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Steering Car-Like Systems with Trailers Using Sinusoids

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Abstract

Methods for steering car-like robots with trailers are investigated. A connection is demonstrated between Murray and Sastry's work of steering with integrally-related sinusoids and Sussmann and Liu's recent work on asymptotic behavior of systems with high-frequency sinusoids as inputs. The merits of coordinate transformations, relative to the convergence properties, are discussed. Simulation results for a car-like robot with two trailers are presented.

1 Introduction

Our interests in this paper are in steering a specific class of systems with so-called "nonholonomic" constraints, namely nonintegrable constraints on the velocities of the configuration variables. In particular, in this paper we are interested in car-like systems with n trailers. There is a great deal of current excitement in control theory, classical mechanics and robotics communities on the use of techniques from the theory of Lie algebras and also so-called reduction techniques to obtain solutions to these problems. In the context of motion planning for mobile robots, these problems were first investigated for the Hilare family of mobile robots by one of us with the research group of Giralt at LAAS, Toulouse (see, for example [9]). Taking up from this work and also drawing from nonholonomic problems arising from the manipulation of objects by fingers rolling on the surface of an object [11] we explored in [12] the use of sinusoidal inputs in steering certain classes of nonholonomic systems. The key to the approach lay in a theorem of Brockett [2] in optimal steering of control systems whose control algebras are Heisenberg algebras. The steering by sinusoids works well for a class of systems which can be converted into a chained form.

At the same time, Sussmann and coworkers (Lafferiere and Liu) have been developing a set of tools for steering general drift free control systems. The results of Sussmann and Lafferiere [8], in particular explained the use of piecewise constant inputs for steering control systems whose Lie Algebras are nilpotent. Also, they brought to the field the machinery of Phillip Hall bases for freely generated Lie algebras. On the other hand, the paper of Sussmann and Liu [16] brought in the use of sinusoids of asymptotically high frequency and amplitude for steering systems which are not necessarily nilpotent. The key to their results were some recent advances by Kurzweil and Jarnik. There have been also several other approaches to nonholonomic motion planning which are too numerous to exhaustively list here (for example, [1]).

In this paper we make the connection between the sinusoidal steering algorithms of Murray and Sastry and the asymptotically high frequency and high amplitude sinusoids of Sussmann and Liu. The connection is made in the context of steering a car like system (mobile robot, Hilare) with 2 trailers. It has been shown in [10] that such systems with n trailers are completely controllable and bases spanning the control Lie algebra generated. It is not, however, possible to transform systems like this into the chained form necessary for applying the algorithm of Murray and Sastry, though a weakening of their theorem presented in this paper enables the system to be put into an order $\rho = 1$ chained form. Once this approximate chained form is obtained, the theory of the use of asymptotically high frequency, high amplitude sinusoids of Sussmann and Liu may be simplified to this situation. Also, the rate of convergence of the transformed system is considerably enhanced. Thus the contributions of the paper are:

1. To give a weakened form of the chained form transformation theorem of Murray and Sastry which may be applied to the car like robot with 2 trailers.
2. To show the enhanced convergence of the asymptotic sinusoidal input steering algorithm when applied to the system transformed into approximate chained form.

The outline of the paper is as follows: in Section 2, we give a description of chained form systems and the transformation theorems for transforming given systems into approximate chained form. We apply this theorem to the car like system with 2 trailers in Section 3. Section 4 gives a review of the use of asymptotic sinusoids for steering controllable systems. Section 5 contains the application of the theory to the car like system with 2 trailers, explaining the details of the choice of the frequencies satisfying noninterference conditions and the enhanced convergence on the transformed system.

2 Chained Form Systems

Many of the results we present require some background in control theory and differential geometry. For definitions of several terms such as vector fields, distributions, Lie brackets, and Lie derivatives, as well as more advanced topics in controllability, the reader unfamiliar with these concepts may wish to consult [6] or [15].

We first present some of the results that can be obtained for nonholonomic systems in a so-called "chained canonical form," *i.e.* systems with two inputs which are in the following form:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{1}$$

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These systems are completely controllable (see [6] or [15]), indeed, a method for steering them is shown here and in [13, 14].

Consider the system (1) with inputs of the form: $u_1 = \alpha \sin \omega t, u_2 = \beta \cos k\omega t$ By directly integrating the system over one period $T = \frac{2\pi}{\omega}$, it is seen that the first $k+1$ states, $x_1, x_2, \ldots, x_{k+1}$, will return to their original values (after completing a closed loop in the state space). However, the state x_{k+2} will change by exactly $x_{k+2}(T) - x_{k+2}(0) = \frac{\alpha^k \beta}{2^k k!} T$.

