Analysis and Control for Manipulators with Both Joint and Link Flexibility

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Abstract

This work is focused on the analysis of manipulators with both joint and link flexibility. Due to the different order of joint and link stiffness, the full-order nonlinear system can be decomposed into different time-scale subsystems, namely, slow subsystem, mid-speed subsystem, and fast subsystem. It is shown that when the link stiffness is much greater than joint stiffness or when the two kinds of stiffness are comparable the vibrations due to joint or link flexibility can be suppressed whatever the control effort is made. Therefore, a composite control law is proposed in the case where the joint stiffness is much greater than the link stiffness to eliminate the structural vibrations while the tracking objective is achieved.

1. Introduction

The problem of controling mechanism with nonrigid links or joints has received widespread attentions in the past decade. Since for some special applications, such as low power-consumption, high motion speed, the need of designing mechanical arms with light weight has been increasing gradually. A lighter arm has much more complex dynamics due to its flexibility distributed along the mechanical beam, which have caused great difficulty in its control task. Consequently, to overcome this difficulty, improved control strategies must be developed, e.g. in [1]-[9], where the modeling and control problem are well addressed. Furthermore, as we have known that most today's industrial robots were equipped with gear-boxes such as harmonic drives that will introduce joint flexibility which is always neglected. This kind of neglects may be acceptable when the operation speed is low, but may be quite devastating when the speed gets high. Thus, when high manipulating performance is needed, the elastic phenomena must be taken into account, [10]-[13].

The nonrigidity of a light-weight flexible arm may consist of distributed link flexibility and lumped joint flexibility. Because of the high complexity of the dynamical equations for a multi-link manipulator with both joint and link flexibility, most of the literatures on the control of flexible manipulator have discussed arms with joint elasticity and with link flexibility separately. In [2], [10], the authors used the singular perturbation technique to reformulate the manipulator dynamics either in flexible-joint or in flexible-link case, however, they must assume either the links or the joints are rigid. Recent work [14] on the control of manipulators having both flexible joints and links had the similar formulations, but it only treated the two time-scale problem.

Hence, in this paper, some discussions on the singular perturbation approaches to the model formulation for the manipulator with both flexible links and joints are given. It will be seen that when the link flexibility and the joint flexibility are comparable, the corresponding subsystems are strongly coupled, indicating singificant interactons between link and joint flexibility. This coupling, however, will vanish when the two kinds of stiffness are widely separated.

This paper is organized as follows: Section 2 discusses the singular perturbation approaches and the properties of a three time-scale system, and provide proofs of the system stability under some conditions. In section 3, we make some discussions on manipulators with both joint and link flexibility under three different cases, respectively, defined as situations where the former dominates the latter, the latter dominates the former, and the both are comparable. In section 4, the controller design of these three cases is given and a conclusion is made that the system is controllable only when the joint stiffness is much larger than the links stiffness, or when the two kinds of stiffness are of the same order. Section 5 shows the simulation results. Finally, some concluding remarks are given in section 6.

2. Preliminaries

In this section, we briefly review some relevant results due to the singular perturbation approach and discuss the characteristics of a multiple time-scale system. Then we propose a two-stage analysis to prove the stability of the overall system. A flexible robotic manipulator with both link and joint flexibility may have different-order of stiffness and, therefore, the two perturbation-parameter system is considered in this section.

Let us consider the three time-scale system as follows

$$\dot{x} = f(x, z_1, z_2, \epsilon, \mu) \tag{1}$$
$$\epsilon \dot{z}_1 = g_1(x, z_1, z_2, \epsilon, \mu) \tag{2}$$
$$\mu \dot{z}_2 = g_2(x, z_1, z_2, \epsilon, \mu) \tag{3}$$

Supposing $\mu \ll \epsilon \ll 1$, we can spilt the full-order system into two subsystems by letting $\mu \to 0$, and use overbar to denote either slow variables or a mixture of slow variables, then $g_2(\bar{x}, \bar{z}_1, \bar{z}_2, \epsilon, 0) = 0$. Assume $\bar{z}_2$ can be represented as a function of $\bar{x}$, and $\bar{z}_1$ as $\bar{z}_2 = h_2(\bar{x}, \bar{z}_1, \epsilon)$ which is the equilibrium state of the boundary layer system, and the boundary layer system, namely, the fast subsystem, is given by defining $\eta_2 = z_2 - \bar{z}_2$, $\tau_2 = \frac{t}{\mu}$, and letting $\mu = 0$, so that (3) can be rewritten as

$$\frac{d\eta_2}{d\tau_2} = g_2(\bar{x}, \bar{z}_1, \eta_2, \epsilon, 0) \tag{4}$$

