathbf{L}_i^{-1} \right) - N \text{tr} \left( \mathbf{D} \mathbf{L}_k^{-1} \right) + M}. \quad (10)$$

TABLE I  
SYSTEM PARAMETERS VALUES

Parameter	Value
Fixed power $P_{\text{FP}}$	5 W
Power for AP local oscillator $P_{\text{LO}}$	0.1 W
Power per AP antenna $P_{\text{AP}}$	0.2 W
Power per UE antenna $P_{\text{UE}}$	0.1 W
Power for data coding $P_{\text{COD}}$	0.01 W/(Gbit/s)
Power for data decoding $P_{\text{DEC}}$	0.08 W/(Gbit/s)
Power for backhaul traffic $P_{\text{BT}}$	0.025 W/(Gbit/s)
AP computational efficiency $L_{\text{AP}}$	750 Gflops/W
Power amplifier efficiency $\alpha_{\text{eff}}$	0.5

*Proof:* Herein, we shall omit the proof of Theorem 1, which is provided in [16] due to limited space. ■

### B. Area Power Consumption

Similar to the approach in [13], [17] but with certain modifications concerning CF massive systems, we have

$$\text{APC} = \lambda_{\text{AP}} \left( \frac{1}{\alpha_{\text{eff}}} P_{\text{TX}} + P_{\text{CPC}} \right), \quad (16)$$

where  $\alpha_{\text{eff}} \in (0, 1]$  is the power amplifier efficiency while  $P_{\text{TX}}$  and  $P_{\text{CPC}}$  are the power usage during the transmission and the circuitry of the system, respectively. In particular,  $P_{\text{TX}}$  includes both the average powers for the uplink pilot and downlink payload transmissions while  $P_{\text{CPC}}$  includes the circuitry dissipation in terms of cooling, power supply, etc.

*Proposition 2:* A generic realistic model for the downlink APC of CF massive MIMO systems is given by

$$\text{APC}(\boldsymbol{\theta}) = \lambda_{\text{AP}} \left( C_0 + C_1 K + C_2 K^2 + D_0 N + D_1 N K - D_2 N K^2 + \mathcal{A} B_w \text{TSE} \right), \quad (17)$$

where  $C_0 = P_{\text{FP}} + P_{\text{LO}}$ ,  $C_1 = \frac{B_w}{7 L_{\text{AP}} \tau_c} - \frac{\xi p_d}{\alpha_{\text{eff}} \zeta \tau_c} + P_{\text{UE}}$ ,  $C_2 = \frac{1}{\alpha_{\text{eff}} \zeta \rho_{\text{tr}} \tau_c}$ ,  $D_0 = P_{\text{AP}}$ ,  $D_1 = \frac{3 B_w}{L_{\text{AP}}} + \frac{3 B_w}{L_{\text{AP}} \tau_c}$ ,  $D_2 = \frac{3 B_w (\xi - 1)}{L_{\text{AP}} \zeta \tau_c}$ , and  $\mathcal{A} = (P_{\text{COD}} + P_{\text{DEC}} + P_{\text{BT}})$ .

*Proof:* See Appendix A. ■

The circuit power parameters, taken from [12], are listed in Table I.

## VI. EE MAXIMIZATION

This section presents the maximization of the constrained EE per unit area under generic hardware and transmission characteristics. Specifically, we focus on  $\boldsymbol{\theta} = (\zeta, \lambda_{\text{AP}})$  that obeys to the problem

$$\boldsymbol{\theta}^* = \arg \max_{\boldsymbol{\theta} \in \Theta} \text{EE}(\boldsymbol{\theta}) = \frac{B_w \text{TSE}(\boldsymbol{\theta})}{\text{APC}(\boldsymbol{\theta})} \quad (18)$$

subject to  $\bar{\gamma}_k(\boldsymbol{\theta}) = \gamma_0$ ,

where  $\text{TSE}(\boldsymbol{\theta})$  is given by (16) with  $\tilde{R}_k$  given by Theorem 1,  $\text{APC}(\boldsymbol{\theta})$  is provided by Proposition 2,  $\bar{\gamma}_k$  is obtained by Theorem 1 while  $\gamma_0 > 0$  is a design parameter. The constraint in (18) does not allow an unacceptable achievable rate under these parameters. Also,  $\lambda_{\text{AP}} \geq 0, \zeta \geq 1, K/\zeta \leq \tau_c$ . The superscript \* represents the optimal values. Note that the maximization with respect to  $K, N$  takes place in the journal version.