The steering algorithm is step-by-step. First, choose u_1 and u_2 to be constant such that they steer x_1 and x_2 to their desired values. The other states will drift in some fashion. Then, starting with x_3, choose inputs of the form above to steer each x_k to its desired value. In general, $n-1$ steps will be needed to steer all n states of the chained form. For more details, see [14].

Many systems can be put into this canonical chained form by a coordinate change and state feedback. A set of sufficient conditions is presented here:

Proposition 1 (Conversion to Chained Form [14]) *Consider a system*

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \tag{2}$$

with input vector fields g_1, g_2 having the following special form:

$$\begin{aligned} g_1(x) &= \frac{\partial}{\partial x_1} + \sum_{i=2}^n g_1^i \frac{\partial}{\partial x_i} \\ g_2(x) &= \sum_{i=2}^n g_2^i \frac{\partial}{\partial x_i} \end{aligned} \tag{3}$$

Define the distributions

$$\begin{aligned} \Delta_0 &= \operatorname{span}\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-2} g_2\} \\ \Delta_1 &= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-3} g_2\} \end{aligned}$$

If for some open set U, $\Delta_0(x) = \mathbf{R}^n$ for all $x \in U$ and Δ_1 is involutive on U, then there exists a local feedback transformation on U: $\xi = \phi(x), u = \beta(x)v$ such that the transformed system is in chained form.

The proof is by construction, and can be found in [14].

Many systems will satisfy the conditions of the proposition, and others can assume this special form after a simple change of coordinates or input (such as dividing through by g_1^1). These systems can then be transformed into the chained canonical form and steered using the Murray and Sastry algorithm presented above. However in some cases the distribution Δ_1 is not involutive and so Proposition 1 does not apply. We can still try a coordinate transformation to put the system into an "approximate" chained form. Here we take some inspiration from the work of Krener [7] and Hauser, Sastry, and Kokotovic [5], and we will give sufficient conditions such that a system of the form (2) can be transformed so that it agrees with the canonical chained form up to terms of some higher order ρ.

Proposition 2 (Approximate Chained Form.) *Consider a system $\dot{x} = g_1(x)u_1 + g_2(x)u_2$ with g_1, g_2 having the following special form:*

$$\begin{aligned} g_1(x) &= \frac{\partial}{\partial x_1} + \sum_{i=2}^n g_1^i \frac{\partial}{\partial x_i} \\ g_2(x) &= \sum_{i=2}^n g_2^i \frac{\partial}{\partial x_i} \end{aligned}$$

Consider some order-ρ *approximation of the input vector fields, $\tilde{g}_1$ and $\tilde{g}_2$, where* [1]

$$g_1(x) = \tilde{g}_1(x) + O(x)^{\rho+1}, g_2(x) = \tilde{g}_2(x) + O(x)^{\rho+1}$$

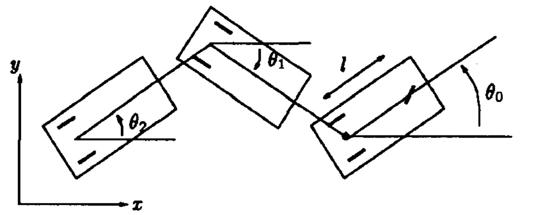

Figure 1: Hilare with 2 Trailers.

Define the distributions

$$\begin{aligned} \tilde{\Delta}_0 &= \operatorname{span}\{\tilde{g}_1, \tilde{g}_2, \mathrm{ad}_{\tilde{g}_1} \tilde{g}_2, \ldots, \mathrm{ad}_{\tilde{g}_1}^{n-2} \tilde{g}_2\} \\ \tilde{\Delta}_1 &= \operatorname{span}\{\tilde{g}_2, \mathrm{ad}_{\tilde{g}_1} \tilde{g}_2, \ldots, \mathrm{ad}_{\tilde{g}_1}^{n-3} \tilde{g}_2\} \end{aligned}$$

If, for some open set U, $\tilde{\Delta}_0(x) = \mathbf{R}^n$ for all $x \in U$ and $\tilde{\Delta}_1$ is involutive on U, then there exists a local feedback transformation on U: $\xi = \phi(x), u = \beta(x)v$ such that the transformed system is in order-ρ chained form, *that is:*

$$\begin{aligned} \dot{\xi}_1 &= v_1 \\ \dot{\xi}_2 &= v_2 + O(\xi)^{\rho+1} \\ \dot{\xi}_3 &= \xi_2 v_1 + O(\xi)^{\rho+1} \\ &\vdots \\ \dot{\xi}_n &= \xi_{n-1} v_1 + O(\xi)^{\rho+1} \end{aligned} \tag{4}$$

The proof is similar to that of Proposition 1 above, see [18].