Note that the slow and mid-speed variables that parametrize the boundary layer system are quasi-static and can be treated as constants. Now, the slow subsystem can be represented as

$$\dot{\bar{x}} = f(\bar{x}, \bar{z}_1, h_2(\bar{x}, \bar{z}_1, \epsilon), \epsilon, 0) \tag{5}$$
$$\epsilon \dot{\bar{z}}_1 = g_1(\bar{x}, \bar{z}_1, h_2(\bar{x}, \bar{z}_1, \epsilon), \epsilon, 0), \tag{6}$$

which, however, still has two different time-scales. Therefore, we denote the above system as a "virtual slow subsystem", and focus on this subsystem in a little more detail.

Consider the virtual slow subsystem (5), (6), and decompose it into a slow subsystem and a fast subsystem again by letting

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$\epsilon = 0$. For the same reason as stated above, we use the notation $\hat{}$ to denote either the slow variables or functions of them, i.e., $g_1(\hat{x}, \hat{z}_1, h_2(\hat{x}, \hat{z}_1, 0), 0, 0) = 0$. Likewise, Assume $\hat{z}_1$ can be solved in terms of $\hat{x}$ to yield $\hat{z}_1 = h_1(\hat{x})$ which is the equilibrium state of the boundary layer of the mid-speed subsystem. After defining $\eta_1 = \bar{z}_1 - \hat{z}_1$, $\tau_1 = \frac{t}{\epsilon}$, we can get that subsystem from (6) by letting $\epsilon = 0$, i.e.,

$$\frac{d\eta_1}{d\tau_1} = g_1(\hat{x}, \eta_1, h_2(\hat{x}, \eta_1, 0)) \tag{7}$$

Similarly, the slow variables are treated as constants in this mid-speed subsystem. Therefore, the really slow subsystem is given by

$$\dot{\hat{x}} = f(\hat{x}) \tag{8}$$

Since the stability issue in a control problem is our most important consideration, in the following we will propose a two-stage analysis inspired by Kokotovic (1986) to show the stability of the full-order multiple time-scale system.

Proposition 1 *Consider the two perturbation - parameters system described by (1), (2), (3). The system can be decomposed into three subsystems, namely, slow subsystem, mid-speed subsystem, fast subsystem, respectively. If the three subsystems are asymptotically stable individually, then the full-order subsystem is ulmately stable in the sense that*

$$\begin{aligned} |x| &\rightarrow O(\epsilon) + O(\mu) \\ |z_1| &\rightarrow h_1(x) + O(\epsilon) + O(\mu) \\ |z_2| &\rightarrow h_2(x, z_1, \epsilon) + O(\mu) \end{aligned} \tag{9}$$

*Proof:*Consider the three subsystems (4), (7), and (8) as follows:

$$\begin{aligned} \frac{d\eta_2}{d\tau_2} &= g_2(\bar{x}, \bar{z}_1, \eta_2, \epsilon) \\ \frac{d\eta_1}{d\tau_1} &= g_1(\hat{x}, \eta_1, h_2(\hat{x}, h_1(\hat{x})) \\ \frac{d\hat{x}}{dt} &= f(\hat{x}) \end{aligned} \tag{10}$$

which are the so-called "high-speed", "mid-speed", and "low-speed" subsystems, respectively. Since the three subsystems are all asymptotically stable, there exist three Lyapunov function candidates V_1, V_2, and V_3 with respect to "low-speed", "mid-speed", and "high-speed" subsystems, respectively, satisfying the following conditions according to a converse theorem of Lyapunov.

$$\begin{aligned} \frac{\partial V_1}{\partial x} f(x) &\leq -\alpha_3 \psi_1^2(x) \\ \frac{\partial V_2}{\partial z_1} g_1(x, z_1, h_2(x, z_1, 0)) &\leq -\alpha_4 \phi_1^2(z_1 - h_1(x)) \\ \frac{\partial V_3}{\partial z_2} g_2(x, z_1, z_2, \epsilon) &\leq -\alpha_5 \phi_2^2(z_2 - h_2(x, z_1)) \end{aligned} \tag{11}$$

for appropriate positive constants α_3, α_4, and α_5.

