Optimal Control of (Min,+) Linear Time-Varying Systems

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Abstract

The class of discrete event dynamic systems involving only synchronization phenomena can be seen as linear time-invariant systems in a particular algebraic structure called $(\min,+)$ *algebra. In the same framework, this paper deals with linear time-varying systems, that is, systems whose parameters may change as functions of time. For example, in a manufacturing system the number of working machines, or the number of trains running in a closed network of railway connections, can vary as functions of time. For such systems, the output tracking problem is optimally solved under just-in-time criterion.*

1. Introduction

A linear system theory analogous to the conventional theory has been developed for a particular class of Discrete Event Dynamic Systems (DEDS) subject to synchronization phenomena. Such systems - usually represented with Timed Event Graphs (TEG) - can be modeled by $(\min,+)$ linear equations. General concepts such as state space, impulse response and transfer function have been introduced [2], [10], [4]. These systems are seen as linear time-invariant (or stationary) systems over the $(\min,+)$ algebraic structure. An optimal solution to the output tracking problem under just in time criterion has also been given [2, §5.6], [7].

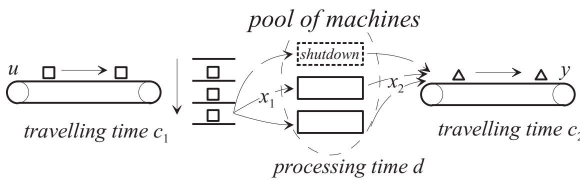
In this paper, we generalize the synthesis of this optimal control to $(\min,+)$ linear *time-varying* systems. We propose a basic example which aims at illustrating the class of systems as well as the 'optimal tracking problem' considered. We study here the simple manufacturing system of figure 1.a which operates as follows. Parts come into a workshop and reach a FIFO storage after a travelling time c_1 on a first conveyor. This storage is located upstream a pool of machines working in parallel. Each part is handled as soon as possible by some machine, and spends d units of time on a machine. The number of machines on running (idle or busy) is a function of time, due for example to planned maintenance or manufacturing resource scheduling. Once processed, parts leave the workshop on a second conveyor (travelling time c_2).

Let us consider the following variables for this system:

- $u(t)$: cumulated number of raw parts released on the first conveyor up to time t,
- $x_1(t)$: cumulated number of parts having left the storage up to time t,
- $x_2(t)$: cumulated number of parts loaded on the second conveyor up to time t,
- $y(t)$: cumulated number of finished parts up to time t,
- $a(t)$: number of working machine(s) at time t.

Notice that u, x_1, x_2, y are non-decreasing functions, usually called *counter* functions [2, §5], whereas a may be not monotonic.

(a)



(b)

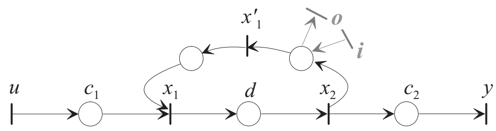


Figure 1: A manufacturing system (a), and its Petri net model (b)

The travelling time c_1 on the first conveyor implies that:

$$\forall t,\ x_1(t) \leq u(t - c_1)\ ,$$

on the other hand, since the processing time of a part is equal to d units of time, and assuming that a machine can be shutdown only when it is idle, we have:

$$\forall t,\ x_1(t) \leq a(t) + x_1(t-d)\ ,$$

hence, considering that a part is handled as soon as possible, we obtain:

$$\forall t,\ x_1(t) = \min\left[a(t) + x_1(t-d)\ ,\ u(t-c_1)\right]\ .$$

Further, we have that $\forall t$,

$$x_2(t) = x_1(t-d), \text{and } y(t) = x_2(t-c_2)\ .$$

Finally, the manufacturing system obeys the following recurrent equations where $\min$ and $+$ are replaced respectively by $\oplus$ and $\otimes$ (as in $(\min,+)$ algebra):

$$\left\{\begin{array}{lcl} x_1(t) & = & a(t) \otimes x_1(t-d) \oplus u(t-c_1) \\ x_2(t) & = & x_1(t-d) \\ y(t) & = & x_2(t-c_2) \end{array}\right. .$$

The question we shall address in this paper can be formulated as follows. Being given:

- the dates of activation and shutdown of the machines (*i.e.*, $a(t)$ is known $\forall t$),
- an output trajectory to be tracked, denoted $\{z(t)\}_{t\in\mathbb{Z}}$ (the customer demand),

what are the *latest dates* of release of raw parts which allow meeting the customer demand? In other words, we will compute the *least input trajectory* $\{u(t)\}_{t\in\mathbb{Z}}$ such that the *output response* $\{y(t)\}_{t\in\mathbb{Z}}$ is *greater* than the *customer demand* $\{z(t)\}_{t\in\mathbb{Z}}$.