3 Systems with Trailers

A car with trailers is a classic example of a system with nonholonomic (nonintegrable) velocity constraints. We will consider a similar system, the Hilare family of mobile robots which reside at LAAS [3].

These robots have two parallel wheels which can be controlled independently. By commanding the same velocity to both wheels, the robot moves in a straight line. By commanding velocities with the same magnitude but opposite direction, the robot pivots about its axis. Although the actual input is the acceleration, we are doing only a kinematic analysis and assume we can control the velocity. It will turn out that all of the velocity controls we generate are smooth functions and can be differentiated to find the acceleration.

Given a system of a car (body 0) with n trailers (bodies $1, \ldots, n$), we can represent its configuration by $n+1$ triplets (x_i, y_i, θ_i) where (x_i, y_i) is the position of the i^{th} body in Cartesian space and θ_i is its orientation with respect to some fixed frame. The configuration space $Q = (\mathbf{R} \times \mathbf{R} \times S^1)^{n+1}$ is then $3n+3$ dimensional.

We show here only the equations for the three-body (two-trailer) system. The reader interested in the general form of these equations may consult [10] We assume that the trailers are attached behind the car, resulting in a system of 3 connected bodies. These connections give us 4 holonomic (integrable) constraints of the form:

$$\begin{aligned} x_i - x_{i-1} &= -\cos \theta_i \\ y_i - y_{i-1} &= -\sin \theta_i, i = 1, 2 \end{aligned}$$

(where we have taken the length of each link to be 1.)

[1] Here and in what follows, $O(x)^\rho$ means terms that are of order ρ or higher in x; more precisely, $f(x)$ is of order ρ in x, or $O(x)^\rho$, if:

$$\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|^\rho} = M, \ |M| < \infty$$

Our feasible configuration space is then reduced to a $9-4=5$ dimensional submanifold of Q. This submanifold can be parameterized by: $(x_0, y_0, \theta_0, \theta_1, \theta_2)$ We will use an alternate parameterization, $x = (x_0, y_0, \theta_0, \varphi_1, \varphi_2)$ where $\varphi_i = \theta_i - \theta_{i-1}$.

Before we define the kinematic model, we first define a simple transformation of the inputs. If v_1, v_2 represent the velocity inputs of the two wheels, respectively, let $u_1 = \frac{1}{2}(v_1 + v_2)$ be the "driving" velocity and $u_2 = \frac{1}{2}(v_1 - v_2)$ be the "turning" velocity. The robot then satisfies the kinematic equations:

$$\begin{aligned} \dot{x}_0 &= u_1 \cos\theta_0 \\ \dot{y}_0 &= u_1 \sin\theta_0 \\ \dot{\theta}_0 &= u_2 \end{aligned}$$

The equations for the trailers are derived from the non-slipping constraints:

$$\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$$

Combining these equations with the previous holonomic constraint equations, we find that the trailer angles satisfy:

$$\begin{aligned} \dot{\theta}_1 &= u_1 \sin(\theta_0 - \theta_1) \\ \dot{\theta}_2 &= u_1 \cos(\theta_0 - \theta_1)\sin(\theta_1 - \theta_2) \end{aligned}$$

or in our alternate coordinates, in which $\varphi_i = \theta_i - \theta_{i-1}$,

$$\begin{aligned} \dot{\varphi}_1 &= u_2 - u_1 \sin\varphi_1 \\ \dot{\varphi}_2 &= u_1(\sin\varphi_1 - \cos\varphi_1 \sin\varphi_2) \end{aligned}$$

We will write this system of equations more compactly as:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

where the input vector fields are

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\sin\varphi_1 \\ \sin\varphi_1 - \cos\varphi_1 \sin\varphi_2 \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

It can be shown that this system is completely controllable; *i.e.* there exists a path between any two states x^1 and x^2 (see [10] for details).

For this system, a feedback transformation to exact chained form is easily found when there are no trailers or one trailer using the method described in Proposition 1. The calculations can be found in [14] or [18] and will be omitted here.

With two or more trailers, the distribution Δ_1 is no longer involutive and the exact transformation will not work. However, the 2-trailer system is reducible to a approximate chained form, as will be shown in Section 5.

4 Asymptotic Sinusoids

The steering-by-sinusoids method presented in Section 2 is important, but as has been noted, it only works for systems in chained form. Also, it can be quite tedious in practice to use so many different steps.