### A. Feasibility

The optimization problem in (18) is feasible for a certain range of values of  $\gamma_0$  because of the multiuser interference.

*Lemma 2:* The feasibility range of values of  $\gamma_0$ , obtained from the maximization problem for CF massive MIMO systems (18), is provided by

$$\gamma_0 < \frac{1}{\lambda_{\text{AP}}}. \quad (19)$$

*Proof:* We simplify the expression of the SINR, which is the inverse of (15) by observing that it is a monotonically increasing function of  $N$ . Thus, we derive its limit as  $N \rightarrow \infty$  as

$$\lim_{N \rightarrow \infty} \bar{\gamma}_k = \frac{\alpha \pi \rho_{\text{tr}} K}{\mathcal{G}}, \quad (20)$$

where  $\mathcal{G} = \alpha \pi K \left( \sum_{j=1}^K |\psi_j \psi_k^H|^2 (K-1) + K \lambda_{\text{AP}} \right) \rho_{\text{tr}} + (\alpha - 2) (K-1) \zeta$ . Given that the upper limit is a decreasing function of the optimizable variable  $\zeta$ , we use the constraint  $\zeta = K/\tau_{\text{tr}}$ , take its minimal value when  $K = 1$ , and, thus, we result in the feasible  $\gamma_0$ . ■

Based on this lemma, the upper limit of the SINR depends only on the AP density  $\lambda_{\text{AP}}$  as  $N \rightarrow \infty$ . The typical value regarding the number of APs in CF massive MIMO systems is 100 – 200 [1], being equivalent to a density  $\lambda_{\text{AP}} \approx 10^{-4} \text{ m}^{-2}$ . Hence, the average SE per user is  $\log_2(1 + 100) \approx 13.29 \text{ b/s/Hz}$ , which is larger than the SE of currently applied systems [20], i.e., (18) is feasible for practical systems.

### B. Optimal Pilot Reuse Factor

Herein, we present of the optimal pilot reuse factor  $\zeta^*$  while the rest of the parameters are fixed.

*Theorem 2:* Let any set of  $\{\lambda_{\text{AP}}, K, N\}$  resulting in the feasibility of the maximization of EE per unit area given by (18). The optimal pilot reuse factor, satisfying the SINR constraint, is obtained by

$$\zeta^* = \frac{\alpha \pi K N \rho_{\text{tr}} \rho_d - \gamma_0 Q_1}{\gamma_0 Q_2}. \quad (21)$$

*Proof:* We consider the constraint and we group the terms including  $\zeta$  in the SINR  $\bar{\gamma}_k = 1/\tilde{\gamma}_k$ , which results in

$$\gamma_o = \frac{\alpha\pi N\rho_{\text{tr}}\rho_{\text{d}}}{Q_1 - \zeta Q_2}, \quad (22)$$

where

$$\begin{aligned} Q_1 &= K\rho_{\text{tr}} \left( (\alpha - 2) \sum_{j=1}^K |\psi_j \psi_k^{\text{H}}|^2 \right. \\ &\quad \left. + \alpha\pi N\rho_{\text{d}} (K - 1) \left( \sum_{j=1}^K |\psi_j \psi_k^{\text{H}}|^2 + \lambda_{\text{AP}} \right) \right), \\ Q_2 &= (\alpha - 1 + N\rho_{\text{d}} (\alpha - 2) (K - 1)) / K, \end{aligned} \quad (23)$$

and we solve (22) with respect to  $\zeta$ . ■

Theorem 2 describes the dependence of  $\zeta^*$  on the rest of the system parameters. In particular, a smaller  $\zeta^*$ , being equivalent to a larger training phase, denotes more precise channel estimation, and subsequently, higher SE, which agrees with (21).

