# A Lower Bound on the Number of Boolean Functions with Median Correlation Immunity 

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#### Abstract

Аннотация The number of $n$-ary balanced correlation immune (resilient) Boolean functions of order $\frac{n}{2}$ is not less than $n^{2^{(n / 2)-2}(1+o(1))}$ as $n \rightarrow \infty$.

Keywords - resilient function, correlation immune function, orthogonal array.


## 1 Introduction

A set $Q_{q}^{n}=\{0,1, \ldots, q-1\}^{n}$ with Hamming metric is called an $n$-dimensional hypercube. A hypercube is called Boolean if $q=2$. A subset of $Q_{q}^{n}$ consisting of $n$-tuples with fixed values in fixed $(n-m)$ coordinates is called $m$-dimensional face ( $m$-face).

A function $f: Q_{q}^{n} \rightarrow\{0,1\}$ is called correlation immune of order $r$ if it takes the value 1 the same number of times for each $(n-r)$-face of the hypercube. A correlation immune function is called balanced (a resilient function) if it takes values 0 and 1 the same number of times.

Applications of correlation immune functions in cryptography and connections between these functions and orthogonal arrays are discussed in [9]. Further we will investigate resilient functions. An asymptotic number (as $n \rightarrow \infty$ ) of resilient Boolean functions of order $r=$ const was found in [3] and [4]. Methods from [3] and [4] are developed in [1] and [5] for the calculation of the asymptotic number of resilient Boolean functions of order $r=O(n / \ln n)$. The resilient Boolean functions of order $n-c$, where $c=$ const, are listed in [8]. The number of resilient Boolean functions of order $\alpha n$ for $0<\alpha<1$ remained completely unknown as $n \rightarrow \infty$. The asymptotic of double logarithm of the number of such functions is unknown. In [2] it is obtained some upper bound $2^{2^{n-\varepsilon(\alpha)}}$ of the number of resilient Boolean functions of order $\alpha n$. This bound is a bit better then a trivial upper bound based on an estimation of algebraic degree of a correlation immune function with some order. Now we consider the case $\alpha=1 / 2$ and obtain a lower bound $n^{2^{(n / 2)-1}(1+o(1))}$ for the number of resilient functions of order $\frac{n}{2}-1$. This bound is a bit better then a well-known lower bound $2^{2^{n / 2}}$.

## 2 Main results

The lower bound $2^{2^{n / 2}}$ follows from a simple construction. Suppose that $n=2 m$. Consider an arbitrary Boolean function $f: Q_{2}^{m} \rightarrow Q_{2}$. Define a function $F: Q_{2}^{2 m} \rightarrow Q_{2}$ by the equation $F(x, y)=f(x) \oplus|y|$, where $|y|$ is the parity of the Hamming weight of $y$. It is clear that $F$ takes values 0 and 1 the same number of times in each face with unfixed coordinate $y_{i}, i=1, \ldots, m$. Consequently, $F$ is a resilient function of order $m-1$. The number of such functions is equal to $2^{2^{m}}$.

In this paper we improve this bound. At first we consider correlation immune functions in $Q_{4}^{n}$. In [7] it was found a sharp asymptotic bound $3^{n+1} 2^{2^{n}+1}(1+o(1))$ of the number of correlation immune functions $f: Q_{4}^{n} \rightarrow Q_{2}$ of order $n-1$ with the frequency of ones $\frac{1}{4}$. But we need the number of correlation immune functions $f: Q_{4}^{n} \rightarrow Q_{2}$ of order $n-1$ with the frequency of ones $\frac{1}{2}$.

Lemma 1 The number of splittings of $Q_{2}^{n}$ into pairwise nonintersecting faces is equal to $n^{2^{n-1}(1+o(1))}$ as $n \rightarrow \infty$.

Proof. A lower bound follows from the estimation of the numbers of perfect matchings in Boolean hypercube (see [6]). Let us prove an upper bound. Consider a splitting of $Q_{2}^{n}$ and arbitrary face $L$ from this splitting. If for each vertex in $L$ with even weight we have a direction to an arbitrary neighbor vertex in $L$ with minimum possible weight then we can recover this face. Consequently, $2^{n-1}$ numbers from the set $\{1, \ldots, n\}$ are sufficient in order to uniquely indicate the splitting. $\Delta$

Suppose that a splitting of $Q_{2}^{n}$ contains 0 -dimensional faces. It is possible to use number 0 for an indication that $L$ is a 0 -dimensional face. In this case the asymptotical bound is the same because $(n+1)^{2^{n-1}}=n^{2^{n-1}(1+o(1))}$ as $n \rightarrow \infty$.

Let us consider $Q_{4}^{n}$ as the Cartesian product $Q_{2}^{n} \times Q_{2}^{n}$. A splitting of $Q_{2}^{n}$ into pairwise nonintersecting faces induces a splitting of $Q_{4}^{n}$ into blocks. One part of coordinates takes two values in a block and another part of coordinates takes four values in a block. For example, consider the block $B=\{0,1\}^{k} \times\{0,1,2,3\}^{n-k}$. Define a function $f: Q_{4}^{n} \rightarrow Q_{2}$ on $B$ by the following equation

$$
\left.f\right|_{B}(x)=\chi_{1}\left(x_{1}\right) \oplus \cdots \oplus \chi_{1}\left(x_{k}\right) \oplus \chi_{2,3}\left(x_{k+1}\right) \oplus \cdots \oplus \chi_{2,3}\left(x_{n}\right),
$$

where $\chi_{2,3}$ and $\chi_{1}$ are indicators of the sets $\{2,3\}$ and $\{1\}$. It is clear that $f$ is a resilient function of order $n-1$. It is not different to verify the following statements.

Lemma 2 Different splittings of $Q_{2}^{n}$ correspond to different resilient functions $f: Q_{4}^{n} \rightarrow Q_{2}$ of order $n-1$.

Theorem 1 There exist at least $n^{2^{(n / 2)-1}(1+o(1))}$ different resilient Boolean functions of order $\frac{n}{2}-1$.

Proof. Define an arbitrary bijection $\varphi: Q_{2}^{2} \rightarrow Q_{4}$. Suppose $f: Q_{4}^{n} \rightarrow Q_{2}$ is a resilient function of order $n-1$. Define function $F: Q_{2}^{2 n} \rightarrow Q_{2}$ by equation $F(x, y)=f\left(\varphi\left(x_{1}, y_{1}\right), \ldots, \varphi\left(x_{n}, y_{n}\right)\right)$. Let us prove that $F$ is a resilient Boolean function of order $n-1$. Consider an arbitrary $(n+1)$ dimensional face $\Gamma$. There exists $i \in\{1, \ldots, n\}$ such that the pair of coordinates $\left(x_{i}, y_{i}\right)$ is not fixed in $\Gamma$. Since $f$ takes each of the values 0 and 1 two times in any 1-dimensional face of $Q_{4}^{n}, F$ takes each of the values 0 and 1 the same number of times in $\Gamma$. It is clear that different resilient functions $f_{1}$ and $f_{2}$ correspond to different resilient functions $F_{1}$ and $F_{2}$. So the Theorem 1 follows from Lemmas 1 and 2. $\Delta$

By the simple construction described in the beginning of this section we can increase together number of variables and correlation immunity of function.

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