However, if one tried to steer in all directions at the same time, even when dealing with a simple chained form system, one would discover that it is not as easy as it seems. If there are any integral relations among the frequencies that were chosen, interference in the form of a zero-frequency component (DC offset) will occur. And it is also important to note that when x_k is steered using the algorithm, $x_1, \ldots, x_{k-1}$ remain unchanged but $x_{k+1}, \ldots, x_n$ will drift in some fashion. Trying to account for all of these extra terms is difficult. Although theoretically possible, such a method would still only work for systems in the exact chained form, or another form that is simple enough to find a closed-form solution.

Some recent work by Sussmann and Liu addresses these problems. They use a parameter j in their inputs, which are sums of sinusoids of frequency $j\omega_k t$, and in the limit as $j \to \infty$, the unwanted interference terms all go to zero.

We formulate the problem as follows. Consider a control system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{5}$$

on some manifold M, and form an extended system,

$$\dot{x} = \textstyle\sum_{k=1}^{m} \hat{g}_k(x)v_k \tag{6}$$

where the new input vector fields $\{\hat{g}_1, \ldots, \hat{g}_m\}$ are related to elements of a degree p P. Hall Basis (described below), and p is chosen to be large enough so that the $\text{span}\{\hat{g}_1, \ldots, \hat{g}_m\} = M$. It is possible to form such an extended system if (5) is completely controllable. The vector $v = (v_1, \ldots, v_m)$ is called the *extended input*.

It is necessary to define a basis for the Lie algebra of vector fields generated by g_1, g_2 that takes into account the fact that Lie brackets satisfy skew symmetry and the Jacobi identity. There are many ways to do this; for this problem we use a P. Hall Basis [4].

First, we need the definition of the *degree* of a Lie bracket B. The degree of $B = [f, g]$, denoted as $\delta(B)$, is defined as $\delta(f) + \delta(g)$. We define $\delta(g_1) = \delta(g_2) = 1$. (Therefore $\delta([g_1, g_2]) = 2$ and $\delta([g_1, [g_1, [g_1, g_2]]]) = 4$.) We can also define the degree of B in g_1, $\delta_1(B)$, and likewise, $\delta_2(B)$. Note that $\delta_1(B) + \delta_2(B) = \delta(B)$.

Let $\mathcal{B}$ represent the (ordered) set of all P. Hall Basis elements. Then $\mathcal{B}$ satisfies the following properties:

(PH1) $g_1, g_2 \in \mathcal{B}$
(PH2) If $\delta(B_1) < \delta(B_2)$ then $B_1 < B_2$
(PH3) $B = [B_1, B_2] \in \mathcal{B}$ if and only if
(a) $B_1, B_2 \in \mathcal{B}$, $B_1 < B_2$ *and*
(b) $B_2 = g_1$ or g_2 or $[B_3, B_4]$ where $B_3, B_4 \in \mathcal{B}$, $B_1 \geq B_3$

For the system of Hilare with 2 trailers, a degree 4 basis will suffice. The degree 4 P. Hall Basis generated by the two vector fields g_1 and g_2 is $\{g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_1, [g_1, [g_1, g_2]]], [g_2, [g_1, g_2]], [g_2, [g_1, [g_1, g_2]]], [g_2, [g_2, [g_1, g_2]]]\}$.

The vector fields $\hat{g}$ used in the extended system are defined as follows: $\hat{g}_1 = g_1$, $\hat{g}_2 = g_2$, $\hat{g}_3 = [g_2, g_1]$, $\hat{g}_4 = [[g_2, g_1], g_2]$, $\hat{g}_5 = [[[g_2, g_1], g_1], g_1]$, $\hat{g}_6 = [[g_2, g_1], g_2]$, $\hat{g}_7 = [[[g_2, g_1], g_1], g_2]$, $\hat{g}_8 = [[[g_2, g_1], g_2], g_2]$ We see that the five vector fields:

$$\{g_1, g_2, g_3, g_4, g_5\} =$$

$$\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\sin\varphi_1 \\ \sin\varphi_1 - \cos\varphi_1 \sin\varphi_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \\ \cos\varphi_1 \\ -\cos\varphi_1 - \sin\varphi_1 \sin\varphi_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 - \cos\varphi_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos\varphi_1 \\ -2\cos\varphi_1 - \cos\varphi_1 \cos\varphi_2 \end{pmatrix}$$

span the entire state space except where $\varphi_1 = \frac{\pi}{2}$. We will use trajectories that are bounded away from this singularity surface.