In figure 1.b, the manufacturing system is modeled by a Petri net (see among others [8] for an exhaustive presentation of Petri net theory). The workshop, which involves exclusively synchronization phenomena, is classically modeled by a TEG model (black part of the graph). The variation of the number of working machines is enabled thanks to the additional transitions labeled i and o (grey part of the graph). In [6], we have shown using the general algebraic model of Petri nets presented in [3] that a subclass of Free Choice Petri Net, and in particular the graph of fig. 1, can be modeled after various manipulations by the following standard linear equations in $(\min,+)$ algebra:

$$\left\{\begin{array}{l} x(t) = A(t) \otimes x(t-1) \oplus B(t) \otimes u(t) \\ y(t) = C(t) \otimes x(t) \end{array}\right. . \qquad (1)$$

If $d = 1$, $c_1 = c_2 = 0$ unit of time, we have:

$$x(t) = [x_1(t)\ \ x_2(t)]^T, u(t) = [u(t)], y(t) = [y(t)]$$

and

$$A(t) = \begin{pmatrix} a(t) & +\infty \\ 0 & +\infty \end{pmatrix}, B(t) = \begin{pmatrix} 0 \\ +\infty \end{pmatrix},$$

$$C(t) = \begin{pmatrix} +\infty & 0 \end{pmatrix}.$$

As an illustration, let us explain briefly how to obtain these evolution equations (see [2],[3] for exhaustive presentations of algebraic modelling of Petri nets). To describe the behavior of the graph, firings of transitions are counted during its evolution. For that, let us define the variable $x_1(t)$ to denote the cumulated number of firings of transition labeled x_1 at time t (and identically for u, x_1', x_2, y, i, o). We have the obvious inequalities:

$$x_1(t) \leq \min\left(u(t), x_1'(t)\right),$$

$$x_2(t) \leq x_1(t-1),$$

$$y(t) \leq x_2(t).$$

Considering that the Petri net operates as soon as possible, *i.e.*, a transition is fired as soon as it is enabled, we have equalities:

$$x_1(t) = \min\left(u(t), x_1'(t)\right) = u(t) \oplus x_1'(t),$$

$$x_2(t) = x_1(t-1),$$

$$y(t) = x_2(t).$$

The equation for x_1' requires more attention because the upstream place has two output transitions, x_1' and o. Such a structure is referred to as a *conflict* and exhibits a nondeterminism. The classical approach for solving conflicts, called *race policy*, comes down to considering that, among the conflicting transitions, the first one to be ready to fire ”wins the race” and fires (the other transition ”loses the race”, and its enabling is preempted by the conflict). Note that in our Petri net, conflicting transitions have null firing times and are not synchronized (only one upstream place), so that the race policy would always result in a tie and could not decide which one is going to fire. With another approach, called *preselection policy* (or *routing policy*), conflicts are solved thanks to a protocol, algorithm or mapping which selects one of the conflicting transitions to fire. We consider here that conflicts are solved according to the preselection policy, assuming moreover that variables $i(t)$ as well as $o(t)$ are known or measured *a priori* (as exogenous data). With this assumption, we can write:

$$x_1'(t) + o(t) = x_2(t) + i(t),$$

or,

$$x_1'(t) = a(t) + x_2(t) = a(t) \otimes x_2(t),$$

where $a(t) = i(t) - o(t)$ is a known parameter.
Finally, we obtain:

$$\begin{aligned} x_1(t) &= x_1'(t) \oplus u(t) = a(t) \otimes x_2(t) \oplus u(t) \\ &= a(t) \otimes x_1(t-1) \oplus u(t), \end{aligned}$$

$$x_2(t) = x_1(t-1),$$

$$y(t) = x_2(t),$$

which yields to Eqs. (1).
The Eqs. (1) only differ from the standard ones of TEG by the fact that elements of matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are functions of time.
The outline of the paper is as follows. In §2, we recall the elements of (min,+) algebra and residuation which we shall use throughout the paper. In §3, we study linear systems over dioids. In particular, the impulse response of time-varying systems is expressed from their state model (1). The optimal control is presented and illustrated through a short example in §4 and §5.

2. Preliminaries

2.1. Dioid

A **dioid** is a set $\mathcal{D}$ endowed with two inner operations ($\oplus$, $\otimes$) such that:

- both $\oplus$ and $\otimes$ are associative and have neutral elements denoted respectively ε and e,
- $\otimes$ is distributive with respect to $\oplus$,
- ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}$, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$),
- $\oplus$ is idempotent ($\forall a \in \mathcal{D}$, $a \oplus a = a$).