Now we begin with the virtual slow subsystem. Similar to Kokotovic (1986), the following conditions are assumed. For some positive constants $\gamma_4 \sim \gamma_6$, the functions f and g_1 satisfy the following conditions.

$$\begin{aligned} \left| \frac{\partial V_1}{\partial x} (f_3 - f_2) \right| &\leq \gamma_4 \psi_1(x) \phi_1(z_1 - h_1(x)) \\ \left| \frac{\partial V_1}{\partial x} (f_2 - f_1) \right| &\leq \gamma_5 \psi_1(x) O(\epsilon) \end{aligned} \tag{12}$$

and

$$\left| \frac{\partial V_2}{\partial z_1} (g_{12} - g_{11}) \right| \leq \gamma_6 O(\epsilon) \phi_1(z_1 - h_1(x)) \tag{13}$$

where f_i and g_{1i} are defined as

$$\begin{aligned} f_1 &\equiv f(x, h_1(x), h_2(x, h_1(x), 0), 0, 0) \\ f_2 &\equiv f(x, h_1(x), h_2(x, h_1(x), \epsilon), 0, \epsilon) \\ f_3 &\equiv f(x, z_1, h_2(x, z_1, \epsilon), 0, \epsilon) \\ g_{11} &\equiv g_1(x, z_1, h_2(x, z_1, 0), 0, 0) \\ g_{12} &\equiv g_1(x, z_1, h_2(x, z_1, \epsilon), 0, \epsilon) \end{aligned}$$

so that, a composite Lyapunov function of the following form

$$V_4 = (1 - d)V_1 + dV_2; \quad d \in (0, 1),$$

which is the first stage composite Lyapunov function. Then, we take the time derivative of V_4 with the above conditions, and get

$$\begin{aligned} &(1 - d)\frac{\partial V_1}{\partial x} f_1 + \frac{d}{\epsilon}\frac{\partial V_2}{\partial z_1} g_{11} + (1 - d)\frac{\partial V_1}{\partial x}[(f_3 - f_2) \\ &+ (f_2 - f_1)] + \frac{d}{\epsilon}\frac{\partial V_2}{\partial z_1}(g_{12} - g_{11}) \\ &\leq -(1 - d)\psi_1 \left[(\alpha_3 - \frac{\gamma_4 \epsilon}{2})\psi_1 - \gamma_5 O(\epsilon)\right] \\ &- \frac{d}{\epsilon}\phi_1 \left[(\alpha_4 - \frac{(1 - d)\gamma_4}{2d})\phi_1 - \gamma_6 O(\epsilon)\right] \end{aligned} \tag{14}$$

which implies that there exist ϵ_0^* and d_0^* such that, when $\epsilon \leq \epsilon_0^*$ and $d_0^* \leq d \leq 1$, $\psi_1(x)$ and $\phi_1(z_1 - h_1(x))$ will converge to a resdual set with size of $O(\epsilon)$ until the solution trajectories of (5), (6) leave a priorly given compact set. Now, we consider (1), (2) together as a subsystem which evolves relatively slow to the subsystem (3), i.e.,

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z}_1 \end{bmatrix} = \begin{bmatrix} f(x, z_1, h_2, \epsilon, 0) \\ g_1(x, z_1, h_2, \epsilon, 0) \end{bmatrix} \tag{15}$$

or, equivalently, $\dot{w} = r(w, h_2, 0)$ where $w = [x^T, \epsilon z_1^T]^T$. Therefore, according to the above description, $\dot{V}_4$ can be interpreted in an alternative form.

$$\begin{aligned} &\frac{\partial V_4}{\partial w} r \leq -\alpha_6 \psi_2(w - \bar{w}) \times \\ &[\psi_2(w - \bar{w}) - O(\epsilon)], \quad \alpha_6 > 0 \end{aligned}$$

where $\bar{w} = (0, \epsilon h_1^T)^T$. Now, further conditions are needed for the stability proof of the full-order system, which are quite similar to (12), and (13) accounting for the interconnection between (15) and the full order system, i.e.,

$$\begin{aligned} \left| \frac{\partial V_4}{\partial w}(r_3 - r_2) \right| &\leq \gamma_7 \psi_2(w - \bar{w}) \phi(z_2 - h_2) \\ \left| \frac{\partial V_4}{\partial w}(r_2 - r_1) \right| &\leq \gamma_8 O(\mu) \psi_2(w - \bar{w}) \\ \left| \frac{\partial V_3}{\partial z_2}(g_{22} - g_{21}) \right| &\leq \gamma_9 \phi_2(z_2 - h_2) O(\mu) \end{aligned} \tag{16}$$

for some positive constants $\gamma_7 \sim \gamma_9$, where r_i and g_{2i} are defined in the following.