We now investigate the convergence properties of the algorithm. For any desired path $\gamma(t)$ on M, an extended input v can be found that steers the system along γ. Indeed, since γ is known,

$$\dot{\gamma}(t) = \textstyle\sum_{k=1}^{m} \hat{g}_k(\gamma(t))v_k(t) \tag{7}$$

can be solved for v_k using an inverse or pseudo-inverse method. Since we need only $\{g_1, \ldots, g_5\}$, we can choose $v_6 = v_7 = v_8 = 0$.

Using ordinary (non-extended) inputs of the form:

$$\begin{aligned} u_1^j &= \eta_{1,0} + j^{1/2}\eta_{1,1}\sin(j\omega_1 t) \\ &\quad + j^{2/3}\eta_{1,2}\sin(j\omega_2 t) + j^{3/4}\eta_{1,3}\sin(j\omega_3 t) \\ u_2^j &= \eta_{2,0} + j^{1/2}\eta_{2,1}\cos(j\omega_1 t) \\ &\quad + j^{2/3}\eta_{2,2}\cos(j2\omega_2 t) + j^{3/4}\eta_{2,3}\cos(j3\omega_3 t) \end{aligned} \tag{8}$$

the trajectories x^j of the system

$$\dot{x}^j = g_1(x^j)u_1 + g_2(x^j)u_2$$

converge to x^∞ satisfying

$$\dot{x}^\infty(t) = \textstyle\sum_{k=1}^m \hat{g}_k(x^\infty(t))v_k(t)$$

where the extended input is given by:

$$\begin{aligned} v &= \left(\eta_{1,0}, \eta_{2,0}, \frac{-\eta_{1,1}\eta_{2,1}}{2\omega_1}, \frac{\eta_{1,2}^2\eta_{2,2}}{8\omega_2^2}, \frac{-\eta_{1,3}^3\eta_{2,3}}{48\omega_3^3}\right) \\ &= (v_1, v_2, v_3, v_4, v_5) \end{aligned} \tag{9}$$

We then say that the inputs u^j converge to the extended input v. In order for this to happen, certain non-interference conditions must be satisfied by the frequencies ω_i (see below). The sense of the convergence is that in the limit as $j \to \infty$, the actual trajectory will come arbitrarily close (in the L_∞ norm sense) to the trajectory x^∞. The algorithm for steering is therefore to first choose a trajectory γ, then find an extended input v_0 using Equation 7, and choose appropriate functions $\eta_{i,k}$ so that the ordinary inputs u^j will converge to this extended input v_0. In general, the v_0 will not be unique, and even for a chosen v_0, many different combinations of the $\eta_{i,k}$ functions will give the desired result.

It is interesting to note the similarities between the extended input above and the corresponding functions in the Murray and Sastry algorithm presented in Section 2. If we formulate the 5-dimensional chained system as:

$$\begin{aligned} g_1 &= (1, 0, x_1, x_2, x_3)^T \\ g_2 &= (0, 1, 0, 0, 0)^T \\ g_3 &= (0, 0, 1, 0, 0)^T &&= \mathrm{ad}_{g_1} g_2 \\ g_4 &= (0, 0, 0, 1, 0)^T &&= \mathrm{ad}_{g_1}^2 g_2 \\ g_5 &= (0, 0, 0, 0, 1)^T &&= \mathrm{ad}_{g_1}^3 g_2 \end{aligned}$$

then we would use the following inputs to get motion in each direction:

$$\begin{array}{ll} u_1 = \alpha & \Delta x_1 = \alpha T \\ u_2 = \beta & \Delta x_2 = \beta T \\ \left\{ \begin{array}{l} u_1 = \alpha \sin \omega t \\ u_2 = \beta \cos \omega t \end{array} \right. & \Delta x_3 = \frac{\alpha\beta}{2\omega} T \\ \left\{ \begin{array}{l} u_1 = \alpha \sin \omega t \\ u_2 = \beta \cos 2\omega t \end{array} \right. & \Delta x_4 = \frac{\alpha^2\beta}{8\omega^2} T \\ \left\{ \begin{array}{l} u_1 = \alpha \sin \omega t \\ u_2 = \beta \cos 3\omega t \end{array} \right. & \Delta x_5 = \frac{\alpha^3\beta}{48\omega^3} T \end{array}$$

Compare these with the extended inputs in (9). The differences in the minus signs result from the definitions of the vector fields $\hat{g}_i$, and the factor of T is subsumed in the extended input v. What is especially striking is that although Murray and Sastry's result was only shown to work for systems in chained form, the result we have stated is for *any* system in the form (5) such that the brackets $\{g_1, g_2, g_3, g_4, g_5\}$ as defined above are a basis.

