

A novel view: edge isoperimetric methods and reliability evaluation of several kinds of conditional edge-connectivity of interconnection networks

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Abstract—Reliability evaluation and fault tolerance of an interconnection network of some parallel and distributed systems are discussed separately under various link-faulty hypotheses in terms of different \mathcal{P} -conditional edge-connectivity. With the help of edge isoperimetric problem's method in combinatorics, this paper mainly offers a novel and unified view to investigate the \mathcal{P} -conditional edge-connectivities of hamming graph K_L^n with satisfying the property that each minimum \mathcal{P} -conditional edge-cut separates the K_L^n just into two components, such as L^t -extra edge-connectivity, t -embedded edge-connectivity, cyclic edge-connectivity, $(L-1)t$ -super edge-connectivity, $(L-1)t$ -average edge-connectivity and L^t -th isoperimetric edge-connectivity. They share the same values in form of $(L-1)(n-t)L^t$ (except for cyclic edge-connectivity), which equals to the minimum number of links-faulty resulting in an L -ary- n -dimensional sub-layer from K_L^n . Besides, we also obtain the exact values of h -extra edge-connectivity and h -th isoperimetric edge-connectivity of hamming graph K_L^n for each $h \leq L^{\lfloor \frac{n}{2} \rfloor}$. For the case $L=2$, $K_2^n = Q_n$ is n -dimensional hypercube. Our results can be applied to more generalized class of networks, called n -dimensional bijective connection networks, which contains hypercubes, twisted cubes,

crossed cubes, Möbius cubes, locally twisted cubes and so on. Our results improve several previous results on this topic.

Index Terms—Edge isoperimetric problem, Conditional connectivity, Edge disjoint path, Reliability, Unified method, bipartite

I. INTRODUCTION

With the emergence of new topological architectures, parallel and distributed processing research is at the heart of developing new systems in order to be able to take advantage of the underlying interconnection network. A thorough understanding of various aspects of parallel and distributed systems is necessary to be able to achieve the performance of the new parallel computers and supercomputers. Fault tolerance and reliability have to be optimized for some obvious reasons when choosing good models for interconnection networks. It is well known that the topological interconnection network of parallel and distributed system is usually modeled as a graph $G = (V, E)$, where the vertices and edges represent processors and physical links between the processors.

There are many topological parameters of G to evaluate the reliability of this interconnection network. It is known that the problem of finding node or edge disjoint paths is closely related to a well-known Menger's theorem. Menger theorem laid a solid foundation for the theory of fault tolerance and reliability evaluation of interconnection network of parallel processing systems. Let x and y be two any distinct vertices of a graph G . The minimum size of an (x, y) edge-cut equals the maximum number of edge disjoint (x, y) -paths. Menger's theorem is a characterization of the edge-connectivity in finite graphs in terms of the maximum number of edge-disjoint paths that can be found between any two distinct pair of vertices. As

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the Menger's theorem is often required to find a maximum of edge-disjoint paths between two given vertices of G , motivated by this, we want to go even further, and consider the cases on many-to-many edge disjoint paths of a connected graph G under additional condition.

Let \mathcal{P} be some graph-theoretic property of a connected graph G . We aim to find the maximum number of edge disjoint paths connecting any two disjoint connected subgraphs with just satisfying the property \mathcal{P} in G . If \mathcal{P} is containing a vertex, then the corresponding version is the classical Menger edge-connectivity. Conditional connectivity introduced by F. Harary in 1983, generalized the theories of connectivity in both vertex and edge versions. Not only does it meet the increasing need of a more accurate measure of reliability of large-scale parallel processing systems, but also theoretically enriches the theory of network connectedness. This paper mainly studies six kinds of conditional edge-connectivities under various link-faulty hypotheses. The relationship between the vertex version and various diagnosability has been investigated [3], [11], [15]–[18], [20], [26].

Instead of focusing on studying the \mathcal{P} -conditional edge-connectivity of interconnection networks under various kinds of links faulty hypotheses case by case, this paper mainly offers a novel and unified edge isoperimetric method to investigate the \mathcal{P} -conditional edge-connectivity of hamming graph K_L^n under six kinds of link-faulty hypotheses. These edge-connectivities share a common bipartite property: each minimum \mathcal{P} -conditional edge-cut separates the K_L^n into two components (or two parts). For L -ary- n -dimension hamming graph, L^t -extra edge-connectivity, t -embedded edge-connectivity, cyclic edge-connectivity, $(L-1)t$ -super edge-connectivity, $(L-1)t$ -average edge-connectivity and L^t -th isoperimetric edge-connectivity share the same values in form of $(L-1)(n-t)L^t$ for $L \geq 2$, $n \geq 1$ and $0 \leq t \leq n-1$. Our results are applied to general class of networks. Reliability evaluation and fault tolerance of an interconnection network of some parallel and distributed systems are discussed separately under various link-faulty hypotheses in terms of different \mathcal{P} -conditional edge-connectivity.

The rest of this paper is organized as follows. In section II, some preliminaries, terminologies, and six kinds of definitions about \mathcal{P} -conditional edge-connectivity will be given. In section III, known results about isoperimetric problem, the exact values of $\xi_m(K_L^n)$ and its construction will be presented. In section IV, some useful properties of the functions $\xi_m(K_L^n)$ and $ex_m(K_L^n)$ will be proved. In section V, unified method for \mathcal{P} -conditional edge-connectivities of hamming graph K_L^n will be introduced. Finally, the conclusion of this paper will be given in section VI.

II. PRELIMINARIES

In a connected graph G , for any edge set $F \subseteq E(G)$, the notation $G - F$ denotes the subgraph obtained after removing the edges in F from G . Given a vertex set $X \subseteq V(G)$, we denote $G[X]$ the subgraph of G induced by X , and $\overline{X} =$

$V(G) \setminus X$ is the complement of X . For two vertex sets X and \overline{X} , we denote $[X, \overline{X}]$ the set edges of G with one end in X and the other end in \overline{X} . A component of graph G is a maximal subgraph in which each pair of vertices is connected with each other via a path.

Definition II.1. [7] Let \mathcal{P} be a property of G . The edge subset $F \subset E(G)$ is defined as a \mathcal{P} -conditional edge-cut of G , if any, $G - F$ is disconnected, if any, and each component satisfies the condition \mathcal{P} . $\lambda(\mathcal{P}, G)$, \mathcal{P} -conditional edge-connectivity of G , is defined as the minimum cardinality \mathcal{P} -conditional edge-cut F of G .

Based on the different properties and the faulty-free set, recently many researchers have investigated the various kinds of \mathcal{P} -conditional edge-connectivity of many classes of networks, such as h -extra edge-connectivity, l -embedding edge-connectivity, cyclic edge-connectivity, k -super edge-connectivity, k -average edge-connectivity and h -isoperimetric edge-connectivity. Let

$$\begin{aligned}\mathcal{P}_1^h &= \{ \text{containing at least } h \text{ vertices} \}, \\ \mathcal{P}_2^l &= \{ \text{lying in an } l\text{-dimensional subnetwork of } G \}, \\ \mathcal{P}_3^c &= \{ \text{containing at least one cycle of } G \}, \\ \mathcal{P}_4^k &= \{ \text{having the minimum degree at least } k \}, \\ \mathcal{P}_5^k &= \{ \text{satisfying the average degree at least } k \}.\end{aligned}$$

Note that if $\mathcal{P} = \mathcal{P}_1^h$, it gives the definition of h -extra edge-connectivity of G first introduced by Fàbrega and Foil in 1996.

Definition II.2. [6] For a connected graph G , an edge subset $F \subseteq E(G)$ is called as an h -extra edge-cut of G , if exists, $G - F$ is disconnected and every component of $G - F$ has not less than h vertices. The h -extra edge-connectivity of G , written as $\lambda_h(G)$, is defined as the minimum cardinality among all the h -extra edge-cuts.

And if $\mathcal{P} = \mathcal{P}_2^l$, it is defined as the l -embedded edge-connectivity of an n -dimensional recursive network G_n , $\eta_l(G_n)$ for $0 \leq l \leq n-1$, first introduced by Yang and Wang in 2012 [28].

A graph G is cyclic edge connected if G can be separated into at least two components with two of them containing a cycle. The cyclic edge-connectivity of G , $c\lambda(G)$, is minimum number of edges whose removal disconnects the graph G satisfying that at least two components with one cycle [22].

Similarly, if $\mathcal{P} = \mathcal{P}_4^k$ and $\mathcal{P} = \mathcal{P}_5^k$, the k -super edge-connectivity and k -average edge-connectivity of G can be defined as the minimum cardinality of k -super edge-cut and the minimum cardinality of k -average edge-cut, such that each component of their removal from G has the minimum degree at least k and the average degree at least k , respectively. The

h -th isoperimetric edge-connectivity $\gamma_h(G)$ of G is defined as

$$\gamma_h(G) = \min\{|[U, \bar{U}]| : U \subset V(G), |U| \geq h, |\bar{U}| \geq h\},$$

where $|[U, \bar{U}]|$ is the number of edges with one end in U and the other end in $\bar{U} = V \setminus U$ [25], [34]. Write

$$\beta_h(G) = \min\{|[U, \bar{U}]| : U \subset V(G), |U| = m\}.$$

A graph G with $\gamma_j(G) = \beta_j(G)$, $j = 1, \dots, k$ is said to be γ_h -optimal. Although the definition of h -th isoperimetric edge-connectivity $\gamma_h(G)$ of the graph G deletes the connectedness of components from that of h -extra edge-connectivity $\lambda_h(G)$, and one does find the situation where two definitions are not equivalent, for the networks involved hamming graph and its related variants in this paper, they do share the same values.

Although the above six kinds of \mathcal{P} -conditional edge-connectivities contain its own properties, they share a fact that removal their each minimum \mathcal{P} -conditional edge-cut results in exact two components (or two parts). The \mathcal{P} -conditional edge-connectivity is called bipartite, if each minimum \mathcal{P} -conditional edge-cut results in exact two components (or two parts).

Let $\mathcal{B} = \{\mathcal{P} \mid \mathcal{P}\text{-conditional edge-connectivity is bipartite}\}$.

Theorem II.3. $\lambda(\mathcal{P}_1^k, G)$, $\lambda(\mathcal{P}_2^k, G)$, $\lambda(\mathcal{P}_3^k, G)$, $\lambda(\mathcal{P}_4^k, G)$, $\lambda(\mathcal{P}_5^k, G)$ and $\lambda(\mathcal{P}_6^k, G)^*$ are bipartite.