### C. Optimal APs Density

After inserting (21) into (18), the optimization problem is expressed as

$$\begin{aligned} \text{EE}(\zeta^*, K, N) &= \frac{B_w \text{ASE}(\zeta^*, K, N)}{\text{APC}(\zeta^*, K, N)} \\ \text{subject to } 1 &\leq \frac{\alpha\pi N\rho_{\text{tr}}\rho_{\text{d}} - \gamma_o Q_1}{\gamma_o Q_2} \leq \frac{K}{\tau_c}. \end{aligned} \quad (24)$$

*Theorem 3:* Let any set of  $\{K, N\}$  keeping the optimization problem (24) feasible. For fixed  $K$  and  $N$ , the EE per unit area is maximized by

$$\lambda_{\text{AP}}^* = \min(\max(\lambda_{\text{AP}_0}, \lambda_{\text{AP}_1}), \lambda_{\text{AP}_2}), \quad (25)$$

where

$$\lambda_{\text{AP}_0} = \frac{(a_1 + a_3)G + \sqrt{a_2 a_3 a_4 (a_1 + a_3)G}}{a_2 a_4 G} \quad (26)$$

with

$$G = a_2(a_4 + a_5 + a_6 K \log(1 + \gamma_o)) \quad (27)$$

while  $\lambda_{\text{AP}_1} = \frac{a_3 - a_1}{a_2}$ ,  $\lambda_{\text{AP}_2} = \frac{\tau_c / (K a_3) + a_1}{a_2}$ , and the parameters  $\{a_i\}$  are provided in Table II.

*Proof:* We observe that TSE and APC include the term  $\zeta^* \tau_c / K$ , which can be rewritten as  $\zeta^* \tau_c / K = \frac{a_2 \lambda_{\text{AP}} - a_1}{a_3}$ . By substituting this term into the objective function of (24), we obtain (28). Following the approach in [17, Lem. 3], it can be shown that (28) is a quasi-concave function of  $\lambda_{\text{AP}}$ . Thus, (26) is obtained by taking the first derivative of (28) and equating to zero. Given that the constraint in (24) depends on  $\lambda_{\text{AP}}$ , we obtain  $\lambda_{\text{AP}_1}$  and  $\lambda_{\text{AP}_2}$ . ■

## VII. NUMERICAL RESULTS

Let a sufficiently large squared area of  $1 \text{ km}^2$ , where the AP locations follow a PPP  $\Phi_{\text{AP}}$  with density  $\lambda_{\text{AP}} = 100 \text{ APs/km}^2$  based on a wraparound topology to keep the translation invariance. Also, let the transmission bandwidth be  $B_w = 20 \text{ MHz}$  and that each coherence block includes  $\tau_c = 200$  samples. In addition, we assume that  $N = 20$  antennas per AP and  $K = 10$  users in total while  $\zeta = 4$ . Moreover, we assume that  $\rho_{\text{tr}} = 100 \text{ mW}$ ,  $\rho_{\text{d}} = 200 \text{ mW}$ ,  $\alpha = 4$ , and  $\xi = 1/3$ . Note that we consider these values unless otherwise stated.

In Fig. 1, we evaluate the EE by varying the pilot reuse factor  $\zeta$  and AP density  $\lambda_{\text{AP}}$  for a given pair of  $K, N$  based on Theorems 2 and 3. In particular, it is shown that the EE is a pseudo-concave function with respect to  $\zeta$  with a unique global maximum at  $\zeta^* = 3$  while the corresponding optimal EE is  $\text{EE}^* = 5.92 \text{ Mbit/Joule}$ . Also, EE is a quasi-concave function with respect to  $\lambda_{\text{AP}}$  according to Theorem 3. Notably, this figure depicts the optimal value of the AP density being  $\lambda_{\text{AP}} = 25 \text{ APs/km}^2$ . For the sake of comparison, in Fig. 2, we have illustrated a conventional ‘‘cellular’’ massive MIMO scenario, where an AP with  $N = 20$  antennas is located per cell and  $K = 10$  users are served in total based on the work in [9]. Notably, we observe that CF massive MIMO systems present higher EE than conventional mMIMO systems while the required AP density is much lower.

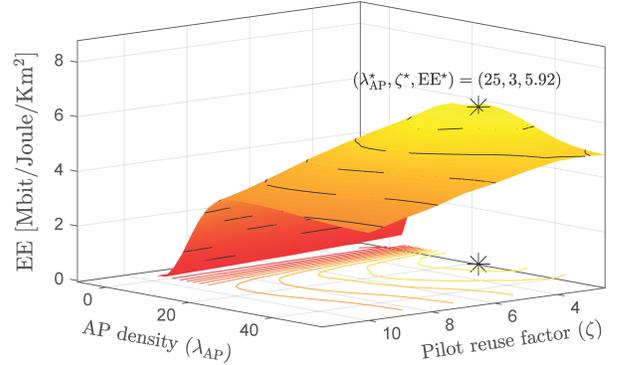


Fig. 1. Energy efficiency per unit area (Mbit/J/km<sup>2</sup>) of CF massive MIMO systems versus the AP density  $\lambda_{\text{AP}}$  and pilot reuse factor  $\zeta$ . The optimal EE per unit area is star-marked and the corresponding parameters are provided.