If $\otimes$ is commutative, $\mathcal{D}$ is called a **commutative** dioid.
In any dioid, a natural **order** is defined by:

$$a \preceq b \Leftrightarrow a \oplus b = b\ .$$

$(\mathcal{D}, \preceq)$ is a **complete dioid** if each subset A of $\mathcal{D}$ admits a least upper bound denoted

$$\bigoplus_{x \in A} x = \sup_{x \in A} x,$$

and if $\otimes$ distributes with respect to infinite sums. In particular,

$$T = \bigoplus_{x \in \mathcal{D}} x = \sup_{x \in \mathcal{D}} x,$$

is the greatest element of $\mathcal{D}$.
In a complete dioid, the greatest lower bound, noted $\wedge$, always exists;

$$a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x.$$

Example 1 Let $\overline{\mathbb{Z}}_{min}$ be the set $\mathbb{Z} \cup \{\pm\infty\}$ endowed with min as $\oplus$ and usual addition as $\otimes$. It is a complete commutative dioid with neutral elements $\varepsilon = +\infty$ and $e = 0$ ($T = -\infty$). Note that order ($\preceq$) in $\overline{\mathbb{Z}}_{min}$ is just reversed with respect to the usual order ($\leq$).

Example 2 In this paper, we consider counter functions, *i.e.*, non-decreasing functions: $\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_{min}$. This set, denoted Σ, can be endowed with

- pointwise $\min$ as $\oplus$

$$(u \oplus v)(t) = u(t) \oplus v(t) = \min(u(t), v(t)),$$

- the sup-convolution as multiplication, noted $*$

$$(u{*}v)(t) = \bigoplus_{s \in \mathbb{Z}} [u(t{-}s) {\otimes} v(s)] = \min_{s \in \mathbb{Z}} [u(t{-}s) {+} v(s)].$$

(Σ, $\oplus$, $*$) is a complete dioid with neutral elements defined by

$$\varepsilon(t) = +\infty, \forall t \in \mathbb{Z}, \text{and } e(t) = \begin{cases} 0 & , \quad t \leq 0 \\ +\infty & , \quad t > 0 \end{cases}.$$

The natural order over this dioid is defined by:

$$u \preceq v \Leftrightarrow u(t) \preceq v(t),\ \forall t \in \mathbb{Z}.$$

Theorem 1 (see [2, §4.5.3]) *In a complete dioid, the particular implicit equation*

$$x = a \otimes x \oplus b$$

admits $a^* b$ as least solution, with

$$a^* = \bigoplus_{i \geq 0} a^i \ \text{ and } \ a^0 = e.$$

Example 3 Starting from a "scalar" dioid $\mathcal{D}$, let us consider $p \times p$ matrices with entries in $\mathcal{D}$. The sum and product of matrices are defined conventionally from the sum and product of scalars. This set of matrices endowed with these two operations is also a dioid denoted $\mathcal{D}^{p \times p}$. Note that n-dimensional row or column vector problems can be handled by embedding such vectors in square matrices with $n-1$ additional arbitrary (identically equal to ε) rows or columns.

2.2. Residuation

Residuation is a general notion in lattice theory which allows defining 'pseudo-inverses' of some isotone maps (f is isotone if $a \preceq b \Rightarrow f(a) \preceq f(b)$).
Laws $\oplus$ and $\otimes$ of a dioid are not invertible in general. Residuation is hence used to 'solve' equations of the type $a \otimes x = b$, $x \otimes a = b$. We will use residuation here to find 'greatest subsolutions' of such equations.

Definition 1 *An isotone map $f : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are ordered sets, is said to be* ***residuated*** *if it exists an isotone map $h : \mathcal{D} \to \mathcal{C}$ such that*

$$f \circ h \preceq Id_{\mathcal{D}} \text{ , and } h \circ f \succeq Id_{\mathcal{C}}.$$

$Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity maps of $\mathcal{C}$ and $\mathcal{D}$ respectively. h is unique and is denoted $f^\sharp$. It is called ***the residual*** *of f.*

If f is residuated then $\forall y \in \mathcal{D}$, the least upper bound of subset $\{x \in \mathcal{C} | f(x) \preceq y\}$ exists and belongs to this subset. This greatest subsolution is equal to $f^\sharp(y)$.

Property 1 (see [2, §4.4]) *Let $\mathcal{C}$ a complete dioid, the isotone map $L_a : x \to a \otimes x$ defined on $\mathcal{C}$ is residuated. The greatest solution of inequation $a \otimes x \preceq b$ therefore exists and is equal to ${L_a}^\sharp(b)$, also denoted $a \backslash b$ or $\frac{b}{a}$.*
The isotone map $R_a : x \to x \otimes a$ is also residuated. The greatest solution of inequation $x \otimes a \preceq b$ will be denoted b / a or $\frac{b}{a}$.
These 'quotients' satisfy the following formulæ

$$a \otimes (a \backslash x) \preceq x \qquad (x / a) \otimes a \preceq x \qquad (f.1)$$

$$a \backslash (x \wedge y) = (a \backslash x) \wedge (a \backslash y) \quad (x \wedge y) / a = (x / a) \wedge (y / a) \quad (f.2)$$

$$(a \otimes b) \backslash x = \frac{a \backslash x}{b} \qquad x / (a \otimes b) = \frac{x / b}{a} \qquad (f.3)$$