Proof. For $i \neq 3$, recall that $\lambda(\mathcal{P}_i^k, G)$ is the minimum number of an edge set of the graph G whose removal disconnects the graph G with all its components satisfying the condition \mathcal{P}_i^k . Suppose F is a \mathcal{P}_i^k -conditional edge-cut, it can be deduced that the optimal number is obtained only when there are exactly two components produced in $G - F$. In fact, if there exists a minimum \mathcal{P}_i^k -conditional edge-cut F_0 , whose removal disconnects the connected graph G with all its components $H_1, H_2, \dots, H_z, z > 2$ satisfying the condition \mathcal{P}_i^k , then there exists an integer $b, 1 \leq b \leq z$, and $[V(H_b), \overline{V(H_b)}]$ is also an \mathcal{P}_i^k -conditional edge-cut of G , where the induced subgraphs by $V(H_b)$ and $\overline{V(H_b)}$ are connected. Because of $F_0 \cap E(H_b) \neq \emptyset$ or $F_0 \cap E(G[\overline{V(H_b)}]) \neq \emptyset$, $|[V(H_b), \overline{V(H_b)}]| < |F_0|$, a contradiction. So for each $i \neq 3$, $\lambda(\mathcal{P}_i^k, G)$ is bipartite. If $i = 3$, $k = c$, for convince, $\lambda(\mathcal{P}_3^c, G)$ means that each component contains one cycle. The similar proof can be done. \square

Given a connected graph, once above six conditional edge-connectivities are well-defined, then set $\{\mathcal{P}_1^h, \mathcal{P}_2^h, \mathcal{P}_3^c, \mathcal{P}_4^k, \mathcal{P}_5^k, \mathcal{P}_6^h\} \subseteq \mathcal{B}$. Their bipartite property is its essential feature for them.

We borrow the conception of Mader's atom [23] to study of \mathcal{P}_i^k -conditional edge-connectivities. A vertex set S is called a $\lambda(\mathcal{P}_i^k)$ -fragment if $[S, \bar{S}]$ is a minimum \mathcal{P}_i^k -conditional edge-cut. Obviously, if S is a $\lambda(\mathcal{P}_i^k)$ -fragment, so is \bar{S} . A minimum $\lambda(\mathcal{P}_i^k)$ -fragment is a $\lambda(\mathcal{P}_i^k)$ -atom. Let A be called a $\lambda(\mathcal{P}_i^k)$ -atom, then $G[A]$ is connected. γ_k -fragment and λ_k -atom are defined similarly without connected condition constrain. Write

$$\alpha(\mathcal{P}_i^k, G) \text{ the cardinality of a } \lambda(\mathcal{P}_i^k)\text{-atom and}$$

$$\theta_{\mathcal{P}_i^k}(G) = \min\{|X| \mid G[X] \text{ satisfying the property } \mathcal{P}_i^k\}.$$

It is easy to see that a graph G is $\lambda(\mathcal{P}_i^k)$ -optimal if and only if $\alpha(\mathcal{P}_i^k, G) = \theta_{\mathcal{P}_i^k}(G)$. The parameters $\alpha(\mathcal{P}_i^k, G)$ and $\theta_{\mathcal{P}_i^k}(G)$ play an important role in obtaining the exact value of $\lambda(\mathcal{P}_i^k, G)$ and determining its optimality.

This idea drives us to study the minimum possible “boundary-size” of a set of a given “size”. The classical isoperimetric problem is usually expressed in the form of an inequality $L_M^2 \geq 4\pi A_M$ that relates the length L_M of a closed curve M and the area A_M of the planar region that it encloses in the Euclidean plane R^2 . Which closed curve minimizes the length L of the curve with a fixed size of the area A ? The equality holds if and only if M is a circle. It was “known” to the ancient Greeks, but it was not until the 19th century that this was proved vigorously by Karl Weierstrass in a series of lectures in 1870, in Berlin. Isoperimetric problems are classical objects of the study in mathematics. In general, they ask for the minimum possible “boundary-size” of a set of a given “size”, where the exact meaning of these words varies according to the problem. Making efforts to obtain the exact value of the \mathcal{P}_i^k -conditional edge-connectivity drives us to study the minimum possible “edge boundary-size” of a set of a given “size” in graph G .

For discrete case, a graph $G = (V, E)$, the given “size” is the “number of vertices”, while the “boundary-size” of the graph is “the size of the edge boundary or the vertex boundary”. Both edge and vertex versions are related to the \mathcal{P} -conditional edge-connectivity and \mathcal{P} -conditional connectivity, respectively. This paper mainly focuses on edge version. Let $G = (V, E)$ be a connected graph and $A \subseteq V(G)$. Recall that the definition of $\beta_m(G) = \min_{A \subseteq V, |A|=m} |[A, \bar{A}]|$ in the h -th isoperimetric edge-connectivity. For a given graph G , the edge isoperimetric problem of G is to find a subset $A^* \subseteq V(G)$ with $|A^*| = m$ such that $\beta_m(G) = |[A^*, \bar{A}^*]|$ for any $1 \leq m \leq |V(G)|$ (introduced by Harper in 1964) [8]. Even if in Harper's seminal paper of edge isoperimetric problem on hypercube $Q_n = K_2^n$, the original definition $G[A^*]$ is not required to be connected, his example does satisfy that $K_2^n[A^*]$ is connected. Let $\xi_m^e(G) = \min\{|[X, \bar{X}]| : X \subset V(G) \text{ with } |X| = m \leq \lfloor |V(G)|/2 \rfloor, \text{ and } G[X] \text{ is connected}\}$. $\xi_m^e(G)$ is unilateral connected for each $1 \leq m \leq \lfloor |V(G)|/2 \rfloor$.

In this paper we want to go further. Let $\xi_m(G) = \min\{|[X, \bar{X}]| : X \subset V(G) \text{ with } |X| = m \leq \lfloor |V(G)|/2 \rfloor, \text{ and both } G[X] \text{ and } G[\bar{X}] \text{ are connected}\}$. $\xi_m(G)$ is bilateral connected for each $1 \leq m \leq \lfloor |V(G)|/2 \rfloor$. More coincidentally,

*Strictly speaking, the removal of a minimum h -th isoperimetric edge-cut, results in two parts rather than two components. The connectedness of these two parts is not necessary.

Harper's seminal example meets the equations $\beta_m(K_2^n) = \xi_m(K_2^n) = \xi_m(K_2^n)$ for each $m \leq 2^{n-1}$. However, it does not always hold for general G . Even if we add the unilateral and bilateral connected conditions, the result $\xi_m(G) = \xi_m^e(G)$ does not always hold for each $1 \leq m \leq \lfloor |V(G)|/2 \rfloor$.

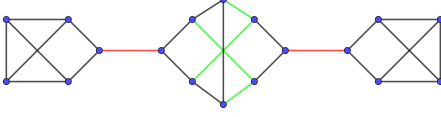


Fig. 1. Graph G^* without satisfying condition $\xi_8(G^*) \neq \xi_8^e(G^*)$.

For example, given a 3-regular graph G^* as shown in Fig. 1, $\xi_8(G^*) = \lambda_1^8(G^*) = 4 \neq 2$. If the red edge cut in graph G^* is removed, G^* is divided into three components instead of two, and the number of vertices of the two components is less than 8. If the green edge cut in graph G^* is removed, G^* is divided into two components, the number of vertices is 8 and 10, respectively. So $\xi_8(G^*) = 4 \neq 2 = \xi_8^e(G^*)$.

Followed Harper's idea, for r -regular graph G , $|A| = m$ by Handshaking Lemma, $\beta_m(G) = r|A| - ex_m(G)$, where $ex_m(G)$ is the densest degree sum among all the m vertices induced subgraphs. For each $m \leq \lfloor |V(G)|/2 \rfloor$, if one can find a subset $X_m^* \subseteq V(G)$ satisfying that $\beta_m(G) = \xi_m^e(G) = \xi_m(G) = |[X_m^*, \overline{X_m^*}]|$, $|X_m^*| = m$, and both $G[X_m^*]$ and $G[\overline{X_m^*}]$ are connected, with $2|E(G[X_m^*])| = ex_m(G)$, then these three definitions are equivalent. Such subsets X_m^* are called optimal for edge isoperimetric problem of G . We say that optimal subsets are nested $X_m^* \subseteq X_{m+1}^*$ if there exists a total order \mathcal{O} on the set $V(G)$ such that for any $m \leq |V(G)|$ the collection of the first m vertices on the order is an optimal subset $X_m^* = \{v_1, v_2, v_3, \dots, v_i\}$,

$$v_1 < v_2 < v_3 < \dots < v_i < v_{i+1} < \dots < v_{p-2} < v_{p-1} < v_p.$$

For the networks involved hamming graph K_L^n and its related variants in this paper, they do satisfy this property. So in the following discussion, conventionally, we use the notation $\xi_m(K_L^n)$. It is quite convenient for us to study bipartite \mathcal{P} -conditional edge-connectivity of K_L^n . In particular, for the h -extra edge-connectivity, in 2018, Zhang et al. obtained that [32], for each $1 \leq h \leq \lfloor |V(G)|/2 \rfloor$,

$$\lambda_h(G) = \min\{\xi_m(G) : h \leq m \leq \lfloor |V(G)|/2 \rfloor\}.$$

where $\xi_m(G) = \min\{|[X, \overline{X}]| : X \subset V(G), |X| = m \leq \lfloor |V(G)|/2 \rfloor, \text{ and both } G[X] \text{ and } G[\overline{X}] \text{ is connected}\}$. While for general cases, in 2017, L. P. Montejano and I. Sau proved that the problem of determining the exact value of $\lambda_h(G)$ is NP-complete [21].

Given two graphs G and H , the Cartesian product of G and H $H \times G$ is defined as $V(H \times G) = \{uv \mid v \in V(G), u \in V(H)\}$, $E(H \times G) = \{ux, vy \mid (u = v, x, y \in E(G)) \text{ or } (x = y, u, v \in E(H))\}$.

The Cartesian product of graphs G_3 and $G_2 \times G_1$ is defined

as $G_3 \times G_2 \times G_1$. Similarly, one can define the Cartesian product of graphs G_n and $G_{n-1} \times \dots \times G_2 \times G_1$ inductively.