Fig. 3 shows the effect of the SINR constraint  $\gamma_o$  on the EE. Especially, we assume that  $\gamma_o \in \{1, 3, 7\}$  resulting in an average SE equal to 1, 2, and 3, respectively. It is shown the decrease of the EE with  $\gamma_o$  which notifies that the achievable SE should stay at a satisfactory level according to the specified requirements. Instead, we will end up in a highly energy-efficient system that will be useless with concern to the user experience because of low SE. Moreover, the gap between the lower and upper bounds is small, which denotes the tightness of the bound proposed by Theorem 1 and validates the various approximations. Moreover, we observe again that the

$$EE(\zeta^*) = \frac{\frac{K\xi}{\lambda_{AP}} \left(1 - \frac{a_3}{a_2\lambda_{AP}-a_1}\right) \log_2(1+\gamma)}{a_4 + a_5 \frac{a_3}{a_2\lambda_{AP}-a_1} + a_6 K\xi \left(1 - \frac{a_3}{a_2\lambda_{AP}-a_1}\right) \log_2(1+\gamma)} \quad (28)$$

TABLE II  
OPTIMIZATION PARAMETERS FOR OPTIMAL AP DENSITY  $\lambda_{AP}$

Parameter	Value
$a_1$	$\rho_{tr}\tau_c K \left( \alpha\pi N\rho_d - ((\alpha-2) + \alpha\pi N\rho_d)\gamma \sum_{j=1}^K  \psi_j \psi_k^H ^2 \right)$
$a_2$	$\alpha\pi\gamma\rho_d\rho_{tr}\tau_c K N (K-1)$
$a_3$	$\gamma K N \rho_d (\alpha-1 + (K-1)(\alpha-2))$
$a_4$	$C_0 + C_{11}K + N(D_0 + D_1K)$
$a_5$	$\frac{1}{\alpha_{eff}} ((\xi-1)\rho_d + K\rho_{tr}) - \frac{3BK\xi}{L_{AP}}$
$a_6$	$P_{COD} + P_{DEC} + P_{BT}$

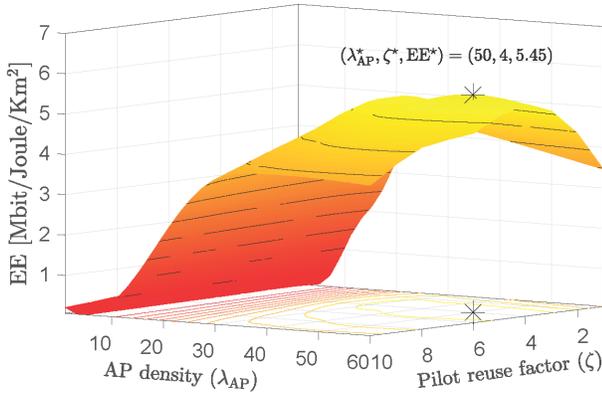


Fig. 2. Energy efficiency per unit area (Mbit/J/km<sup>2</sup>) of “cellular” mMIMO systems versus the AP density  $\lambda_{AP}$  and pilot reuse factor  $\zeta$ . The optimal EE is star-marked and the corresponding parameters are provided.

EE increases with  $\lambda_{AP}$ , and then, it decreases after a specific value of the AP density around  $\lambda_{AP} = 30$  APs/km<sup>2</sup> which is equivalent to a reasonable distance for practical deployments among the APs being 103 m approximately.

## VIII. CONCLUSION

We investigated the EE per unit area in CF mMIMO systems with multiple-antenna APs by accounting for their irregular spatial randomness. In particular, we considered that the APs are PPP distributed, and we introduced a realistic power consumption model for this architecture. Notably, we derived a closed-form lower bound on the ASE and achieved to obtain closed-form expressions regarding the optimal EE per unit area in terms of system variables being the pilot reuse factor and APs density. Hence, we provided valuable fundamental conclusions regarding the impact of these parameters on the EE per unit area.