Here we should note that we have not exactly followed Sussmann and Liu's algorithm, but rather used their idea of high frequency, high amplitude inputs and letting the parameter $j \to \infty$ to eliminate the interference caused by trying to steer using sinusoids all at once. We have used only 2 frequencies for each direction in the extended input; their algorithm uses n frequencies to generate motion in the direction of a degree n Lie bracket. Although it is not clear that a simplification such as ours would work for every case, it does work for the systems discussed here. Their complete theory is quite general and complex and cannot be completely examined in this paper. For a much more exhaustive treatment, see [17].

The frequencies ω that are used must satisfy certain non-interference conditions. In Sussmann and Liu's paper, these are formulated as independence relations among various sets. The conditions stated here come from the coefficients of the Chen-Fliess expansion of the inputs. This is a somewhat simplistic, but hopefully insightful description.

Consider the inputs described by equation 8. Let

$$\begin{aligned} \Omega_1 &= \{\pm\omega_1, \pm\omega_2, \pm\omega_3, \pm\omega_4\} \\ \Omega_2 &= \{\pm\omega_1, \pm 2\omega_2, \pm 3\omega_3, \pm 4\omega_4\} \end{aligned}$$

Then Ω_k is the set of all frequencies contained in u_k. In order to generate motion in a bracket direction B, where $\delta_1(B) = k_1$ and $\delta_2(B) = k_2$, the sum of k_1 of the frequencies in Ω_1 and k_2 of the frequencies in Ω_2 must equal 0. For example, we want motion in $[g_1[g_1, g_2]]$, so $\omega_2 + \omega_2 - 2\omega_2 = 0$. However, we must check that there is no other combination of frequencies $\nu_1, \nu_2 \in \Omega_1, \nu_3 \in \Omega_2$ such that $\nu_1 + \nu_2 + \nu_3 = 0$. Likewise, a combination $\{2\omega, \omega\}$ would give motion in $[g_2, [g_2, g_1]]$. Since we do *not* want to move in this direction, we must check that there are no frequencies $\nu_1 \in \Omega_1, \nu_2, \nu_3 \in \Omega_2$ such that $\nu_1 + \nu_2 + \nu_3 = 0$. A similar check is made for all the bracket directions up to order p.

It should be noted that, by following this method, one might think that a frequency combination of $\{2\omega, 2\omega\}$ would give motion in the direction of the bracket $[g_2, [g_1, [g_1, g_2]]]$. This is not true, however, as this combination is really the same as $\{\omega, \omega\}$ and will give motion in the direction $[g_1, g_2]$. If motion in the direction $[g_2, [g_1, [g_1, g_2]]]$ is desired, our simplification will not suffice and the more complex theory must be used.

5 Applications

We have chosen to try this algorithm on the problem of parallel-parking a mobile robot with two trailers attached. Anyone who has ever watched luggage carts unloading baggage from airplanes can appreciate the significance of finding a path-planning scheme for such a system. Parallel-parking, or moving perpendicular to the orientation of the wheels, is a difficult enough trajectory on which to test the theory. The convergence results of Sussmann and Liu were proven for arbitrary systems, but our simulations show that in practice, the convergence can be unacceptably slow. On the other hand, if the system is first transformed into the approximate chained form, the convergence is much faster.

We start with the equations above for Hilare with 2 trailers,

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

where the input vector fields are:

$$g_1 = \begin{pmatrix} 1 \\ \frac{\sin\theta}{\cos\theta} \\ 0 \\ \frac{-\sin\varphi_1}{\cos\theta} \\ \frac{\sin\varphi_1 - \cos\varphi_1 \sin\varphi_2}{\cos\theta} \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Note that we have divided g_1 by $(\cos\theta)$ to get the vector fields in the form required by Proposition 2 , and that this transformation is valid whenever $\theta \neq \frac{\pi}{2}$. An approximation of this system can be found by Taylor-expanding the sine and cosine functions about 0 and truncating the higher-order terms. The order-1 approximation of g_1 and g_2 are:

$$\bar{g}_1 = \begin{pmatrix} 1 \\ \theta \\ 0 \\ -\varphi_1 \\ \varphi_1 - \varphi_2 \end{pmatrix} \quad \bar{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

where we have eliminated terms in the Taylor series that are of order 2 and higher. We then find the distributions:

$$\begin{aligned} \tilde{\Delta}_0 &= \operatorname{span}\{\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4, \bar{g}_5\} \\ \tilde{\Delta}_1 &= \operatorname{span}\{\bar{g}_2, \bar{g}_3, \bar{g}_4\} \end{aligned}$$