Let us note that formula $f.1$ is a simple deduction from definition 1:

$$L_a \circ {L_a}^\sharp \preceq Id_{\mathcal{C}} \Leftrightarrow \left(L_a \circ {L_a}^\sharp\right)(x) = a \otimes (a \backslash x) \preceq x,$$

and,

$$R_a \circ {R_a}^\sharp \preceq Id_{\mathcal{C}} \Leftrightarrow \left(R_a \circ {R_a}^\sharp\right)(x) = (x / a) \otimes a \preceq x.$$

Let us recall a necessary and sufficient condition for a map defined on complete dioids to be residuated.

Theorem 2 (see [2, §4.4.2, th. 4.50]) *Let f be an isotone mapping from a complete dioid $\mathcal{C}$ into a complete dioid $\mathcal{D}$. The map f is residuated if, and only if, $f(\varepsilon) = \varepsilon$, and for every subset X of $\mathcal{C}$*

$$f\left(\bigoplus_{x \in X} x\right) = \bigoplus_{x \in X} f(x) \; .$$

3. Linear systems

3.1. Representation of linear systems

A system $\mathcal{S}$ is a mapping from the set of admissible input signals to the set of admissible output signals. In this paper, the signals of interest are 'counters', *i.e.*, non-decreasing functions: $\mathbb{Z} \to \overline{\mathbb{Z}}_{min}$. In example 2, we have denoted Σ this set of signals. So that it constitutes a set of admissible signals, this set must be endowed with a kind of vector space structure by defining the two following operations:

- pointwise minimum (*i.e.*, addition in $\overline{\mathbb{Z}}_{min}$) of time functions, which plays the role of inner addition of signals:

$$\forall t, \; (u \oplus v)(t) \triangleq u(t) \oplus v(t) = \min\left(u(t), v(t)\right);$$

- addition of (*i.e.*, product in $\overline{\mathbb{Z}}_{min}$ by) a constant, which plays the role of external product of a signal with a scalar:

$$\forall t, \; (a \cdot u)(t) \triangleq a \otimes u(t) = a + u(t).$$

In [4], [9], a theory for systems defined on these structures of signals has been developed by analogy with conventional system theory. In the following, we recall some results fitted to our framework.

Definition 2 *A system $\mathcal{S}$ is called linear over $\overline{\mathbb{Z}}_{min}$, or* **$(\min,+)$** ***linear****, if*

$$\mathcal{S}(u_1 \oplus u_2) = \mathcal{S}(u_1) \oplus \mathcal{S}(u_2) = \min\left(\mathcal{S}(u_1), \mathcal{S}(u_2)\right)$$

and, $\forall a \in \mathbb{Z}$,

$$\mathcal{S}(a \cdot u(\cdot)) = a \otimes \mathcal{S}(u(\cdot)) = a + \mathcal{S}(u(\cdot)).$$

Definition 3 *A system is said to be* ***continuous*** *if, for any finite or infinite collection $\{u_i\}_{i \in I}$, it satisfies*

$$\mathcal{S}\left(\bigoplus_{i \in I} u_i\right) = \bigoplus_{i \in I} \mathcal{S}(u_i).$$

Definition 4 *A system is said to be* ***causal*** *if $\forall u_1, u_2$,*

$$u_1(t) = u_2(t) \text{ for } t < \tau \Rightarrow [\mathcal{S}(u_1)](t) = [\mathcal{S}(u_2)](t) \text{ for } t < \tau.$$

The notion of **impulse response** has also been introduced in [4, chap. V], [9], [2]. In particular, for systems defined on Σ, we have the following characterization [4, chap. V, §3.2].

Theorem 3 *Let $\mathcal{S}$: $\Sigma \to \Sigma$ be a linear continuous system, then there exists a unique mapping $h : \mathbb{Z}^2 \to \overline{\mathbb{Z}}_{min}$ (called* ***impulse response****) such that*

1. $\forall s \in \mathbb{Z}, t \to h(t,s) \in \Sigma$
2. $\forall t \in \mathbb{Z}, s \to h(t,-s) \in \Sigma$
3. *$y = \mathcal{S}(u)$ can be obtained by $y(t) = \bigoplus_{s \in \mathbb{Z}} [h(t,s) \otimes u(s)], \forall t \in \mathbb{Z}$.*

For systems defined on Σ, the **impulse** is the signal denoted e_0 and defined by

$$e_0(t) = \begin{cases} e\ (=0) & ,\quad t \leq 0 \\ \varepsilon\ (=+\infty) & ,\quad t > 0 \end{cases} .$$

When the system is modeled by a Petri net (as in introduction), such an input comes down to firing the source transition u an infinity of times after time 0 (so that an infinity of tokens are released).