Each vertex $G_n \times G_{n-1} \times \dots \times G_2 \times G_1$ can be denoted by $x_n x_{n-1} \dots x_2 x_1$, $0 \leq x_i < |V(G_i)|$, $1 \leq i \leq n$. If $G_k = G$, for each $1 \leq k \leq n$, then $G_n \times G_{n-1} \times \dots \times G_2 \times G_1 = \underbrace{G \times G \times \dots \times G \times G}_n$, denoted by G^n , and it is n -th cartesian

product power graph of G . If $G = K_L$, the n -th cartesian product power graph of K_L is called L -ary n -dimensional hamming graph, denoted by K_L^n . The vertex set of K_L^n can be denoted by $X_n X_{n-1} \dots X_2 X_1 = \{x_n x_{n-1} \dots x_2 x_1 \mid x_i \in \{0, 1\}, 0 \leq i \leq 1\}$, $0 \leq x_i \leq L-1$, $1 \leq i \leq n$. For any two vertices $u = u_n u_{n-1} \dots u_2 u_1$ and $v = v_n v_{n-1} \dots v_2 v_1$ in the graph K_L^n , there is an edge between u and v if and only if exactly one integer j satisfies $u_i \neq v_i$ if $i = j$; $u_i = v_i$ if $i = j$, $j \in \{1, 2, \dots, n\}$. By the definition of the K_L^n , for given integers $0 \leq k \leq L-1$, the subgraph $K_L^n[kX_{n-1} \dots X_2 X_1]$ induced by $kX_{n-1} \dots X_2 X_1$ is an L -ary $(n-1)$ -dimensional sub-layer of K_L^n . Let $z_n z_{n-1} \dots z_{a+2} z_{a+1} X_a \dots X_2 X_1$ be the set $\{z_n z_{n-1} \dots z_{a+2} z_{a+1} x_a \dots x_2 x_1 : x_i \in \{0, 1, 2, \dots, L-1\}, i \leq 1, 2, \dots, a, z_j \text{ is fixed for each } j = a+1, a+2, \dots, n\}$. By the definition of the K_L^n , similarly, one can write $K_L^n[z_n z_{n-1} \dots z_{a+2} z_{a+1} X_a \dots X_2 X_1]$ the subgraph induced by the vertex set $z_n z_{n-1} \dots z_{a+2} z_{a+1} X_a \dots X_2 X_1$. It is an L -ary a -dimensional sub-layer of K_L^n . Thus we also use $z_n z_{n-1} \dots z_{a+1} X_a \dots X_2 X_1$ to denote this L -ary a -dimensional sub-layer of K_L^n if no confusion arises. The lexicographic ordering \preceq on $\Pi_{i=1}^n [k_i]$ is defined as follows: $x_1 x_2 \dots x_n$ precedes $y_1 y_2 \dots y_n$ if for some i we have $x_i < y_i$ and $x_j = y_j$ for $j < i$.

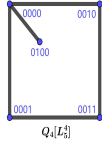
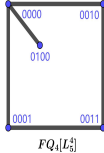
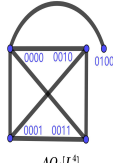
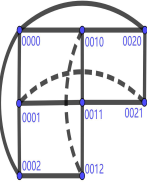
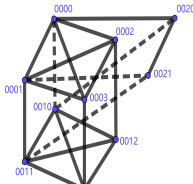
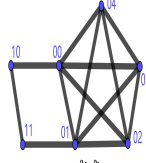
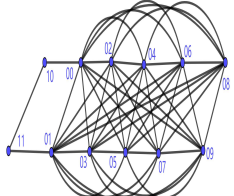
III. THE EXACT VALUES OF $\xi_m(K_L^n)$ AND ITS

CONSTRUCTION

Recall that a string that contains only 0 's, 1 's, \dots , $(L-1)$'s is called an L -base n -string. For a positivity integer $x = \sum_{i=1}^n x_i L^{i-1}$, $0 \leq i \leq n$, $x_i \in \{0, 1, 2, \dots, L-1\}$. Let $x = x_n x_{n-1} \dots x_1$ be the form of L -base n -string. The L -base n -string denotes the L -base string of length n . For any positive integer m , it has a L -base decomposition $m = \sum_{i=0}^s a_i L^{b_i} \leq \lfloor L^n/2 \rfloor$, $a_i \in \{1, 2, \dots, L-1\}$, $b_0 > b_1 > \dots > b_s$. Let $S_m = \{0, 1, \dots, m-1\}$ (under decimal representation), $|S_m| = m$, $m \leq \lfloor L^n/2 \rfloor$, and the corresponding set L_m^n be the corresponding set represented by L -base n -string. In Table I, given a positive integer m as an example, the calculation process of $\xi_m(G)$ is explained more concretely and vividly. At the same time, the corresponding induced subgraphs and the functions of $O(K_L) = L^{\lfloor \frac{n}{2} \rfloor}$, for graphs Q_4 , FQ_4 , AQ_4 , K_3^n , K_4^n and K_{10}^n are given. In the expression of $ex_m(G^n)$, the specific values of I_i and δ_i are shown in the Table II.

Lemma III.1. [31] Given d -regular connected graph G , the number of vertices is $|V(G)| = L$. If the lexicographic order provides the optimal solution of the edge isoperimetric

TABLE I
THE EXACT VALUES OF $\xi_m(K_L^n)$ AND $ex_m(K_L^n)$ AND THEIR CONSTRUCTION

G^n	Q_4	FQ_4	AQ_4	K_3^4	K_4^4	K_5^2	K_{10}^2
m	5	5	5	8	10	7	12
S_m	{0, 1, 2, 3, 4}	{0, 1, 2, 3, 4}	{0, 1, 2, 3, 4}	{0, 1, 2, 3, 4, 5, 6, 7}	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}	{0, 1, 2, 3, 4, 5, 6}	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
L_m^n	{0000, 0001, 0010, 0011, 0100}	{0000, 0001, 0010, 0011, 0100}	{0000, 0001, 0010, 0011, 0100}	{0000, 0001, 0002, 0010, 0011, 0012, 0020, 0021}	{0000, 0001, 0002, 0003, 0010, 0011, 0012, 0013, 0020, 0021}	{00, 01, 02, 03, 04, 10, 11}	{00, 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11}
Decomposition of m	$5 = 2^2 + 2^0$	$5 = 2^2 + 2^0$	$5 = 2^2 + 2^0$	$8 = 2 \times 3^1 + 2 \times 3^0$	$10 = 2 \times 4^1 + 2 \times 4^0$	$7 = 5^1 + 2 \times 5^0$	$12 = 10^1 + 2 \times 10^0$
$ex_m(G^n)$	$2 \times 2^2 + 2 \times 2^0 = 10$	$2 \times 2^2 + 2 \times 2^0 = 10$	$3 \times 2^2 - 2^0 + 4 \times 2^0 + 1 = 16$	$2(2 \times 3^1 + 3^1 + 3^0) + 2 \times 2 \times 3^0 = 28$	$3 \times 2 \times 4^1 + 2 \times 4^1 + 2 \times 2 \times 4^0 + 2 \times 2 \times 2 \times 4^0 = 42$	$4 \times 5^1 + 2 \times 4^1 + 2 \times 5^0 + 2 \times 2 \times 5^0 = 42$	$9 \times 10^1 + 2 \times 4^1 + 2 \times 1 \times 10^0 + 2 \times 2 \times 10^0 = 96$
$\xi_m(G^n)$	$4 \times 5 - 10 = 10$	$5 \times 5 - 10 = 15$	$(2 \times 4 - 1) \times 5 - 16 = 19$	$2 \times 4 \times 8 - 28 = 36$	$3 \times 4 \times 10 - 42 = 78$	$4 \times 2 \times 7 - 42 = 14$	$9 \times 2 \times 12 - 96 = 120$
Induced graph $G^n[L_m^n]$							
$O(K_L)$	$2^{\lfloor \frac{n}{2} \rfloor}$	$2^{\lfloor \frac{n}{2} \rfloor}$	$2^{\lfloor \frac{n}{2} \rfloor}$	$3^{\lfloor \frac{n}{2} \rfloor}$	$4^{\lfloor \frac{n}{2} \rfloor}$	$5^{\lfloor \frac{n}{2} \rfloor}$	$10^{\lfloor \frac{n}{2} \rfloor}$
$I(K_L)$	$I_0 = 0, I_1 = 0$	$I_0 = 0, I_1 = 0$	$I_0 = 0, I_1 = 0$	$I_0 = 0, I_1 = 0, I_2 = 1$	$I_0 = 0, I_1 = 0, I_2 = 1, I_3 = 3$	$I_0 = 0, I_1 = 0, I_2 = 1, I_3 = 3, I_4 = 6$	$I_0 = 0, I_1 = 0, \dots, I_9 = 36$
$\delta(K_L)$	$\delta_1 = 0, \delta_2 = 1$	$\delta_1 = 0, \delta_2 = 1$	$\delta_1 = 0, \delta_2 = 1$	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 2$	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 2, \delta_4 = 3$	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 2, \delta_4 = 3, \delta_5 = 4$	$\delta_1 = 0, \delta_2 = 1, \dots, \delta_{11} = 10$

*Strictly speaking, $ex_m(AQ_n)$ does not conform to this regular pattern, $ex_m(AQ_n) = \sum_{i=0}^s (2i-1)2^{t_i} + \sum_{i=0}^s 4i2^{t_i} + \delta$, where if m is even, then $\delta = 0$; if m is odd, then $\delta = 1$.

TABLE II
THE FUNCTIONS OF I_i AND δ_i CORRESPONDING TO K_L .

$I_0 = 0$	$I_1 = 0$	$I_2 = 1$	$I_3 = 3$	$I_4 = 6$	\dots	$I_i = i(i-1)/2$	\dots	$I_{L-1} = (L-2)(L-1)/2$	$I_L = L(L-1)/2$
	$\delta_1 = 0$	$\delta_2 = 1$	$\delta_3 = 2$	$\delta_4 = 3$	\dots	$\delta_i = i-1$	\dots	$\delta_{L-1} = L-2$	$\delta_L = L-1$

problem on the power graph and $I_0(G) = 0, I_1(G) = 0, I_m(G) = ex_m(G)/2, \delta(G) = I_m(G) - I_{m-1}(G), 1 \leq m \leq L$, then for any $1 \leq m = \sum_{i=0}^s a_i L^{b_i} \leq L^n, b_0 > b_1 > \dots > b_s, a_i \in \{1, 2, \dots, L-1\}, 0 \leq i \leq s, ex_m(G^n) = \sum_{i=0}^s [\delta_L(G) a_i b_i L^{b_i} + 2I_{a_i}(G) L^{b_i}] + 2 \sum_{i=0}^{s-1} \sum_{k=i+1}^s \delta_{a_i+1}(G) a_k L^{b_k}$, and $\xi_m^e(G^n) = \delta_L(G) nm - ex_m(G^n)$.