$$\begin{aligned} r_1 &\equiv r(w, h_2(w), 0) \\ r_2 &\equiv r(w, h_2(w), \mu) \\ r_3 &\equiv r(w, z_2, \mu) \\ g_{21} &\equiv g_2(w, z_2, 0) \\ g_{22} &\equiv g_2(w, z_2, \mu) \end{aligned}$$

Given These conditions we can define the second stage composite Lyapunov function as follows

$$V = (1 - \bar{d})V_4 + \bar{d}V_3, \quad \bar{d} \in (0, 1)$$

and, then, obtain the same results as the former by taking time derivative of V, i.e,

$$(1 - \bar{d})\frac{\partial V_4}{\partial w} r_1 + \frac{\bar{d}}{\mu}\frac{\partial V_3}{\partial z_2} g_{21} + (1 - \bar{d})\frac{\partial V_4}{\partial w}[(r_3 - r_2)$$

$$+(r_2 - r_1)] + \frac{\bar{d}}{\mu}\frac{\partial V_3}{\partial z_2}(g_{22} - g_{21})$$
$$\leq -(1-\bar{d})\psi_2\left[(\alpha_6 - \frac{\gamma_7\mu}{2})\psi_2 - O(\epsilon) - \gamma_8 O(\mu)\right]$$
$$-\frac{\bar{d}}{\mu}\phi_2[(\alpha_5 - \frac{1-\bar{d}}{2d}\gamma_7)\phi_2 - \gamma_9 O(\mu) \quad (17)$$

Again, this implies that there exist μ^* and $\bar{d}^*$ such that for all $\mu \leq \mu^*$ and $\bar{d}^* \leq \bar{d} \leq 1$ we have $\psi_2(w-\bar{w})$ and $\phi_2(z_2 - h_2)$ will be converging to a resdual set of size $O(\epsilon) + O(\mu)$ and $O(\mu)$, respectively, so long as the solution trajectories of the overall system remains bounded inside a priorly given compact set. Repeatedly using the robustness argument of each subsystem, we can then conclude that, in fact, all signal trajectories will never leave that aforementioned compact set, and, hence, all signals remain uniformly bounded. □□□

3. Applications to Flexible Manipulators

3.1 Dynamic model

Consider an n-link robotic manipulator with both joint and link flexibility. The deflection of link i, $i = 1 \leq i \leq n$, can be considered as:

$$y_i(x,t) = \sum_{j=1}^{\infty} \phi_{ij}\delta_{ij}$$

which is governed by the Euler-Bernoulli beam equation

$$EI\frac{\partial^4 y_i(x,t)}{\partial x^4} + \rho\frac{\partial^2 y_i(x.t)}{\partial t^2} = 0$$

subject to some appropriate boundary conditions, where ϕ_{ij} is the mode shape function for mode j of link i, E is the Young's modulus, I is the moment of inertia, and ρ is the mass density per unit length. Here, we assume all links have the same I and ρ. Furthermore, the flexible joints can be modeled as linear torsional springs with constant spring stiffness. Thus, by using Lagrangian-Euler formulation, the equations of the dynamic model can be shown in the following:

$$\begin{aligned} M\begin{bmatrix} \ddot{q}_1 \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_1\delta \end{bmatrix} &= \begin{bmatrix} K_2(q_2 - q_1) \\ 0 \end{bmatrix} \\ J\ddot{q}_2 + K_2(q_2 - q_1) &= u, \end{aligned} \quad (18)$$

where

- $q_1 \in R^n$: a vector joint angles associated with links,
- $\delta \in R^m$: a vector of m flexible modes of all links,
- $q_2 \in R^n$: a vector of shaft angles associated eith each actuator,
- $M \in R^{n\times n}$: inertia matrix,
- $f_1 \in R^n, f_2 \in R^m$: nonlinear coupling terms, including Coriolis, centrifugal, and gravitational forces.
- $K_1 \in R^{m\times m}$: equivalent spring constant matrix of the links,
- $K_2 \in R^{n\times n}$: torsional spring constant matrix of the joints,
- $J \in R^{n\times n}$: motor inertia matrix,
- $u \in R^n$: a vector of control input torques from motor actuators.