Corollary 1 *A linear continuous system $\mathcal{S}$ over Σ is causal if, and only if, its impulse response h satisfies*

$$h(t,s) = h(t,t), \text{ for } s > t.$$

Remark 1 As in conventional linear theory, the impulse response $h(t,j)$ of a *time-invariant* (or *stationary*) system only depends upon the difference $t-j$.

From now on, we will only consider causal continuous linear systems, and for sake of briefness, we will most of times only write 'linear systems'.

3.2. Input/output relationship of $(\min$, +) linear time-varying systems

Starting from the standard state model (1), we will here explicit the input/output relationship and identify the impulse response of such $(\min,+)$ linear time-varying systems.
The first equation of (1) can also be written,

$$x(t) = \Phi(t,t_0)x(t_0) \oplus \bigoplus_{j=t_0+1}^{t} \Phi(t,j)B(j)u(j)\ ,\ t \geq t_0$$

where the *state-transition matrix* $\Phi(t,i)$ is given by

$$\Phi(t,i) = \begin{cases} \text{not defined} & ,i > t \\ Id & ,i = t \\ A(t) \otimes A(t-1) \otimes \cdots \otimes A(i+1) & ,i < t \end{cases}$$

Then we have, for $t \geq t_0$

$$y(t) = C(t)\Phi(t,t_0)x(t_0) \oplus \bigoplus_{j=t_0+1}^{t} C(t)\Phi(t,j)B(j)u(j). \quad (2)$$

Remark 2 The state-transition matrix satisfies the composition property

$$\Phi(t,i) = \Phi(t,k) \otimes \Phi(k,i)\ , \text{ where } t \geq k \geq i\ ,$$

and in particular for $t \geq i+1$

$$\begin{aligned} \Phi(t,i) &= A(t)\Phi(t-1,i), \\ &= \Phi(t,i+1)A(i+1). \end{aligned}$$

Proposition 1 *The least solution of Eqs. (1) is given by*

$$\forall t \in \mathbb{Z}, \quad \overline{y}(t) = \bigoplus_{j \leq t} h(t,j)u(j) \quad (3)$$

with

$$h(t,j) = C(t)\Phi(t,j)B(j), \text{ for } j \leq t. \quad (4)$$

Proof By tending t_0 towards $-\infty$ in Eq. (2), it is clear that any solution is greater than $\overline{y}$.
Setting $\overline{y}(t) = C(t)\overline{x}(t)$ with

$$\overline{x}(t) = \bigoplus_{j \leq t} \Phi(t,j)B(j)u(j),$$

we show that $\overline{x}$ satisfies the first equation of (1):

$$\begin{aligned} \overline{x}(t) &= \textstyle\bigoplus_{j \leq t} \Phi(t,j)B(j)u(j) \\ &= \textstyle\bigoplus_{j \leq t-1} \Phi(t,j)B(j)u(j) \oplus B(t)u(t) \\ &= A(t)\left[\textstyle\bigoplus_{j \leq t-1} \Phi(t-1,j)B(j)u(j)\right] \oplus B(t)u(t) \\ &\qquad \text{(thanks to rem. 2)} \\ &= A(t)\overline{x}(t-1) \oplus B(t)u(t) \end{aligned}$$

$\square$

The expression of impulse response h can be extended in a causal manner (see corollary 1) by setting

$$h(t,j) = C(t)B(t) \text{ for } j > t,$$

which yields to:

$$\overline{y}(t) = \bigoplus_{j \leq t} h(t,j)u(j) = \bigoplus_{j \in \mathbb{Z}} h(t,j)u(j) \quad (3')$$

since for $j > t$, $u(j) \preceq u(t)$, and by isotony of $\otimes$:

$$h(t,j)u(j) = h(t,t)u(j) \preceq h(t,t)u(t)$$

Remark 3 For conventional discrete-time linear time-varying systems [5], [1], the input/output relationship is given by:

$$y(k) = \sum_{j=-\infty}^{k} h(k,j)u(j)\ .$$

The analogy with formula (3) should be clear.

Given an input trajectory $\{u(t)\}_{t \in \mathbb{Z}}$, the least solution of Eqs. (1) is the output response for which the number of events having occurred up to time $t \in \mathbb{Z}$ is the greatest (remember that the order in $\overline{\mathbb{Z}}_{min}$ is reversed with respect to the usual). Selecting this solution corresponds therefore to consider the least constraining conditions for the evolution of the system. This means not only that the system operates 'as soon as possible' (think of parts handled as soon as a machine is available in example), but also that we have selected the initial conditions which generate the greatest possible output (also called *canonical initial conditions*). When considering causal signals of Σ, *i.e.*, for $t < 0$ $u(t) = u(0)$, $x(t) = x(0)$ and $y(t) = y(0)$, as well as $A(t) = A(0)$, $B(t) = B(0)$, $C(t) = C(0)$, the first equation of (1) is implicit, *i.e.*,

$$x(0) = A(0)x(0) \oplus B(0)u(0) .$$

So the least solution (greatest with respect to usual order), $x(0) = [A(0)]^* B(0)u(0)$ (thanks to th. 1), is the *canonical* initial state.