Let $m = \sum_{i=0}^s a_i L^{b_i}, a_i \in \{1, 2, \dots, L-1\}, n \geq b_0 > b_1 > \dots > b_s \geq 0$. In 2018, Zhang [31] constructed $(s+1)$ L -ary sub-layer family whose vertex sets do not intersect each other. For $0 \leq i \leq s$, the i -th L -ary sub-layer family of K_L^n is denoted by C^i , which contains a_i L -ary b_i -dimensional sub-layers $C^{i,1}, C^{i,2}, \dots, C^{i,a_i}$ of K_L^n . For $0 \leq i \leq s, 1 \leq j_i \leq a_i \leq L-1$, the j_i -th L -ary b_i -dimensional sub-layer of K_L^n is denoted by C^{i,j_i} . Given C^0 , which contains a_0 L -ary b_0 -dimensional sub-layers, then the j_0 -th L -ary b_0 -dimensional sub-layer of K_L^n is C^{0,j_0} :

$$00 \dots 00 j_0 - 1 X_{b_0} X_{b_0-1} \dots X_2 X_1.$$

The induced subgraphs of each L -ary b_0 -dimensional sub-layer are isomorphic to the power graph G^{b_0} of a graph G . They do not intersect each other and are $\delta_L(G) \times b_0$ -regular. There are $I_{a_i}(G) L^{b_0}$ edges in the 0-th L -ary sub-layer family.

The structure process is as follows:

$$\begin{aligned}
 C^{0,j_0}, 1 \leq j_0 \leq a_0 : & \overbrace{00 \dots 00 j_0 - 1 X_{b_0} X_{b_0-1} \dots X_2 X_1}^{n} \\
 & \underbrace{\text{if any}}_{b_0} \\
 C^{1,j_1}, 1 \leq j_1 \leq a_1 : & \overbrace{00 \dots 00 a_0 00 \dots 00 j_1 - 1 X_{b_1} X_{b_1-1} \dots X_2 X_1}^{n} \\
 & \underbrace{\text{if any}}_{b_0} \underbrace{\text{if any}}_{b_1} \\
 C^{2,j_2}, 1 \leq j_2 \leq a_2 : & \overbrace{00 \dots 00 a_0 00 \dots 00 a_1 00 \dots 00 j_2 - 1 X_{b_2} X_{b_2-1} \dots X_2 X_1}^{n} \\
 & \underbrace{\text{if any}}_{b_0} \underbrace{\text{if any}}_{b_1} \underbrace{\text{if any}}_{b_2} \\
 & \underbrace{\phantom{00 \dots 00 a_0 00 \dots 00 a_1 00 \dots 00 j_2 - 1 X_{b_2} X_{b_2-1} \dots X_2 X_1}}_{b_1} \\
 & \underbrace{\phantom{00 \dots 00 a_0 00 \dots 00 a_1 00 \dots 00 j_2 - 1 X_{b_2} X_{b_2-1} \dots X_2 X_1}}_{b_0} \\
 & \dots
 \end{aligned}$$

For $C^{i,j_i}, i > 0$, the $(b_{i-1} + 1)$ -th coordinate $(j_{i-1} - 1)$ of

C^{i-1, a_i-1} is changed to a_{i-1} . Let the $(b_i + 1)$ -th coordinate of C^{i, j_i} be $(j_i - 1)$. And let the coordinate of the $(b_i + 2)$ -th to the n -th be 0 except for the $(b_{i-1} + 1)$ -th, if any. Through this construction method, $L_m = \bigcup_{0 \leq i \leq s, 1 \leq j_i \leq a_i} V(C^{i, j_i})$ is obtained.

Let $V(C^i) = \bigcup_{1 \leq j_i \leq a_i} V(C^{i, j_i})$ and $G_m^{1*} =: G^n[L_m]$.

Each L -ary sub-layer C^{i, j_i} has $\delta_L(G)b_i L^{b_i}/2$ edges. For $0 \leq i \leq s$, $1 \leq j_i \leq a_i \leq L - 1$, given i , if $a_i = 1$, C^i contains only one L -ary b_i -sub-layer; if $a_i = 2$, then between C^{i, a_i} and C^{i, a_i-1} has a matching of size L^{b_i} ; and if $a_i > 2$, a_i and a_t are adjacent in the graph G , $a_t < a_i$, then between C^{i, a_i} and C^{i, a_t} has a matching of size L^{b_i} . There are $I_{a_i}(G)$ groups of such matches exist in C^i L -ary b_i -dimensional sub-layer of L -ary b_i -dimensional sub-layer family.

For each $0 \leq i \leq k \leq s$, there are $\delta_{a_i+1}(G)a_k L^{b_k}$ edges between sub-layer family $V(C^i)$ and sub-layer family $V(C^k)$. It can be counted that C_m^{1*} contains $\sum_{i=0}^s [0.5\delta_L(G)b_i L^{b_i} + (I_{a_i}(G))L^{b_i}]$ edges inside C^i . The number of edges between C^i is $\sum_{i=0}^{s-1} \sum_{k=i+1}^s \delta_{a_i+1}(G)a_k L^{b_k}$. Since G^n is $\delta_L(G)n$ -regular, from the Handshaking lemma, $\xi_m(G^n) = \delta_L(G)nm - ex_m(G^n)$. The number of edges of $G^n[L_m]$ is $\sum_{i=0}^s [0.5\delta_L(G)a_i b_i L^{b_i} + I_{a_i}(G)L^{b_i}] + \sum_{i=0}^{s-1} \sum_{k=i+1}^s \delta_{a_i+1}(G)a_k L^{b_k}$.

As $G = K_L$ is $L-1$ -regular connected graph, the number of vertices is $|V(K_L)| = L$. The lexicographic order provides the optimal solution of the edge isoperimetric problem on K_L^n and $I_0(K_L) = 0$, $I_1(K_L) = 0$, $2I_m(K_L) = ex_m(G) = m(m-1)$, $\delta_m(K_L) = I_m(K_L) - I_{m-1}(K_L) = m-1$, $1 \leq m \leq L$. Note that $\delta_L(K_L)L - 1 = r$.

Corollary III.2. [31] For any integers $1 \leq m = \sum_{i=0}^s a_i L^{b_i} \leq L^n$, $b_0 > b_1 > \dots > b_s$, $a_i \in \{1, 2, \dots, L-1\}$, $0 \leq i \leq s$, $ex_m(K_L^n) = \sum_{i=0}^s [(L-1)a_i b_i L^{b_i} + (a_i-1)a_i L^{b_i}] + 2\sum_{i=0}^{s-1} \sum_{k=i+1}^s a_i a_k L^{b_k}$, and $\xi_m(K_L^n) = (L-1)nm - ex_m(K_L^n)$.

Corollary III.3. (1) (Harper 1964 [8] and Li and Yang [19] 2013) $\xi_m(K_2^n) = nm - ex_m(K_2^n) = nm - [\sum_{i=0}^s b_i 2^{b_i} + \sum_{i=0}^s 2 \cdot i \cdot 2^{b_i}]$, for $m = \sum_{i=0}^s 2^{b_i} \leq 2^n$, $b_0 > b_1 > \dots > b_s$.

(2) $\xi_m(K_3^n) = 2nm - ex_m(K_3^n) = 2nm - \{\sum_{i=0}^s [2a_i b_i 3^{b_i} + 2(a_i-1)3^{b_i}] + 2\sum_{i=0}^{s-1} \sum_{j=i+1}^s a_i a_j 3^{b_j}\}$, for $1 \leq m = \sum_{i=0}^s a_i 3^{b_i} \leq 3^n$, $b_0 > b_1 > \dots > b_s$, $a_i \in \{1, 2\}$, $0 \leq i \leq s$.

The n -dimensional bijective connection networks \mathcal{B}_n (also called hypercube-like networks) are a class of cube-based networks including several well known interconnection networks like hypercubes (denoted by Q_n), twisted cubes [9] (denoted by TQ_n), crossed cubes [5] (denoted by CQ_n), spined cubes SQ_n , parity cubes PQ_n , Z -cubes ZQ_n [33], varietal cubes VQ_n , Möbius cubes [4] (denoted by MQ_n), locally twisted cubes [27] (denoted by LTQ_n), restricted

hypercube-like networks $RHLLN_n$, generalized cubes GQ_n , generalized twisted cubes [1] (denoted by GTQ_n) and Mcubes [24] (denoted by MCQ_n) as members.

In 2013, Zhang et al. obtained that $\xi_m(\mathcal{B}_n) = \xi_m(K_2^n)$ [30].

Corollary III.4. $\xi_m(\mathcal{B}_n) = \xi_m(K_2^n)$ for each $m \leq 2^n$.

So our main results can be used for the n -dimensional bijective connection networks.

IV. SOME USEFUL PROPERTIES OF THE FUNCTIONS

$$\xi_m(K_L^n) \text{ AND } ex_m(K_L^n)$$

Before given our main results on \mathcal{P}_i^k -conditional edge-connectivity of K_L^n , we will give some interesting properties of functions $\xi_m(K_L^n)$ and $ex_m(K_L^n)$. If there is no specific explanation in the following, we assume that n is a positive integer $n \geq 2$.

Lemma IV.1. Let $1 \leq h = h_1 + h_2 \leq L^{\lfloor \frac{n}{2} \rfloor}$, where $h_1 = \sum_{i=0}^t a_i L^{b_i}$, $h_2 = \sum_{i=0}^s a'_i L^{b'_i}$, $a_i, a'_i \in \{1, 2, \dots, L-1\}$, and $b_0 > b_1 > \dots > b_t > b'_0 > \dots > b'_s$. Then $ex_h(K_L^n) = ex_{h_1}(K_L^n) + ex_{h_2}(K_L^n) + 2(\sum_{i=0}^t a_i)h_2$.