where $\bar{g}_3, \bar{g}_4, \bar{g}_5$ are as defined in the P. Hall Basis. We note that $\tilde{\Delta}_0 = \mathbf{R}^5$ for $U = \{(x, y, \theta, \varphi_1, \varphi_2) : \varphi_1 \neq \frac{\pi}{2}\}$ and that $\tilde{\Delta}_1$ involutive. Now it can be seen that the function $h = y - 2\theta + 2\varphi_1 + \varphi_2$ satisfies the conditions in Proposition 2, and so a change of coordinates defined as:

$$\begin{aligned} \xi_1 &= x \\ \xi_2 &= L^3_{\bar{g}_1} h = \varphi_1 - \varphi_2 \\ \xi_3 &= L^2_{\bar{g}_1} h = \varphi_2 \\ \xi_4 &= L_{\bar{g}_1} h = \theta - \varphi_1 - \varphi_2 \\ \xi_5 &= h = y - 2\theta + 2\varphi_1 + \varphi_2 \end{aligned}$$

and a state feedback of the form:

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= (L^4_{\bar{g}_1} h) u_1 + (L_{\bar{g}_2} L^3_{\bar{g}_1} h) u_2 \\ &= (-2\varphi_1 + \varphi_2) u_1 + u_2 \end{aligned}$$

will put the system into order-ρ chained form, $\rho = 1$.

In these coordinates, the differential equations look like:

$$\begin{aligned} \dot{\xi}_1 &= v_1 \\ \dot{\xi}_2 &= v_2 + v_1 (2\xi_2 + \xi_3 + \\ &\quad \frac{\cos(\xi_2 + \xi_3)\sin\xi_3 - 2\sin(\xi_2 + \xi_3)}{\cos(\xi_2 + 2\xi_3 + \xi_4)}) \\ \dot{\xi}_3 &= v_1 \frac{\sin(\xi_2 + \xi_3) - \cos(\xi_2 + \xi_3)\sin\xi_3}{\cos(\xi_2 + 2\xi_3 + \xi_4)} \\ \dot{\xi}_4 &= v_1 \frac{\cos(\xi_2 + \xi_3)\sin\xi_3}{\cos(\xi_2 + 2\xi_3 + \xi_4)} \\ \dot{\xi}_5 &= v_1 \frac{\sin(\xi_2 + 2\xi_3 + \xi_4) - \sin(\xi_2 + \xi_3) - \cos(\xi_2 + \xi_3)\sin\xi_3}{\cos(\xi_2 + 2\xi_3 + \xi_4)} \end{aligned}$$

which look quite complicated, but agree with the chained form to first order.

For our simulations, we chose a parallel-parking trajectory, corresponding to moving sideways. We start at $x_0 = (0, 1, 0, 0, 0)$ and try to go to $x_f = (0, 0, 0, 0, 0)$. See Figure 2 for the chosen trajectory. We use Sussmann and Liu's modified algorithm as described above for Hilare with two trailers. The frequencies that we chose to satisfy the non-interference conditions were: $\omega_1 = \frac{5}{6}, \omega_2 = \frac{6}{7}, \omega_3 = 1$. We chose these by checking all possible combinations of the input frequencies, in the bracket directions up to order 4, for no interference. When we were choosing our desired trajectory, we wanted to be sure that we avoided the point of singularity, $\varphi_1 = \frac{\pi}{2}$. Therefore, to keep our inputs small enough, we chose a linear parameterization of the straight path from x_0 to x_f along 100 seconds. We scaled all the frequencies by $\frac{2\pi}{10}$ which resulted in ω_3 going through an integral number of periods in 100 seconds.

We simulated the system in both the original coordinates and in the order-1 chained form coordinates. We expected that the convergence properties would be improved by using the approximate chained form, since the bracket directions that we are not trying to move in consist of only higher order terms. The results that we have obtained confirm that hypothesis. See Figures 3 and 4.

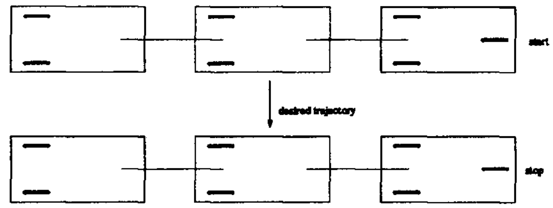

Figure 2: The Desired Trajectory.

6 Discussion

In an exact chained form system, all the Lie brackets of the input vector fields are zero except those of the form: $\mathrm{ad}^k_{g_1} g_2, k = 1, \ldots, n-2$. In an approximate chained form system, all Lie brackets of the input vector fields except those in this special form will contain only terms of order-ρ and higher. Therefore, although the theory states that all the interference terms go to zero as $j \to \infty$, when the system is in the approximate chained form the interference terms are already small at the start. This is seen dramatically when comparing the standard system and the system in transformed coordinates in Figures 5 and 5. The standard system simulation does not even stay close to the desired path (the trailer angles are especially bad), whereas the system in transformed coordinates tracks it reasonably well.