In the next section, we compute a control input u_{opt} (optimal under just in time criterion) assuming that matrices $A(t)$, $B(t)$, $C(t)$ are known for all t and define a system on Σ. If elements of these given matrices are constant or non-decreasing, solutions x and y of Eqs. (1) are obviously non-decreasing (in other words, these given matrices actually define a system on Σ). But, as noticed in introduction, we consider that the elements of these matrices may not be monotonic functions (possibly decreasing on an interval). One should then be aware that being given u, a non-decreasing input, some given $\{A(t), B(t), C(t)\}_{t\in\mathbb{Z}}$ may generate non-monotonic state x and output y. So, before computing the control input, it will be necessary to check that the given matrices $\{A(t), B(t), C(t)\}_{t\in\mathbb{Z}}$ define a system on Σ.
Conditions *1.* and *2.* of theorem 3 characterize impulse responses of systems defined on Σ. Using expression 4, we just have to check that the given matrices satisfy the following conditions:
$\forall t, j \in \mathbb{Z},\ t \geq j;$

$$C(t+1)\Phi(t+1,j)B(j) \preceq C(t)\Phi(t,j)B(j),$$

and,

$$C(t)\Phi(t,j-1)B(j-1) \preceq C(t)\Phi(t,j)B(j).$$

established from those of theorem 3. If the given matrices $\{A(t), B(t), C(t)\}_{t\in\mathbb{Z}}$ satisfy these conditions, for all non-decreasing input u, solution y of Eq.(1) is non-decreasing (in other words, these matrices define a system on Σ). To ensure furthermore that the state x is also non-decreasing, the given matrices must satisfy:
$\forall t, j \in \mathbb{Z},\ t \geq j;$

$$\Phi(t+1,j)B(j) \preceq \Phi(t,j)B(j),$$

and,

$$\Phi(t,j-1)B(j-1) \preceq \Phi(t,j)B(j).$$

4. Optimal control

Let $\mathcal{S}$ a linear system defined on Σ, the output y can be written

$$y = \mathcal{H}(u),$$

where $\mathcal{H}$: $(\Sigma, \oplus, *) \to (\Sigma, \oplus, *)$ is defined by:

$$[\mathcal{H}(u)](t) = \bigoplus_{s\in\mathbb{Z}} h(t,s)u(s)$$

($h(t,s)$ is the impulse response of $\mathcal{S}$).

Denoting $z \in \Sigma$ the output signal to be tracked, the *optimal control*, denoted u_{opt}, is defined by

$$u_{opt} = Sup\{u \in \Sigma \mid y \preceq z\} .$$

u_{opt} is the greatest solution of inequation $\mathcal{H}(u) \preceq z$. Remembering that the order $\preceq$ is just reversed with the usual order, $\{u_{opt}(t)\}_{t\in\mathbb{Z}}$ is the least input trajectory such that for all t the output response $\{y(t)\}_{t\in\mathbb{Z}}$ is greater than the output to be tracked $\{z(t)\}_{t\in\mathbb{Z}}$. For a manufacturing system, this control input, which gives the latest dates of release of raw parts such that the customer demand is satisfied, fulfills the so-called *just-in-time criterion*.
This greatest solution exists if map $\mathcal{H}$ is residuated. $(\Sigma, \oplus, *)$ being a complete dioid (see example 2), we only need to show that $\mathcal{H}$ satisfies the conditions of theorem 2:

- $\forall t \in \mathbb{Z}, \quad [\mathcal{H}(\varepsilon)](t) = \bigoplus_{s\in\mathbb{Z}} h(t,s)\varepsilon(s) = \varepsilon(t)$
- $\forall t \in \mathbb{Z},$

$$\begin{aligned}\big[\mathcal{H}(\bigoplus_i u_i)\big](t) &= \bigoplus_{s\in\mathbb{Z}} h(t,s) \bigoplus_i u_i(s)\\ &= \bigoplus_i \bigoplus_{s\in\mathbb{Z}} h(t,s)u_i(s)\\ &= \big[\bigoplus_i \mathcal{H}(u_i)\big](t)\end{aligned}$$

Control input u_{opt} therefore exists and is defined by:

$$u_{opt} = \mathcal{H}^\sharp(z) .$$

Proposition 2 *Controls $u_{opt}(t)$, $t \in \mathbb{Z}$, are defined by,*

$$u_{opt}(t) = \big[\mathcal{H}^\sharp(z)\big](t) = \bigwedge_{i\geq t} h(i,t) \backslash\!\!\!\! \circ\, z(i)$$