Proof. According to the expression of $ex_m(K_L^n)$, we have $ex_{h_1}(K_L^n) = \sum_{i=0}^t [(L-1)a_i b_i L^{b_i} + 2I_{a_i} L^{b_i}] + 2\sum_{i=0}^{t-1} \sum_{k=i+1}^t a_i a_k L^{b_k}$ and $ex_{h_2}(K_L^n) = \sum_{i=0}^s [(L-1)a'_i b'_i L^{b'_i} + 2I_{a'_i} L^{b'_i}] + 2\sum_{i=0}^{s-1} \sum_{k=i+1}^s a'_i a'_k L^{b'_k}$. Let $a_{t+1} = a'_0$, $a_{t+2} = a'_1$, \dots , $a_{t+s+1} = a'_s$, $b_{t+1} = b'_0$, $b_{t+2} = b'_1$, \dots , $b_{t+s+1} = b'_s$. Then $h_2 = \sum_{i=t+1}^{t+s+1} (a'_i L^{b'_i})$ and $ex_{h_2}(K_L^n) = \sum_{i=t+1}^{t+s+1} [(L-1)a'_i b'_i L^{b'_i} + 2I_{a'_i} L^{b'_i}] + 2\sum_{i=t+1}^{t+s} \sum_{k=i+1}^{t+s+1} a'_i a'_k L^{b'_k}$. Note that

$$\begin{aligned} ex_h(K_L^n) &= ex_{h_1+h_2}(K_L^n) \\ &= \sum_{i=0}^t [(L-1)a_i b_i L^{b_i} + 2I_{a_i} L^{b_i}] + \sum_{i=t+1}^{t+s+1} [(L-1)a'_i b'_i L^{b'_i} + 2I_{a'_i} L^{b'_i}] + 2\sum_{i=0}^{t-1} \sum_{k=i+1}^t a_i a_k L^{b_k} \\ &\quad + 2\sum_{i=t+1}^{t+s} \sum_{k=i+1}^{t+s+1} a'_i a'_k L^{b'_k} + 2\sum_{i=0}^t a_i \\ &\quad \times \sum_{i=t+1}^{t+s+1} a'_i L^{b'_i} \\ &= \sum_{i=0}^t ((L-1)a_i b_i L^{b_i} + 2I_{a_i} L^{b_i}) + 2\sum_{i=0}^{t-1} \sum_{k=i+1}^t a_i a_k L^{b_k} \\ &\quad + \sum_{i=t+1}^{t+s+1} [(L-1)a'_i b'_i L^{b'_i} + 2I_{a'_i} L^{b'_i}] \\ &\quad + 2\sum_{i=t+1}^{t+s} \sum_{k=i+1}^{t+s+1} a'_i a'_k L^{b'_k} + 2\sum_{i=0}^t a_i \\ &\quad \times \sum_{i=t+1}^{t+s+1} a'_i L^{b'_i} \\ &= ex_{h_1}(K_L^n) + ex_{h_2}(K_L^n) + 2(\sum_{i=0}^t a_i)h_2. \end{aligned}$$

Then, the proof is completed. \square

Lemma IV.2. $\xi_{m+1}(K_L^n) - \xi_m(K_L^n) \geq 0$ for $1 \leq m < L^{\lfloor \frac{n}{2} \rfloor}$.

Proof. Let $m = \sum_{i=0}^s a_i L^{b_i} < L^{\lfloor \frac{n}{2} \rfloor}$, $1 \leq a_i \leq L$. So $m = \sum_{i=0}^s a_i < \lfloor \frac{n}{2} \rfloor$. Then

$$\begin{aligned}
& \xi_{m+1}(K_L^n) - \xi_m(K_L^n) \\
&= \delta_L(K_L^n)n(m+1) - ex_{m+1}(K_L^n) - [\delta_L(K_L^n)nm \\
&\quad - ex_m(K_L^n)] \\
&= (L-1)n(m+1) - ex_{m+1}(K_L^n) - [(L-1)nm \\
&\quad - ex_m(K_L^n)] \\
&= (L-1)n - [ex_{m+1}(K_L^n) - ex_m(K_L^n)] \\
&= (L-1)n - 2 \sum_{i=0}^s a_i \\
&\geq (L-1)n - 2 \times \lfloor \frac{n}{2} \rfloor (L-1) \\
&= (L-1)(n - 2 \times \lfloor \frac{n}{2} \rfloor) \geq 0.
\end{aligned}$$

The proof is completed. \square

Lemma IV.3. For any $0 \leq g \leq L-2$, $0 \leq t \leq n-2$, then $\xi_{(g+1)L^t}(K_L^n) - \xi_{gL^t}(K_L^n) \geq 0$.

Proof. $\xi_{(g+1)L^t}(K_L^n) - \xi_{gL^t}(K_L^n) = (L-1)nL^t - (L-1)tL^t - 2gL^t = (L-1)(n-t)L^t - 2gL^t > (L-1)(n-t-2)L^t$. The last inequality holds because of $g < L-1$ and $n-t \geq 2$. So the result holds. \square

Lemma IV.4. For any $0 \leq g \leq L-1$, $0 \leq t \leq n-2$, $h_0 < L^t$, then $\xi_{gL^t+h_0}(K_L^n) - \xi_{gL^t}(K_L^n) \geq 0$.

Proof. $\xi_{gL^t+h_0}(K_L^n) - \xi_{gL^t}(K_L^n) = (L-1)nh_0 - 2gh_0 - ex_{h_0}(K_L^n) \geq (L-1)(n-2)h_0 - ex_{h_0}(K_L^{n-2}) = \xi_{h_0}(K_L^{n-2}) \geq 0$. \square

Lemma IV.5. For any positive integers $0 \leq t \leq n-2$, then $\xi_{L^{t+1}}(K_L^n) - \xi_{L^t}(K_L^n) \geq 0$.

Proof. $\xi_{L^{t+1}}(K_L^n) - \xi_{L^t}(K_L^n)$
 $= (L-1)n(L-1)L^t - ex_{L^{t+1}}(K_L^n) + ex_{L^t}(K_L^n)$
 $= (L-1)^2nL^t - [(L-1)(t+1)L^{t+1} - (L-1)tL^t]$
 $= (L-1)L^t(nL - n - tL - L - t)$
 $= (L-1)[(n-t)(L-1) - L]L^t$
 $\geq (L-1)[2(L-1) - L]L^t$
 $= (L-1)(L-2)L^t$
 ≥ 0 . \square

Lemma IV.6. For some integers L, m, n, g and t , $n \geq 2$, $L \geq 2$, $1 \leq g \leq L-1$ and $gL^t \leq m \leq L^{n-1}$, then $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$.

Proof. **Case 1:** If $g = 1$, $t = n-1$, then $\xi_{L^{n-1}}(K_L^n) \geq \xi_m(K_L^n) = \xi_{L^{n-1}}(K_L^n)$.

Case 2: For $1 \leq g \leq g_0 \leq L-1$, $t = n-2$, by **Lemma IV.3**, we have $\xi_{(g_0+1)L^{n-2}}(K_L^n) \geq \xi_{g_0L^{n-2}}(K_L^n)$ for $g_0 \leq L-2$. And by **Lemma IV.4**, $\xi_{g_0L^{n-2}+h_0}(K_L^n) \geq \xi_{g_0L^{n-2}}(K_L^n)$ for $g_0 \leq L-1$ is be obtained. So we can have $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$ for any $gL^t \leq m \leq L^{n-1}$.

Case 3: For $0 \leq t < n-2$, we divide the $gL^t \leq m \leq L^{n-1}$ into two parts $m < L^{t+1}$ and $L^{t+1} \leq m \leq L^{n-1}$.

Subcase 3.1: By **Lemma IV.5**, we can obtain that $\xi_{L^{t_0+1}}(K_L^n) \geq \xi_{L^{t_0}}(K_L^n)$ for any integer $t+1 \leq t_0 \leq n-2$. And by **Lemma IV.3** and **IV.4**, we can obtain that $\xi_m(K_L^n) \geq \xi_{L^{t+1}}(K_L^n)$ for any $L^{t+1} \leq m \leq L^{n-1}$.

Subcase 3.2: For any $gL^t \leq m < L^{t+1}$, $g \leq g_0 \leq L-1$, $\xi_{(g_0+1)L^t}(K_L^n) \geq \xi_{g_0L^t}(K_L^n)$ is be obtained by **Lemma IV.3** and $\xi_{g_0L^t+h_0}(K_L^n) \geq \xi_{g_0L^t}(K_L^n)$ for $h_0 < L^t$ is be obtained by **Lemma IV.4**. So one can have $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$ for any $gL^t \leq m < L^{t+1}$.

Combined Subcase 3.1 and Subcase 3.2, we can deduce that $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$ for any $gL^t \leq m \leq L^{n-1}$.

In summary, $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$ for $n \geq 2$, $L \geq 2$, $1 \leq g \leq L-1$, $gL^t \leq m \leq L^{n-1}$. The proof is done. \square

Lemma IV.7. For $L^{n-1} \leq m \leq \lfloor L^n/2 \rfloor$, we have $\xi_m(K_L^n) \geq \xi_{L^{n-1}}(K_L^n)$.

Proof. Since the function $\xi_m(K_L^n)$ is highly relied on the decomposition of the integer m and on the parity of L , we give the decomposition of the integer $\lfloor L^n/2 \rfloor$ as follows:

$$\lfloor L^n/2 \rfloor = \begin{cases} \frac{L}{2}L^{n-1} & \text{for even } L; \\ \sum_{i=0}^{n-1} \lfloor \frac{L}{2} \rfloor L^{n-1-i} & \text{for odd } L. \end{cases} \quad (1)$$

In order to obtain our result, three cases of decomposition of m are discussed (if L is even, there are only **Case 1** and **Case 2**).