3.2 Problem Formulation

In this section, we reformulate the original dynamic equations in singular perturbation forms. Let

$$M^{-1} = \begin{bmatrix} h_1 & h_2 \\ h_2^T & h_3 \end{bmatrix}$$

and multiply it to both sides so that we can rewrite the dynamical equations as:

$$\begin{aligned} \ddot{q}_1 &= -h_1 f_1 - h_2 f_2 - h_2 K_1\delta + h_1 K_2(q_2 - q_1) \\ \ddot{\delta} &= -h_2^T f_1 - h_3 f_2 - h_3 K_1 \delta + h_2^T K_2(q_2 - q_1) \\ & J\ddot{q}_2 + K_2(q_2 - q_1) = u \end{aligned} \quad (19)$$

Further define $K_1\delta = w_1', K_2(q_2 - q_1) = w_2'$ where we let $K_1 = \bar{k}_1/\mu_1, K_2 = \bar{k}_2/\mu_2$, with $\bar{k}_1 \in R^{m\times m}$, $\bar{k}_2 \in R^{n\times n}$, and $\mu_1, \mu_2 \in R$. If we define $w_1 = \bar{k}_1^{-1}w_1'$, $w_2 = \bar{k}_2^{-1}w_2'$ and $x_1 = q_1$, $x_2 = \dot{q}_2$, $z_1 = w_1$, $z_2 = \dot{w}_2$, $z_3 = w_2$, $z_4 = \dot{w}_2$, and $\sqrt{\mu_1} = \epsilon_1$, $\sqrt{\mu_2} = \epsilon_2$, then equation (19) can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1 f_1 - h_2 f_2 - h_2 z_1 + h_1 z_3 \\ \epsilon_1 \dot{z}_1 &= z_2 \\ \epsilon_1 \dot{z}_2 &= -h_2^T f_1 - h_3 f_2 - h_3 z_1 + h_2^T z_3 \\ \epsilon_2 \dot{z}_3 &= z_4 \\ \epsilon_2 \dot{z}_4 &= h_1 f_1 + h_2 f_2 + h_2 z_1 - (h_1 + J^{-1})z_3 \\ & + J^{-1}u \end{aligned} \quad (20)$$

which is clearly a three time-scale system. In the following discussion, we decomposed the full-order system into three subsystems whose time-scales are t, $\frac{t}{\epsilon_1}$, $\frac{t}{\epsilon_2}$, respectively.

3.3 Discussions on Three Cases

In the following, we discuss the two perturbation parameters system in three different conditions. These three conditions reveal how the flexible manipulator behaves when the ratio of the flexibility of the link to that of the joint varies, and are shown as follows:

Case I:($\epsilon_1^2 \ll \epsilon_2^2 \ll 1$) In this case, the link stiffness is much larger than that of the joint so that ϵ_1 is the smallest perturbation parameter, and thus the subsystem associated with ϵ_1 can be regarded as the fast subsystem. Then we set $\epsilon_1 = 0$ in equation (20) and use variables with overbar to denote the resulting slow variables to yield

$$\begin{aligned} \bar{z}_2 &= 0 \\ \bar{z}_1 &= h_3^{-1}(-h_2^T f_1 - h_3 f_2 + h_2^T \bar{z}_3) \end{aligned}$$

which is the equilibrium state. The fast subsystem can thus be found by defining $\tau_1 = \frac{t}{\epsilon_1}$, $\eta_1 = z_1 - \bar{z}_1$, $\eta_2 = z_2$, and, then, by setting $\epsilon_1 = 0$ to obtain

$$\begin{aligned} \frac{d\eta_1}{d\tau_1} &= \eta_2 \\ \frac{d\eta_2}{d\tau_1} &= -h_3\eta_1 \end{aligned} \quad (21)$$

which is the boundary layer of the fast subsystem. Note that the slow and the mid-speed variables in that subsystem are quasi-static, and, hence, can be regarded as constants. Apparently, (21) becomes a linear system which is uncontrollable since there is no direct external input to the subsystem.

Now to derive both the slow and the mid-speed subsystems, we let $\epsilon_2 = 0$ in equation (29), and denote $\hat{z}_3$, $\hat{z}_4$ as slow variables with respect to the slow subsystem, i.e,

$$\begin{aligned} \hat{z}_4 &= 0 \\ \hat{z}_3 &= (h_1 - h_2 h_3^{-1} + J^{-1})^{-1} \\ & \left[(h_1 - h_2 h_3^{-1} h_2^T) f_1 + J^{-1}\hat{u}\right] \\ &\equiv H_3^{-1} H_2 f_1 + H_3^{-1} J^{-1}\hat{u} \end{aligned} \quad (22)$$

to get

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= -H_1 f_1 + H_1 \hat{z}_3 \\ &= (-H_1 + H_1 H_3^{-1} H_2) f_1 + H_1 H_3^{-1} J^{-1}\hat{u} \end{aligned} \quad (23)$$

which clearly corresponds to the rigid slow subsystem, and then we define $\tau_2 = \frac{t}{\epsilon_2}$, $\eta_3 = \bar{z}_3 - \hat{z}_3$, $\eta_4 = \bar{z}_4$, and let $\epsilon_2 = 0$ so that

$$\begin{aligned}\frac{d\eta_3}{d\tau_2} &= \eta_4 \\ \frac{d\eta_4}{d\tau_2} &= -H_3\eta_3 + J^{-1}(\bar{u} - \hat{u}) \qquad (24)\end{aligned}$$

which is the mid-speed subsystem as required. Similarly, all the slow variables involved in (24) are all treated as constants.