Proof We denote w the signal defined by:

$$\forall t \in \mathbb{Z}, w(t) = \bigwedge_{i\geq t} h(i,t) \backslash\!\!\!\! \circ\, z(i).$$

1. Let x satisfying
$$\mathcal{H}(x) \preceq z$$
or equivalently,
$$\forall t \in \mathbb{Z}, \quad \bigoplus_{s\in\mathbb{Z}} h(t,s)x(s) = \bigoplus_{s\le t} h(t,s)x(s) \preceq z(t)$$
$$\forall t, s \in \mathbb{Z},\ s \le t; \quad h(t,s)x(s) \preceq z(t)$$
$$\forall t, s \in \mathbb{Z},\ s \le t; \quad x(s) \preceq h(t,s) \backslash z(t)$$
$$\forall s \in \mathbb{Z}, \qquad x(s) \preceq \bigwedge_{t\ge s} h(t,s) \backslash z(t) = w(s)$$

2. $\forall t \in \mathbb{Z}$,
$$\bigoplus_{s\in\mathbb{Z}} h(t,s)w(s) = \bigoplus_{s\in\mathbb{Z}} h(t,s)\Big[\bigwedge_{i\ge s} \frac{z(i)}{h(i,s)}\Big] \preceq$$
$$\bigoplus_{s\in\mathbb{Z}} h(t,s)\frac{z(t)}{h(t,s)} \preceq \bigoplus_{s\in\mathbb{Z}} z(t) = z(t),$$
which shows that w is solution of $\mathcal{H}(x) \preceq z$. □

In the following, we show that u_{opt} is solution of a system of recurrent equations which proceed backwards in time. These equations offer a strong analogy with the adjoint-state equations of optimal control theory. These equations are an extension to the time-varying case of an existing result for $(\min,+)$ linear time-invariant systems [2, §5.6]. Firstly, let us remark that

$$u_{opt}(t) = \bigwedge_{i\ge t} \frac{z(i)}{h(i,t)} = \bigwedge_{i\ge t} \frac{z(i)}{C(i)\Phi(i,t)B(t)}$$
$$= \frac{\bigwedge_{i\ge t} C(i)\Phi(i,t) \backslash z(i)}{B(t)}$$
(thanks to *f*.2 and *f*.3)
$$= \frac{\overline{\xi}(t)}{B(t)}$$
setting $\overline{\xi}(t) = \bigwedge_{i\ge t} \frac{z(i)}{C(i)\Phi(i,t)}$.

Proposition 3 *The greatest solution of equation*
$$\xi(t) = \frac{\xi(t+1)}{A(t+1)} \wedge \frac{z(t)}{C(t)} \tag{5}$$
is given by $\overline{\xi}(t) = \bigwedge_{i\ge t} \frac{z(i)}{C(i)\Phi(i,t)}$

Proof

1. Let us first show that $\overline{\xi}$ is solution of Eq. (5).

$\forall t \in \mathbb{Z}$,

$$\frac{\overline{\xi}(t+1)}{A(t+1)} \wedge \frac{z(t)}{C(t)} = \frac{\left(\bigwedge_{i\ge t+1} C(i)\Phi(i,t+1)\right) \backslash z(i)}{A(t+1)} \wedge \frac{z(t)}{C(t)}$$
$$= \bigwedge_{i\ge t+1} \frac{z(i)}{C(i)\Phi(i,t+1)A(t+1)} \wedge \frac{z(t)}{C(t)}$$
(thanks to *f*.2 and *f*.3)
$$= \bigwedge_{i\ge t+1} \frac{z(i)}{C(i)\Phi(i,t)} \wedge \frac{z(t)}{C(t)\Phi(t,t)}$$
$(\Phi(t,t) = Id)$
$$= \bigwedge_{i\ge t} \frac{z(i)}{C(i)\Phi(i,t)}$$
$$= \overline{\xi}(t)$$

2. Let $\{\xi(t)\}_{t\in\mathbb{Z}}$ a solution of Eq. (5), we have $\forall t \in \mathbb{Z}$
$$\xi(t) = \frac{\xi(t+t_0)}{\Phi(t+t_0,t)} \wedge \bigwedge_{j=t}^{t+t_0-1} \frac{z(j)}{C(j)\Phi(j,t)}, \quad t_0 \ge 1\,.$$
With $t_0 \to +\infty$, it is clear that $\forall t$, $\xi(t) \preceq \overline{\xi}(t)$.