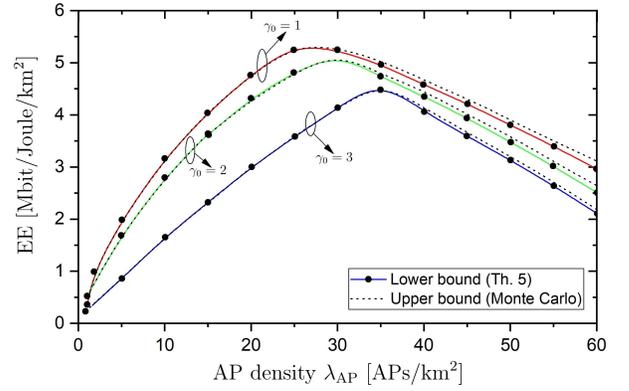


Fig. 3. Energy efficiency per unit area (Mbit/J/km<sup>2</sup>) of CF mMIMO systems versus the AP density  $\lambda_{AP}$  for different SINR constraints. “Solid-bullet” and “dashed” lines correspond to the lower bound due to the Theorem 1 and upper bound due to Monte Carlo simulation of the average SE.

## APPENDIX A PROOF OF PROPOSITION 2

The proof includes two parts. It starts with the expression of  $P_{TX}$  by means of a lemma, and then continues with the presentation of  $P_{CPC}$ .

*Lemma 3:* The total average transmit power consumption due to uplink pilot and downlink data transmissions of an arbitrary AP is

$$P_{TX} = K \frac{K/\zeta\rho_{tr} + \tau_d\rho_d}{\tau_c}, \quad (29)$$

where  $\tau_d = \xi(\tau_c - \tau_{tr})$ .

*Proof:* In each coherence block, each user transmits pilot symbols for a fraction of  $\tau_{tr}/\tau_c$  with power  $\rho_{tr}$ , where  $\tau_{tr} = K/\zeta$ , while each AP transmits data symbols for a fraction of  $\tau_d/\tau_c$  with power  $\rho_d$ . ■

Based on [13], the second part of (16), concerning the  $P_{CPC}$  of an arbitrary AP, is given by

$$P_{CPC} = P_{FP} + P_{TC} + P_{C-BC} + P_{CE} + P_{LP}, \quad (30)$$

where these terms correspond to the power consumptions of circuitry parts. Specifically,  $P_{FP}$  expresses the power consumed for site-cooling and control signaling and the traffic-independent mixed power consumption of each backhaul,  $P_{TC}$  for the transceiver chain,  $P_{C-BC}$  for coding and load-dependent backhauling cost, while  $P_{CE}$  and  $P_{LP}$  describe the powers consumed for the processes of channel estimation process and linear processing. Actually, each term depends on the system parameters. In particular, we have that  $P_{TC} = NP_{AP} + P_{LO} + KP_{UE}$ , where  $P_{AP}$ ,  $P_{LO}$ , and  $P_{UE}$  are the powers per AP antenna, AP local oscillator, and the power per user antenna. Furthermore, we have  $P_{C-BC} = B_w ASE(P_{COD} + P_{DEC} + P_{BT})$ , where the terms from left to right denote the bandwidth, the powers for data coding and decoding as well as the total power for the backhaul traffic. Concerning the computation of  $P_{CE}$ , we have that the MMSE estimation involves  $N\tau_d$  and  $N$  operations for the calculations of  $\psi_k^H \tilde{y}_m^{tr}$  and  $\hat{\mathbf{h}}_{mk}$ . In total, the MMSE estimation requires  $KN(\tau_{tr} + 1)$  operations needing 3 flops per operation with AP computational efficiency  $\alpha_{eff}$ . Since this procedure takes  $\frac{B_w}{\tau_c}$  coherence blocks per second and  $\tau_{tr} = \frac{K}{\zeta}$ , we have

$$P_{CE} = \frac{3}{L_{AP}} \frac{B_w}{\tau_c} KN \left( \frac{K}{\zeta} + 1 \right). \quad (31)$$

The linear processing power  $P_{LP}$  is a sum of the powers consumed by precoding/transmitting the data and computation of the precoder, i.e.,  $P_{LP_t}$  and  $P_{LP_p}$ , respectively. Hence, we have

$$P_{LP} = P_{LP_t} + P_{LP_p}, \quad (32)$$

where  $P_{LP_t} = \frac{3}{L_{AP}} \frac{B_w}{\tau_c} KN \xi(\tau_c - \tau_{tr})$  with  $\tau_{tr} = \frac{K}{\zeta}$ , and the power consumed by the conjugate beamformer is given by [12], [13] as  $P_{LP_p} = \frac{B_w K}{7\tau_c L_{AP}}$ . Substituting (29) and the power expressions in (30) into (16), we conclude the proof.