Case II: ($\epsilon_2^2 \ll \epsilon_1^2 \ll 1$) Different from the previous case, here the link flexibility is the dominant one, and, hence the dynamics associated with the joint flexibility becomes the fast subsystem. Now, by letting $\epsilon_2 = 0$ in (20), we obtain the following equilibrium state.

$$\begin{aligned}\bar{z}_4 &= 0 \\ \bar{z}_3 &= (h_1 + J^{-1})^{-1}(h_1 f_1 + h_2 f_2 + h_2 \bar{z}_1 + J^{-1}\bar{u})\end{aligned}$$

Similar to the discussion above, the fast subsystem can thus be derived by defining $\tau_2 = \frac{t}{\epsilon_2}$, $\eta_1 = z_3 - \bar{z}_3$, $\eta_2 = z_4$, and by letting $\epsilon_2 = 0$ as:

$$\begin{aligned}\frac{d\eta_1}{d\tau_2} &= \eta_2 \\ \frac{d\eta_2}{d\tau_2} &= -(h_1 + J^{-1})\eta_1 + J^{-1}(u - \bar{u}) \\ &= -(h_1 + J^{-1})\eta_1 + J^{-1}u_{f1} \qquad (25)\end{aligned}$$

which is the boundary layer of the fast subsystem that obviously can be controlled under state-feedback. Again, the variables corresponding to the slow and the mid-speed subsystems are all treated as constants in this fast subsystem.

To derive the slow subsystem, we then let $\epsilon_1 = 0$ to yield

$$\begin{aligned}\hat{z}_2 &= 0 \\ \hat{z}_1 &= -f_2 - T_5^{-1}T_4 f_1 - T_5^{-1}T_6\hat{u} \qquad (26)\end{aligned}$$

and then substitute (26) into the virtual slow subsystem to obtain

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= (T_1 - T_2T_5^{-1}T_4)f_1 \\ &\quad + (T_3 - T_2T_5^{-1}T_6)\hat{u} \qquad (27)\end{aligned}$$

which is known as the slow subsystem. Finally the boundary layer of the mid-speed subsystem can be obtained by defining $\tau_1 = \frac{t}{\epsilon_1}$, $\eta_3 = \bar{z}_1 - \hat{z}_1$, $\eta_4 = \bar{z}_2$, and by letting $\epsilon_1 \to 0$

$$\begin{aligned}\frac{d\eta_3}{d\tau_1} &= \eta_4 \\ \frac{d\eta_4}{d\tau_1} &= T_2\eta_3 + T_6(\bar{u} - \hat{u}) \\ &\equiv T_2\eta_3 + T_6 u_{f2} \qquad (28)\end{aligned}$$

where all the slow-variables that parametrize the system are regarded as constant as before.

(iii) Case III: ($\epsilon_1^2 \approx \epsilon_2^2$) In this case, ϵ_1 and ϵ_2 are of the same order. Therefore, we spilt the full-order system into two subsystems only, namely, a slow subsystem and a fast subsystem.

Consider (20) again, and let $\epsilon_1 = \epsilon_2 = 0$ so that

$$\begin{aligned}\bar{z}_1 &= h_3^{-1}[-h_2^T f_1 - h_3 f_2 + \\ &\quad h_2^T B_1^{-1}(B_2 f_1 + B_3\bar{u})] \\ \bar{z}_2 &= 0 \\ \bar{z}_3 &= B_1^{-1}(B_2 f_1 + B_3\bar{u}) \\ \bar{z}_4 &= 0 \qquad (29)\end{aligned}$$

where

$$\begin{aligned}B_1 &= h_1 + h_2 h_3^{-1} h_2^T + J^{-1} \\ B_2 &= h_1 - h_2 h_3^{-1} h_2^T \\ B_3 &= h_2 h_3^{-1} J^{-1}\end{aligned}$$

so that the slow subsystem can then be obtained by substituting equation (29) into the original full-order system (25):

$$\begin{aligned}\dot{\bar{x}}_1 &= \bar{x}_2; \\ \dot{\bar{x}}_2 &= -h_1 f_1 - h_2 f_2 - h_2\bar{z}_1 + h_1\bar{z}_3 \qquad (30) \\ &= -(h_1 + h_2h_3^{-1}h_2^T - h_2h_3^{-1}h_2^T B_1^{-1}B_2)f_1 \\ &\quad - (h_2h_3^{-1}h_2^T B_1^{-1}B_3 - h_1B_1^{-1}B_3)\bar{u}\end{aligned}$$

which is recognized as the rigid slow subsystem.