□

Finally, u_{opt} is the greatest solution of
$$\begin{cases} \xi(t) = \dfrac{\xi(t+1)}{A(t+1)} \wedge \dfrac{z(t)}{C(t)} \\ u(t) = \dfrac{\xi(t)}{B(t)} \end{cases}, \forall t \in \mathbb{Z}\,. \tag{6}$$

The initial conditions of recurrence of these equations may be:
$$\exists\, T_f \text{ such that } \forall t > T_f,$$

1. $z(t) = z(T_f)$, $\xi(t) = \xi(T_f)$,
2. $A(t) = A(T_f)$, $B(t) = B(T_f)$ et $C(t) = C(T_f)$.

For $t > T_f$, the first equation is hence implicit, *i.e.*,
$$\xi(T_f) = \frac{\xi(T_f)}{A(T_f)} \wedge \frac{z(T_f)}{C(T_f)}\,,$$
and we select the greatest solution:
$$\xi(T_f) = \frac{z(T_f)}{C(T_f)A(T_f)^* B(T_f)}\,.$$

For a manufacturing system, assumption 1. means that production must be controlled over a finite temporal horizon. Beyond a final instant T_f, the output to be tracked and the 'co-state' ξ are in fact supposed to remain constant. Assumption 2. comes down to considering that the parameters of the system are also constant after T_f.

5. Example

We consider the manufacturing system of figure 1, described in section 1, with $d = 1$ (handling time of a part) and $c_1 = c_2 = 0$ (travelling times). The upper part of figure 2 represents the trajectories of the output to be tracked z, of the optimal control u_{opt} computed with Eqs. (6), and of the output response y to u_{opt} computed with Eqs. (1).
The lower curve represents the evolution of $a(t)$ which gives the number of working machines at time t. It has been supposed that two machines normally work in the workshop. One of these has to be shutdown (*e.g.* due to planned maintenance) between instants ten and fourteen.
We see that the response output y to the computed optimal control u_{opt} is greater than the trajectory z. In term of manufacturing system, the customer demand is always satisfied.

Between instants ten and fourteen, only one machine is running, and the customer demand rate is two parts per unit of time (sequence in the grey box labeled **(b)**). The control input u_{opt} (one raw part released per unit of time) is then such that the machine works at its maximum rate, but the resulting production remains slower than the customer demand. This slowing down has been anticipated between instants three and seven (sequence in the grey box labeled **(a)**). In fact, the customer demand is then equal to one part per unit of time whereas two machines are running. The control input is then such that finished parts are produced ahead of customer demand. The anticipated production of finished parts is not possible between instants seven and ten since the desired production rate (two parts per unit of time) is then equal to the maximum production rate of the workshop.
More generally note that the release of raw parts always occurs 'at the latest' so that the customer demand is achieved. In other words, the control input u_{opt} satisfies the just-in-time criterion.

6. Conclusion

We have considered the class of DEDS involving exclusively synchronization phenomena for which parameters may vary as functions of time. For a manufacturing system, these varying parameters are typically the number of working machines (often considered as 're-usable resources'), or the number of withdrawn parts for conformance test ('non-re-usable resources'). A linear state model with time-varying coefficients or an input/output relationship (the coefficients of the impulse response depend both upon the instant of observation, and upon the instant of application of the unit pulse) can be obtained in $(\min,+)$ algebra. We propose an optimal output tracking solution under just in time criterion. The proposed optimal control, based on residuation theory, is a simple extension of an existing result for $(\min,+)$ linear time-invariant systems.
The linear system theory over dioids offers an interesting property in that: systems can be studied both in the time domain with the $(\min,+)$ algebraic structure and in the event domain with a dual algebraic structure - the $(\max,+)$ algebra. We are besides trying to develop the ideas of this paper in the $(\max,+)$ algebraic structure. In a manufacturing system, the parameters we may then allow to vary would be for example the processing times or the transportation times.

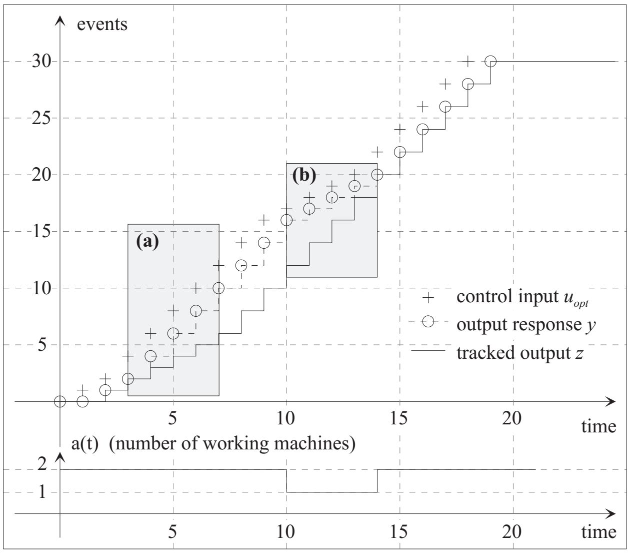


Figure 2: Application of the optimal control

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