Case 1: For any $1 \leq g < \lfloor \frac{L}{2} \rfloor$, one can obtain that

$$\begin{aligned}
& \xi_{(g+1)L^{n-1}}(K_L^n) - \xi_{gL^{n-1}}(K_L^n) \\
&= (L-1)nL^{n-1} - (L-1)(n-1)L^{n-1} - 2gL^{n-1} \\
&\geq (L-1)L^{n-1} - 2(\lfloor \frac{L}{2} \rfloor - 1)L^{n-1} \\
&= (\lceil \frac{L}{2} \rceil - \lfloor \frac{L}{2} \rfloor + 1)L^{n-1} \\
&\geq L^{n-1} \\
&> 0.
\end{aligned}$$

Case 2: For any $1 \leq g < \lfloor \frac{L}{2} \rfloor$, $h_0 < L^{n-1}$, we have

$$\begin{aligned}
& \xi_{gL^{n-1}+h_0}(K_L^n) - \xi_{gL^{n-1}}(K_L^n) \\
&= (L-1)nh_0 - 2gh_0 - ex_{h_0}(K_L^n) \\
&\geq (L-1)nh_0 - 2(\lfloor \frac{L}{2} \rfloor - 1)h_0 - ex_{h_0}(K_L^n) \\
&= (L-1)n - (L-1)h_0 + (\lceil \frac{L}{2} \rceil - \lfloor \frac{L}{2} \rfloor + 1)h_0 \\
&\quad - ex_{h_0}(K_L^n) \\
&= (L-1)(n-1)h_0 - ex_{h_0}(K_L^n) + (\lceil \frac{L}{2} \rceil - \lfloor \frac{L}{2} \rfloor + 1)h_0 \\
&= \xi_{h_0}(K_L^{n-1}) + (\lceil \frac{L}{2} \rceil - \lfloor \frac{L}{2} \rfloor + 1)h_0 \\
&\geq 0.
\end{aligned}$$

Actually, by the **Case 1** and **Case 2**, one can already prove that for $L^{n-1} \leq m \leq \lfloor \frac{L}{2} \rfloor L^{n-1}$, $\xi_m(K_L^n) \geq \xi_{L^{n-1}}(K_L^n)$. But if L is odd, for $\lfloor \frac{L}{2} \rfloor L^{n-1} \leq m \leq \sum_{i=0}^{n-1} \lfloor \frac{L}{2} \rfloor L^{n-1-i}$, two subcases are still needed to discuss.

Case 3:

Subcase 3.1: For odd L , $h_0 < \frac{1}{2}L^{n-2-j}$, then

$$\begin{aligned}
& \xi_{\sum_{i=0}^j \lfloor \frac{L}{2} \rfloor L^{n-1-j} + h_0}(K_L^n) - \xi_{\sum_{i=0}^j \lfloor \frac{L}{2} \rfloor L^{n-1-j}}(K_L^n) \\
&= (L-1)nh_0 - 2\sum_{i=0}^j \lfloor \frac{L}{2} \rfloor h_0 - ex_{h_0}(K_L^n) \\
&= (L-1)nh_0 - 2(\lfloor \frac{L}{2} \rfloor - 1)(j+1)h_0 - ex_{h_0}(K_L^n) \\
&= (L-1)(n-j-1)h_0 - ex_{h_0}(K_L^n) \\
&= (L-1)(n-j-1)h_0 - ex_{h_0}(K_L^{n-j-1}) \\
&= \xi_{h_0}(K_L^{n-j-1}) \\
&\geq 0.
\end{aligned}$$

Subcase 3.2:

$$\begin{aligned}
& \xi_{\sum_{i=0}^{j+1} \lfloor \frac{L}{2} \rfloor L^{n-1-j}}(K_L^n) - \xi_{\sum_{i=0}^j \lfloor \frac{L}{2} \rfloor L^{n-1-j}}(K_L^n) \\
&= (L-1)n\frac{L}{2}L^{n-1-j-1} - 2\sum_{i=0}^j \lfloor \frac{L}{2} \rfloor L^{n-2-j} \\
&\quad - ex_{\frac{L}{2}L^{n-2-j}}(K_L^n) \\
&= (L-1)n\lfloor \frac{L}{2} \rfloor L^{n-2-j} - (L-1)(j+1)\frac{L}{2}L^{n-2-j} \\
&\quad - ex_{\frac{L}{2}L^{n-2-j}}(K_L^n) \\
&= (L-1)(n-j-1)\frac{L}{2}L^{n-2-j} - ex_{\frac{L}{2}L^{n-2-j}}(K_L^{n-1-j}) \\
&= \xi_{\frac{L}{2}L^{n-2-j}}(K_L^{n-j-1}) \\
&\geq 0.
\end{aligned}$$

If L is odd, for $\lfloor \frac{L}{2} \rfloor L^{n-1} \leq m \leq \sum_{i=0}^{n-1} \lfloor \frac{L}{2} \rfloor L^{n-1-i}$, based on above two subcases, one can obtain that $\xi_m(K_L^{n-1}) \geq \xi_{\lfloor \frac{L}{2} \rfloor L^{n-1}}(K_L^{n-1}) \geq \xi_{L^{n-1}}(K_L^n)$. All in all, the proof is done. \square

Lemma IV.8. For some nonnegative integer $t \leq n-1$, $1 \leq g \leq L-1$, $gL^t \leq m \leq \lfloor L^n/2 \rfloor$, then $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$.

Proof. **Case 1:** For $t < n-1$, by **Lemma IV.7** and **Lemma IV.6**, the inequality $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$ holds for any $gL^t \leq m \leq \lfloor L^n/2 \rfloor$, $1 \leq g \leq t-1$.

Case 2: For $t = n-1$, we need to prove that $\xi_m(K_L^n) \geq \xi_{gL^{n-1}}(K_L^n)$ for any $gL^{n-1} \leq m \leq \lfloor L^n/2 \rfloor$, $1 \leq g \leq \lfloor L/2 \rfloor$. Although the integer satisfies $1 \leq g \leq L-1$, because of $gL^{n-1} \leq \lfloor L^n/2 \rfloor$, one can obtain that $g \leq \lfloor L/2 \rfloor$. By the proof of **Case 1** and **Case 2** in **Lemma IV.7**, $\xi_m(K_L^n) \geq \xi_{gL^{n-1}}(K_L^n)$ for any $gL^{n-1} \leq m \leq \lfloor L/2 \rfloor L^{n-1}$. If L is even, $\lfloor L^n/2 \rfloor = \frac{L}{2}L^{n-1}$, we can obtain that $\xi_m(K_L^n) \geq \xi_{gL^{n-1}}(K_L^n)$ for any $gL^{n-1} \leq m \leq \lfloor L/2 \rfloor L^{n-1} = \lfloor L^n/2 \rfloor$, our result holds. While for odd L , because of $\lfloor L^n/2 \rfloor = \sum_{i=0}^{n-1} \lfloor L/2 \rfloor L^{n-1-i}$, by the proof of Subcase 3.1 and Subcase 3.2 in **Lemma IV.7**, we have $\xi_m(K_L^n) \geq \xi_{\lfloor L/2 \rfloor L^{n-1}}(K_L^n)$ for any $\lfloor L/2 \rfloor L^{n-1} \leq m \leq \lfloor L^n/2 \rfloor$. So for odd L , $\xi_m(K_L^n) \geq \xi_{gL^{n-1}}(K_L^n)$ for any $gL^{n-1} \leq m \leq \lfloor L^n/2 \rfloor$ and $1 \leq g \leq \lfloor L/2 \rfloor$.

In summary, for any integers $0 \leq t \leq n-1$, $1 \leq g \leq L-1$, $gL^t \leq m \leq \lfloor L^n/2 \rfloor$, $\xi_m(K_L^n) \geq \xi_{gL^t}(K_L^n)$. The proof is completed. \square

Lemmas IV.3-IV.8 imply the complex layered-increasing properties and self-similarity of the function $\xi_m(K_L^n)$. It is crucial to \mathcal{P} -conditional edge-connectivities of hamming graph K_L^n . Analysing the properties of the function $\xi_m(K_L^n)$ might be of interest in their own right.

V. UNIFIED METHOD FOR \mathcal{P} -CONDITIONAL

EDGE-CONNECTIVITIES OF HAMMING GRAPH K_L^n

Theorem V.1. Let n, g, t, L be four integers, $n \geq 1$, $L \geq 2$, $0 \leq t \leq n-1$, $1 \leq g \leq L-1$ and $gL^t \leq \lfloor L^n/2 \rfloor$. If \mathcal{P} -conditional edge-connectivity is bipartite and $\theta_{\mathcal{P}}(K_L^n) = gL^t$. We obtain that $\lambda(\mathcal{P}, K_L^n) = \xi_{gL^t}(K_L^n) = \delta_L(K_L)n g L^t - \delta_L(K_p) g t L^t - g(g-1)L^t = g[(L-1)(n-t) - (g-1)]L^t$.

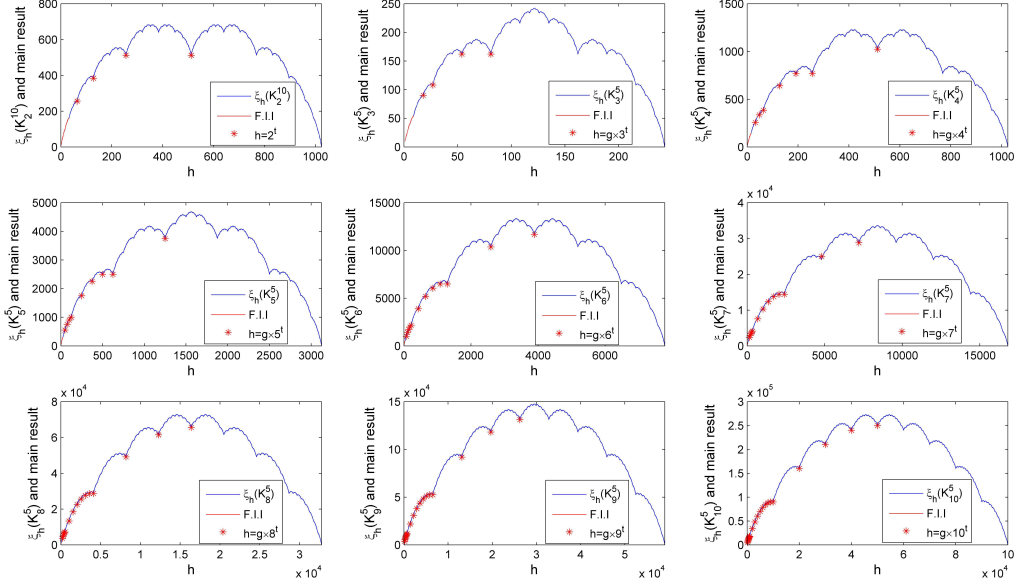


Fig. 2. The main results on $\lambda(\mathcal{P}_i^k, K_L^n)$ with $\theta_{\mathcal{P}}(K_L^n) = gL^t$ and $\theta_{\mathcal{P}}(K_L^n) \leq L^{\lfloor \frac{n}{2} \rfloor}$, $\xi_h(K_L^n)$ and $\xi_{gL^t}(K_L^n)$ for $2 \leq L \leq 10$.