This result could be expected from looking at the directions in the P. Hall basis in which we are not steering. Two of the three directions, g_6 and g_8 have terms of low order $(1, \theta, \varphi_1, \varphi_2)$. (We do not have to worry about g_7 in this case since it is zero.) The interference terms associated with these two directions are going to zero slowly, as $\frac{1}{j^k}$ where $|k| < 1$. In the transformed system, the unwanted directions consist only of higher order terms.

Arguably, the path we have chosen to approximate is poor. No self-respecting luggage-cart driver in his right mind would try to move sideways! However, this issue should be addressed when choosing the desired trajectory γ. The beauty of the theory is that it guarantees convergence to *any* chosen trajectory. If the robot is moving in an obstacle-ridden environment, a preliminary path-planner can choose an obstacle-free path without regard for the nonholonomic constraints of the system. The procedure presented here can then be used to track this path closely enough so that obstacles are avoided.

Looking at the equations that specify the inputs (8), we see as j increases to give us closer tracking, that both the frequency and the magnitude of the inputs also increase. This is another reason to choose the approximately chained form system, since better convergence can be achieved with smaller inputs. Of course, one way to scale down the frequencies needed is to scale time when choosing the desired trajectory.

The authors are aware of no other methods which find a closed-form path from x_0 to x_f for a car-like robot with two trailers. Barraquand and Latombe [1] discretize the state space and do a search through feasible trajectories; this method grows in complexity as the number of trailers increases. Lafferiere and Sussmann [8] have a method that is exact only for nilpotent systems (Hilare with two trailers is not nilpotent).

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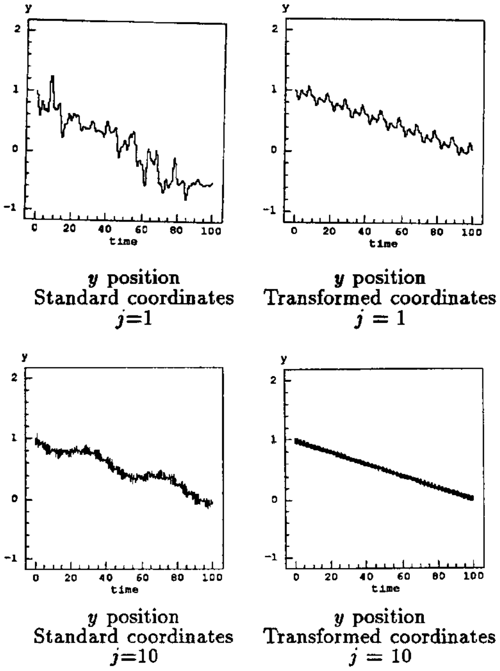

Figure 3: A comparison of the y coordinate values for $j = 1, 10$ and in standard and transformed coordinates. The desired trajectory is a straight line from 1 to 0. The time scale is 100 seconds. Note how much closer the transformed system plots are to the desired trajectory.

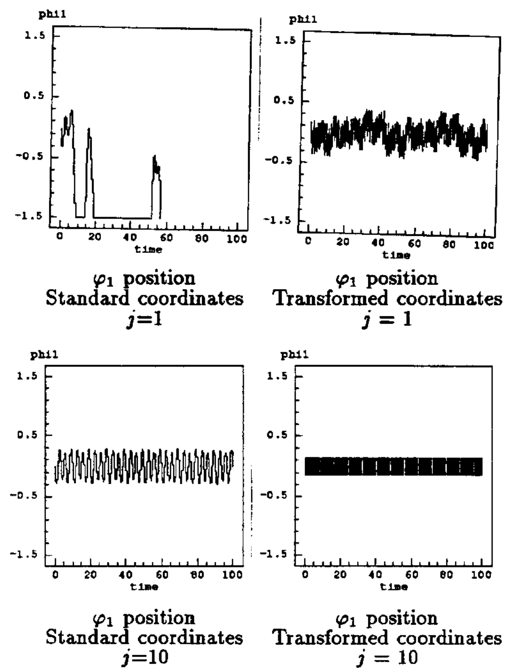

Figure 4: A comparison of the φ_1 coordinate values for $j = 1, 10$ and in standard and transformed coordinates. The desired trajectory is a straight line at 0. The time scale is 100 seconds. Note that in the standard coordinates, $j = 1$, the φ_1 coordinate becomes greater that $\frac{\pi}{2}$ and is off the scale of the graph.

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