## REFERENCES

- [1] H. Q. Ngo *et al.*, "Cell-free massive MIMO versus small cells," *IEEE Trans. Wireless Commun.*, vol. 16, no. 3, pp. 1834–1850, 2017.
- [2] T. Marzetta, "Noncooperative cellular wireless with unlimited numbers of base station antennas," *IEEE Trans. Wireless Commun.*, vol. 9, no. 11, pp. 3590–3600, November 2010.
- [3] S. Shamai and B. M. Zaidel, "Enhancing the cellular downlink capacity via co-processing at the transmitting end," in *IEEE VTS 53rd Vehicular Technology Conference, VTC 2001 Spring*, vol. 3, pp. 1745–1749.
- [4] Z. Chen and E. Björnson, "Channel hardening and favorable propagation in cell-free massive MIMO with stochastic geometry," *IEEE Trans. Commun.*, vol. 66, no. 11, pp. 5205–5219, 2018.
- [5] H. Q. Ngo *et al.*, "On the total energy efficiency of cell-free massive MIMO," *IEEE Trans. Green Commun. Net.*, vol. 2, no. 1, pp. 25–39, 2018.
- [6] L. D. Nguyen *et al.*, "Energy efficiency in cell-free massive MIMO with zero-forcing precoding design," *IEEE Commun. Lett.*, vol. 21, no. 8, pp. 1871–1874, Aug 2017.
- [7] M. Alonzo *et al.*, "Energy-efficient power control in cell-free and user-centric massive MIMO at millimeter wave," *IEEE Trans. Green Commun. Net.*, 2019.
- [8] A. Papazafeiropoulos *et al.*, "Performance analysis of cell-free massive MIMO systems: A stochastic geometry approach," *IEEE Trans. Veh. Tech.*, vol. 69, no. 4, pp. 3523–3537, 2020.
- [9] E. Björnson, L. Sanguinetti, and M. Kountouris, "Deploying dense networks for maximal energy efficiency: Small cells meet massive MIMO," *IEEE J. Sel. Areas Commun.*, vol. 34, no. 4, pp. 832–847, 2016.
- [10] S. N. Chiu *et al.*, *Stochastic geometry and its applications*. John Wiley & Sons, 2013.
- [11] M. Haenggi, R. K. Ganti *et al.*, "Interference in large wireless networks," *Foundations and Trends® in Networking*, vol. 3, no. 2, pp. 127–248, 2009.
- [12] E. Björnson, J. Hoydis, and L. Sanguinetti, "Massive MIMO networks: Spectral, energy, and hardware efficiency," *Foundations and Trends® in Signal Processing*, vol. 11, no. 3–4, pp. 154–655, 2017.
- [13] A. Pizzo *et al.*, "Network deployment for maximal energy efficiency in uplink with multislope path loss," *IEEE Trans. Green Commun. Net.*, vol. 2, no. 3, pp. 735–750, 2018.
- [14] S. Verdú, *Multuser detection*. Cambridge university press, 1998.
- [15] M. Medard, "The effect upon channel capacity in wireless communications of perfect and imperfect knowledge of the channel," *IEEE Trans. Inf. Theory*, vol. 46, no. 3, pp. 933–946, May 2000.
- [16] A. Papazafeiropoulos *et al.*, "Towards optimal energy efficiency in Cell-Free Massive MIMO systems," *arXiv preprint arXiv:2005.07459*, 2020.
- [17] E. Björnson *et al.*, "Optimal design of energy-efficient multi-user MIMO systems: Is massive MIMO the answer?" *IEEE Tran. Wireless Commun.*, vol. 14, no. 6, pp. 3059–3075, June 2015.
- [18] A. K. Papazafeiropoulos and T. Ratnarajah, "Deterministic equivalent performance analysis of time-varying massive MIMO systems," *IEEE Trans. Wireless Commun.*, vol. 14, no. 10, pp. 5795–5809, 2015.
- [19] J. Hoydis, S. ten Brink, and M. Debbah, "Massive MIMO in the UL/DL of cellular networks: How many antennas do we need?" *IEEE J. Select. Areas Commun.*, vol. 31, no. 2, pp. 160–171, February 2013.
- [20] H. Holma and A. Toskala, *LTE for UMTS: Evolution to LTE-Advanced*, Wiley, Ed., 2011.