To derive the fast subsystem in this case, we refer to (20), set the time scale as $\tau = \frac{t}{\epsilon}$, and define $\eta_1 = z_1 - \bar{z}_1$, $\eta_2 = z_2$, $\eta_3 = z_3 - \bar{z}_3$, $\eta_4 = z_4$, so that the fast system can be expressed in a manner similar to the previous deviation.

$$\begin{aligned}\frac{d\eta_1}{d\tau} &= \eta_2 \\ \frac{d\eta_2}{d\tau} &= -h_3\eta_1 + h_2^T\eta_3 \qquad (31) \\ \frac{d\eta_3}{d\tau} &= \eta_4 \\ \frac{d\eta_4}{d\tau} &= h_2\eta_1 - (h_1 + J^{-1})\eta_3 + J^{-1}(u - \bar{u}) \qquad (32)\end{aligned}$$

or, equivalently, $\frac{d}{d\tau}\eta = A\eta + Bu_f$, where $\eta^T = [\eta_1^T\ \eta_2^T\ \eta_3^T\ \eta_4^T]^T$ and $u_f = u - \bar{u}$. Equation (31) constitutes the so-called boundary-layer condition of the fast subsystem.

4. Controller Design

When we encounter a system which is required to perform tasks in a desired manner, intuitive questions will arise, such as " Can it be controlled? " and " What is the difficulty if the previous answer is affirmative? ". In fact, we are interested mainly in controlling the low-speed, gross behavior of a robot system. Thus, of primary importance to us is the stability or stabilizability of the fast subsystem. Hence, in this section, we take the stability problem into consideration and make an attempt to design the controller for the robotic system. According to the aformentioned formulation, it is likely that we can design the controller of the three subsystems separately. The controller design procedure will be stated case by case as follows.

(i) Case I: ($\epsilon_1^2 \ll \epsilon_2^2 \ll 1$) In this case, the fast subsystem, mid-speed subsystem, and slow subsystem are shown in equations (21), (24), (23), respectively. As shown in (21), the boundary layer of the fast subsystem is uncontrollable due to a lack of fast controller. If there exist some structural damping in the original system, then the boundary layer system will be asymptotically stable, and the control efforts we made will be concentrated on the mid-speed and the slow subsystem. If there is no assumed damping in the original full-order system, the boundary layer of the fast subsystem will then be oscillatory.

(ii) Case II: ($\epsilon_2^2 \ll \epsilon_1^2$) In this case, the three subsystems, namely, high-speed, mid-speed, and low-speed subsystem are discussed as shown in equations (25), (28), (27), respectively.

First we consider the slow subsystem

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= (T_1 - T_2T_5^{-1}T_4)f_1 \\ &\quad + (T_3 - T_2T_5^{-1}T_6)\hat{u} \qquad (33)\end{aligned}$$

where $T_1 \sim T_6$ are defined as previous ones. Since the slow subsystem are equivalent to the rigid robot system which is always feedback linearizable [2], $\hat{u}$ may be designed as

$$\hat{u} = (T_3 - T_2T_5^{-1}T_6)^{-1}[-(T_1 - T_2T_5^{-1}T_4)f_1 + v]$$

where $v = [-(T_1 - T_2T_5^{-1}T_4)f_1 + \ddot{q}_d - k_1\dot{e} - k_2 e]$ and q_d is the desired trajectory and $\dot{e}, e$ are the velocity and position tracking errors, respectively. Thus, the slow subsystem will result in $\ddot{e} + k_1\dot{e} + k_2 e = 0$, which assures $\dot{e}, e \to 0$ as $t \to \infty$.