Proof. Lower bound: For any conditional \mathcal{P} , let F be any minimum \mathcal{P} -conditional edge-cut of K_L^n , and $K_L^n - F$ is disconnected. And its deletion from K_L^n results in p components $C_1, C_2 \dots C_p$, $p \geq 2$. Each component satisfies the condition \mathcal{P} . As \mathcal{P} -conditional edge-connectivity of K_L^n is bipartite, and the minimality of $\lambda(\mathcal{P}, K_L^n)$ edge-cut of K_L^n , we have $p = 2$. Let C^* be the resulting minimum component. Because of $\delta_{\mathcal{P}}(K_L^n) = gL^t$, by the **Lemma IV.8**, we have $|F| \geq [|V(C^*), \overline{V(C^*)}] = \xi_{gL^t}(K_L^n) = g[(L-1)(n-t) - (g-1)]L^t$.

The above equality holds because of $\xi_{gL^t}(K_L^n) = \delta_{L-1}(K_L^n)ngL^t - (L-1)gL^t - g(g-1)L^t = g[(L-1)(n-t) - (g-1)]L^t$.

Upper bound: It is sufficient to show that, there exists a \mathcal{P} -conditional edge-cut of K_L^n with size $\xi_{gL^t}(K_L^n) = g[(L-1)(n-t) - (g-1)]L^t$, $K_L^n[L_{gL^t}^n] = \overbrace{0 \dots 00}^{n-t} X_t \dots X_2 X_1 \cup \overbrace{0 \dots 01}^{n-t} X_t \dots X_2 X_1 \cup \dots \cup \overbrace{0 \dots 0g-1}^{n-t} X_t \dots X_2 X_1$. Each $\overbrace{0 \dots 0f}^{n-t} X_t \dots X_2 X_1$ for $0 \leq f \leq g-1$ is isomorphic to L -ary t -dimensional sub-layers and $(L-1)t$ regular. The minimum degree of them happens to be $(L-1)t$. By the definition of K_L^n , there exists at least one edge

between these L -ary t -dimensional sub-layers. So, the induced subgraph $K_L^n[L_{gL^t}^n]$ is connected with gL^t vertices, and contains a cycle (If $L = 2$, $t \geq 2$; if $L \geq 3$, $t \geq 1$). On the other hand, $|\overline{L_{gL^t}^n}| = L^n - gL^t = (L-1)L^{n-1} + (L-1)L^{n-2} + \dots + (L-1)L^{t+1} + (L-g)L^t$.

$$K_L^n[\overline{L_{gL^t}^n}] = \bigcup_{t+1 \leq k \leq n-1, 1 \leq b \leq L-1} \overbrace{0 \dots 0b}^{n-k} X_k \dots X_2 X_1 \cup \bigcup_{g \leq o \leq L-1} \overbrace{0 \dots 0o}^{n-t} X_t \dots X_2 X_1.$$

For $t+1 \leq k \leq n-1, 1 \leq b \leq L-1$, each $\overbrace{0 \dots 0b}^{n-k} X_k \dots X_2 X_1$ is L -ary k -dimensional sub-layer of K_L^n , which is $(L-1)k$ regular and isomorphic to K_L^k . There is at least one edge between different sub-layers. For each $g \leq o \leq L-1$, $\overbrace{0 \dots 0o}^{n-t} X_t \dots X_2 X_1$ is L -ary t -dimensional sub-layer of K_L^n , which is $(L-1)t$ regular and isomorphic to K_L^t . Also there is at least one edge between different sub-layers. And $\overbrace{0 \dots 0l}^{n-t-1} X_{t+1} \dots X_2 X_1$ are connected with $\overbrace{0 \dots 0L-1}^{n-t} X_t \dots X_2 X_1$ by the definition of K_L^n . So $K_L^n[\overline{L_{gL^t}^n}]$ is connected with $L^n - gL^t$ vertices ($\geq gL^t$), and contains at least a cycle. As \mathcal{P} -conditional edge-connectivity

is bipartite and $\theta_{\mathcal{P}}(K_L^n) = gL^t$. Thus, $[L_{gL^t}^n, \overline{L_{gL^t}^n}]$ is a \mathcal{P} -conditional edge-cut of K_L^n . So $|[L_{gL^t}^n, \overline{L_{gL^t}^n}]| = \xi_{gL^t}(K_L^n) = g[(L-1)(n-t) - (g-1)L^t]$. \square

By **Theorem V.1**, the exact values of L^t -extra edge-connectivity, t -embedded edge-connectivity, cyclic edge-connectivity, $(L-1)t$ -super edge-connectivity, $(L-1)t$ -average edge-connectivity and L^t -th isoperimetric edge-connectivity of K_L^n share the same values in form of $(L-1)(n-t)L^t$ because they are all bipartite.

Based on the **Theorem V.1** and the following **Theorem V.4**, an algorithm can be designed to calculate bipartite \mathcal{P} -conditional edge-connectivity of K_L^n with $\theta_{\mathcal{P}(K_L^n)}(K_L^n) = gL^t$ or $\theta_{\mathcal{P}(K_L^n)}(K_L^n) \leq L^{\lfloor \frac{n}{2} \rfloor}$ in the **Algorithm.1**. In this algorithm, $O(K_L) = L^{\lfloor \frac{n}{2} \rfloor}$. The time complexity of the algorithm is $O(\log_L(N))$, where $N = L^n$.

Corollary V.2. For any integers $0 \leq t \leq n-1$, $\lambda(\mathcal{P}_1^t, K_L^n)$, $\lambda(\mathcal{P}_2^t, K_L^n)$, $\lambda(\mathcal{P}_4^{(L-1)t}, K_L^n)$, $\lambda(\mathcal{P}_5^{(L-1)t}, K_L^n)$ and $\lambda(\mathcal{P}_6^t, K_L^n)$ share the same value $\xi_{L^t}(K_L^n) = (L-1)(n-t)L^t$. And $\lambda(\mathcal{P}_3^c, K_L^n) = g[(L-1)(n-t) - (g-1)L^t]$.

Proof. As the hamming graph K_L^n is $(L-1)n$ regular, with L^n vertices, for any integers $0 \leq t \leq n-1$, $\lambda(\mathcal{P}_1^t, K_L^n)$, $\lambda(\mathcal{P}_2^t, K_L^n)$, $\lambda(\mathcal{P}_3^c, K_L^n)$, $\lambda(\mathcal{P}_4^{(L-1)t}, K_L^n)$, $\lambda(\mathcal{P}_5^{(L-1)t}, K_L^n)$ and $\lambda(\mathcal{P}_6^t, K_L^n)$ are well-defined, and then the set $\{\mathcal{P}_1^t, \mathcal{P}_2^t, \mathcal{P}_3^c, \mathcal{P}_4^{(L-1)t}, \mathcal{P}_5^{(L-1)t}, \mathcal{P}_6^t\} \subseteq \mathcal{B}$. By **Theorem V.1**, the exact values of $\lambda(\mathcal{P}_1^t, K_L^n)$, $\lambda(\mathcal{P}_2^t, K_L^n)$, $\lambda(\mathcal{P}_4^{(L-1)t}, K_L^n)$, $\lambda(\mathcal{P}_5^{(L-1)t}, K_L^n)$ and $\lambda(\mathcal{P}_6^t, K_L^n)$ share the same value $\xi_{L^t}(K_L^n) = (L-1)(n-t)L^t$, which equal to the minimum number of links-faulty resulting in an L -ary t -dimensional sub-layer with L^t vertices from K_L^n . Note that the value of g is 1, but for the case of cyclic edge-connectivity, we have $g \leq 3$. Actually, for K_2^n , the shortest cycle is C_4 with length $L^2 = 2^2$ for $g = 1$, $t = 2$, and for K_3^n , the shortest cycle is C_3 with length $L^1 = 3^1$ for $g = 1$, $t = 1$, while for K_L^n , $L \geq 4$, the shortest cycle is C_3 with length $3L^0 = 3 \times L^0$ for $g = 3$, $t = 0$. So $\lambda(\mathcal{P}_3^c, K_L^n) = g[(L-1)(n-t) - (g-1)L^t]$. The proof is finished. \square

Corollary V.3. For any integers $0 \leq t \leq n-1$,

1). [29](2021) $\lambda(\mathcal{P}_2^t, K_3^n) = \eta_t(K_3^n) = \xi_{3^t}(K_3^n) = 2(n-t)3^t$;

Algorithm 1: \mathcal{P} -conditional edge-connectivity of K_L^n

Result: Calculating the h -extra edge-connectivity of K_L^n

Input : Given positive integers n , L , i , k and K_L^n , $\theta_{\mathcal{P}}(K_L^n), O(K_L)$

Output: The \mathcal{P} conditional edge connectivity of K_L^n is S

```

1  $I_0 \leftarrow 0$ ;  $v \leftarrow 1$ ; while  $v \leq L$  do
2    $I_v \leftarrow \frac{v(v-1)}{2}$ ;  $v \leftarrow v + 1$ ;  $\delta_v \leftarrow v - 1$ ;
3 end
4 if  $\theta_{\mathcal{P}}(K_L^n) \leq O(K_L)$  then
5    $m \leftarrow \theta_{\mathcal{P}}(K_L^n)$ ;
6 end
7 if  $\theta_{\mathcal{P}}(K_L^n) == gL^t$  then
8    $m \leftarrow gL^t$ ;
9 end
10  $t \leftarrow m$ ;
11  $s' \leftarrow \lfloor \log_L t \rfloor + 1$ ;
12 while  $s' == s' - 1$  do
13   if  $t \leq L$  then
14      $c(s') \leftarrow t$ ; break
15   else
16      $c(s') \leftarrow \text{mod}(t, L)$ ;  $t \leftarrow \lfloor \frac{t}{L} \rfloor$ ;  $s' \leftarrow s' - 1$ ;
17   end
18 end
19  $k \leftarrow 1$ ;  $z \leftarrow 1$ ;
20 while  $k \leq \lfloor \log_L m \rfloor + 1$  do
21   if  $c(k) \neq 0$  then
22      $a(z) \leftarrow c(k)$ ;  $b(z) \leftarrow \lfloor \log_L m \rfloor + 1 - k$ ;
23      $z \leftarrow z + 1$ ;
24   else
25      $k \leftarrow k + 1$ ;
26   end
27  $q \leftarrow 1$ ;  $S \leftarrow \delta_L n m$ ;
28 while  $q \leq z$  do
29   if  $q \leftarrow 1$  or  $q \leftarrow z$  then
30      $S \leftarrow S - \delta_L a(q) b(q) L^{b(q)} - I_{a(q)} L^{b(q)}$ ;
31      $q \leftarrow q + 1$ ;
32   else
33      $S \leftarrow S - \delta_L a(q) b(q) L^{b(q)} - I_{a(q)} L^{b(q)}$ ;
34      $w \leftarrow q + 1$ ;
35     while  $w \leq z$  do
36        $S \leftarrow S - 2\delta_{a(w)} a(w) L^{b(w)}$ ;  $w \leftarrow w + 1$ ;
37     end
38    $q \leftarrow q + 1$ ;
39 end

```

2). [13](2012) $\lambda(\mathcal{P}_2^t, K_2^n) = \eta_t(K_2^n) = \xi_{2^t}(K_2^n) = (n-t)2^t$;

3). [14](2013) $\lambda(\mathcal{P}_2^t, \mathcal{B}_n) = \eta_t(\mathcal{P}_2) = \xi_{2^t}(\mathcal{B}_n) = (n-t)2^t$.

For $2 \leq L \leq 10$, the main results of \mathcal{P}_i^k -conditional edge-connectivity of K_L^n with $\theta_{\mathcal{P}}(K_L^n) = gL^t$ and $\theta_{\mathcal{P}}(K_L^n) \leq$

TABLE III
 \mathcal{P}_i^k -CONDITIONAL EDGE-CONNECTIVITY $\lambda(\mathcal{P}_i^k, K_L^n)$.

$\lambda(\mathcal{P}_1^k, K_L^n)$	t	$K_2^n = Q_n$	K_3^n	K_4^n	\dots	K_L^n
$\lambda(\mathcal{P}_1^{L^t}, K_L^n)$	$0 \leq t \leq n-1$	$(n-t)2^t$	$2(n-t)3^t$	$3(n-t)4^t$	\dots	$(L-1)(n-t)L^t$
$\lambda(\mathcal{P}_2^t, K_L^n)$	$0 \leq t \leq n-1$	$(n-t)2^t$	$2(n-t)3^t$	$3(n-t)4^t$	\dots	$(L-1)(n-t)L^t$
$\lambda(\mathcal{P}_3^n, K_L^n)$		$(n-2)2^2$	$2(n-1)3^1$	$3[(n-0)-2]4^0$	\dots	$3[(L-1)(n-0)-2]L^0$
$\lambda(\mathcal{P}_4^{(L-1)^t}, K_L^n)$	$0 \leq t \leq n-1$	$(n-t)2^t$	$2(n-t)3^t$	$3(n-t)4^t$	\dots	$(L-1)(n-t)L^t$
$\lambda(\mathcal{P}_5^{(L-1)^t}, K_L^n)$	$0 \leq t \leq n-1$	$(n-t)2^t$	$2(n-t)3^t$	$3(n-t)4^t$	\dots	$(L-1)(n-t)L^t$
$\lambda(\mathcal{P}_6^{L^t}, K_L^n)$	$0 \leq t \leq n-1$	$(n-t)2^t$	$2(n-t)3^t$	$3(n-t)4^t$	\dots	$(L-1)(n-t)L^t$

$L^{\lfloor \frac{n}{2} \rfloor}$, $\xi_h(K_L^n)$ and $\xi_{g_L^t}(K_L^n)$ are shown in the Fig.2, where F.I.I represents the exact values of function $\xi_h(K_L^n)$ in the first increasing interval $1 \leq h \leq L^{\lfloor \frac{n}{2} \rfloor}$. Once the \mathcal{P} -conditional edge-connectivity of K_L^n is bipartite and $\theta_{\mathcal{P}}(K_L^n) = \min\{|X| | K_L^n[X] \text{ satisfying the property } \mathcal{P}\}$ is less than $L^{\lfloor \frac{n}{2} \rfloor}$, the exact value of \mathcal{P} -conditional edge-connectivity of K_L^n is $\xi_{\theta_{\mathcal{P}}(K_L^n)}(K_L^n)$. We also investigate the cases of bipartite \mathcal{P} -conditional edge-connectivity of K_L^n with $\theta_{\mathcal{P}}(K_L^n) = g_L^t$. The exact values of \mathcal{P}_i^k -conditional edge-connectivity of K_L^n , $\lambda(\mathcal{P}_i^k, K_L^n)$, are shown in Table III.

Theorem V.4. Let n and L be two positive integers. If \mathcal{P} -conditional edge-connectivity of K_L^n is bipartite with $h = \theta_{\mathcal{P}}(K_L^n) \leq L^{\lfloor \frac{n}{2} \rfloor}$. Then, one can obtain that $\lambda(\mathcal{P}, K_L^n) = \xi_{\theta_{\mathcal{P}}(K_L^n)}(K_L^n) = \xi_h(K_L^n) = (L-1)nh - ex_h(K_L^n)$ where $h = \sum_{i=0}^s a_i L^{b_i}$, $ex_h(K_L^n) = \sum_{i=0}^s [(L-1)a_i b_i L^{b_i} + (a_i - 1)a_i L^{b_i}] + 2 \sum_{i=0}^{s-1} \sum_{j=i+1}^s a_i a_j L^{b_j}$.

Proof. As for each $1 \leq m \leq \lfloor L^n/2 \rfloor$, one can find a subset $X_m^* = L_m^n \subset V(K_L^n)$ satisfying that $\xi_m(K_L^n) = |\lfloor L_m^n, \overline{L_m^n} \rfloor|$, $|L_m^n| = m$ and both $K_L^n[L_m^n]$ and $K_L^n[\overline{L_m^n}]$ are connected with $2|E(K_L^n[L_m^n])| = ex_m(K_L^n)$. Because \mathcal{P} -conditional edge-connectivity is bipartite, for any minimum \mathcal{P} -conditional edge-cut of K_L^n with m^* vertices for smaller component for some $m^* \leq \lfloor L^n/2 \rfloor$. By **Lemma IV.2**, **Lemma IV.8** and $\theta_{\mathcal{P}}(K_L^n) = h \leq L^{\lfloor \frac{n}{2} \rfloor}$, we can deduce that $\lambda(\mathcal{P}, K_L^n) \geq \xi_{\theta_{\mathcal{P}}(K_L^n)}(K_L^n) = \xi_h(K_L^n)$. Because $[\lfloor L_{\theta_{\mathcal{P}}(K_L^n)}^n, \overline{L_{\theta_{\mathcal{P}}(K_L^n)}^n} \rfloor]$ is a \mathcal{P} -conditional edge-cut, $\lambda(\mathcal{P}, K_L^n) \leq |\lfloor L_{\theta_{\mathcal{P}}(K_L^n)}^n, \overline{L_{\theta_{\mathcal{P}}(K_L^n)}^n} \rfloor| = |\lfloor L_h^n, \overline{L_h^n} \rfloor| = \xi_h(K_L^n) = (L-1)nh - ex_m(K_L^n)$. So $\lambda(\mathcal{P}, K_L^n) = \xi_{\theta_{\mathcal{P}}(K_L^n)}(K_L^n) = \xi_h(K_L^n) = (L-1)nh - ex_h(K_L^n)$. The result holds. \square

Corollary V.5. Let n , h and L be three positive integers, $h \leq L^{\lfloor \frac{n}{2} \rfloor}$. The h -extra edge-connectivity of K_L^n is $\lambda_h(K_L^n) = \lambda_h(K_L^n) = (L-1)nh - ex_h(K_L^n)$ where

$$h = \sum_{i=0}^s a_i L^{b_i}, \quad ex_h(K_L^n) = \sum_{i=0}^s (L-1)a_i b_i L^{b_i} + (a_i - 1)a_i L^{b_i} + 2 \sum_{i=0}^{s-1} \sum_{j=i+1}^s a_i a_j L^{b_j}.$$

Our results improve the case for $L = 2, 3$, $h \leq 2^{\lfloor \frac{n}{2} \rfloor}$.

Corollary V.6. For any integer $0 \leq t \leq n-1$, $L = 2, 3$, $h \leq L^{\lfloor \frac{n}{2} \rfloor}$.

- (1) $[31](2018)\lambda(\mathcal{P}_1^h, K_3^n) = \lambda_h(K_3^n) = \xi_h(K_3^n)$;
- (2) $[19](2013)\lambda(\mathcal{P}_1^t, K_2^n) = \lambda_h(K_2^n) = \xi_h(K_2^n)$;
- (3) $[30](2014)\lambda(\mathcal{P}_1^h, \mathcal{B}_n) = \lambda_h(\mathcal{B}_n) = \xi_h(\mathcal{B}_n)$;
- (4) $[10](2013)\lambda(\mathcal{P}_1^4, \mathcal{B}_n) = \lambda_4(\mathcal{B}_n) = \xi_4(\mathcal{B}_n) = 4n - 8$.

VI. CONCLUSION

Reliability evaluation and fault tolerance of an interconnection network of some parallel and distributed systems are discussed separately under various link-faulty hypotheses in terms of different \mathcal{P} -conditional edge-connectivity, where \mathcal{P} is some graph-theoretic property of a connected graph G . This paper deals with the \mathcal{P} -conditional edge-connectivities of hamming graph K_L^n with satisfying the property that each minimum \mathcal{P} -conditional edge-cut separates the K_L^n just into two components. And these \mathcal{P} -conditional edge-connectivity is called bipartite. We show that for hamming graph K_L^n , the L^t -extra edge-connectivity, t -embedded edge-connectivity, $(L-1)t$ -super edge-connectivity, $(L-1)t$ -average edge-connectivity and L^t -th isoperimetric edge-connectivity share the same values in form of $(L-1)(n-t)L^t$. Besides, we also obtain the exact values of h -extra edge-connectivity and h -th isoperimetric edge-connectivity of hamming graph K_L^n for each $h \leq L^{\lfloor \frac{n}{2} \rfloor}$. Among these six kinds of connectivity, the solution of h -extra edge-connectivity is crucial for solving the other five or even more general \mathcal{P} -conditional edge-connectivity. Because if any abstract \mathcal{P} -conditional edge-connectivity with bipartite property, the smallest cut must be divided into two components, which is composed of a component and its complement. The exact value of the \mathcal{P} -conditional edge-connectivity can be determined according to the size of the $\lambda(\mathcal{P})$ -atom. The value of h -extra edge-connectivity is just a description of the number vertices in the component. Our results improve several previous results on this topic and can be applied to bijective connection

networks, which contain hypercubes, twisted cubes, crossed cubes, Möbius cubes, locally twisted cubes and so on.

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