Secondly, the mid-speed subsystem is taken into account. Consider

$$\begin{aligned} \frac{d\eta_3}{d\tau_1} &= \eta_4 \\ \frac{d\eta_4}{d\tau_1} &= T_2\eta_3 + T_6 u_{f2} \end{aligned} \tag{34}$$

which is parametrized by the slow variables and hence T_2, T_6 can be treated as constants. If the subsystem (33) is a completely controllable pair, we can always use state feedback technique to stabilize the system, i.e,

$$u_{f2} = -K[\eta_3^T, \eta_4^T]^T \tag{35}$$

Therefore, $\eta_3, \eta_4 \to 0$ as $t \to \infty$ will be concluded.

In the fast subsystem,

$$\begin{aligned} \frac{d\eta_1}{d\tau_2} &= \eta_2 \\ \frac{d\eta_2}{d\tau_2} &= -(h_1 + J^{-1})\eta_1 + J^{-1}u_{f1} \end{aligned} \tag{36}$$

since $h_1 + J^{-1}$ is positive definite in all configurations, regulation can be achieved with the addition of some damping terms. Therefore,

$$u_{f1} = -JK_d\eta_2 \tag{37}$$

which will result in $\eta_1, \eta_2 \to 0$ as $t \to \infty$.

Proposition 2 *Consider the full order system (20), if $u = \hat{u} + u_{f1} + u_{f2}$ as designed above, then (20) is ulmately stable in the sense that*

$$\begin{aligned} \|e\|, \|\dot{e}\| &\to O(\epsilon) + O(\mu) \\ \|\eta_1\|, \|\eta_2\| &\to O(\epsilon) \\ \|\eta_3\|, \|\eta_4\| &\to O(\epsilon) + O(\mu) \end{aligned}$$

Proof: The proof can be directly derived from the argumants provided in section 2.

(iii) Case III: ($\epsilon_1^2 \approx \epsilon_2^2$) In this case, the joint stiffness and the link stiffness are of the same order, and we can consider it as a standard two-time scale system.

First consider the fast subsystem (31),

$$\frac{d\eta}{d\tau} = A\eta + Bu_f$$

Since $\{A, B\}$ is a completely controllable pair, we can conclude that the full order system is controllable, and hence obtained the following theorem.

Proposition 3 *If the two perturbation parameters of the full order system (20) are of the same order, then this system is controllable, and the controller can be designed similarly as in case (ii). Besides, if the real parts of all the eigenvalues of A are negative, namely, $\eta \to 0$ as $t \to \infty$ automatically, the asymptotic bounded tracking performance can also be achieved by only applying the rigid part controller.*

Proof: One can easily check the controllability index matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & h_2^T J^{-1} \\ 0 & 0 & h_2^T J^{-1} & 0 \\ 0 & J^{-1} & 0 & -(h_1 + J^{-1}) \\ J^{-1} & 0 & -(h_1 + J^{-1}) & 0 \end{bmatrix}$$

which is always non-singular. The rest of the proof is also straightforward from the discussion provided in section 2.

Remark: Since matrix A is parametrized by the slow variables q_1, appropriate choice of the desired trajectories q_{1d} becomes an important issue.

5. Simulation Results

In this section, we demonstrate the simulation result of the controller proposed in case (ii) and case (iii). The model for simulation is shown in [8] with additional joint flexibility. In case (ii), the joint flexibility ϵ_2 is 0.01 and the link flexibility ϵ_1 is 0.1. The controller gains are $k_1 = 3$, $k_2 = 4$, and $k_d = 3$. The desired trajectory is chosen as $q_d = 3 + 0.5sin(t)$.

Fig.1 and Fig.2 show the tracking errors of angular position and angular velocity, whereas Fig.3, shows the flexible mode, which is under fast controller. Obviously, the flexible mode damp out while the joint tracking the desired trajectory.

In case (iii), both joint and link flexibility are $\epsilon_1 = \epsilon_2 = 0.1$, and the fast controller gains are chosen as k=[0 0 -3.2 -5.1 0 -3.8]. Fig.4, Fig.5 show the tracking performance for both position and velocity, whereas, Fig.6 demonstrates the suppression of vibration.

6. Conclusions

In this paper, singular perturabtion approaches are applied to multiple time-scale systems. The stability of the full order system has been directly addressed. Estimates of domain of attraction and the upper bound of the perturbation parameters were also sought. Furthermore, some discussions are made about the flexible manipulator with both joint and link flexibility when the previous results are applied. In case (i), the link stiffness is much greater than the joint stiffness so that the perturbation parameter ϵ_2 dominates ϵ_1, which makes the link flexibility uncontrollable. In case (ii), as opposed to case (i), the joint stiffness is much greater than the link stiffness, and hence the perturbation parameter ϵ_1 dominates ϵ_2 so that controllers for three subsystems can be developed separately to achieve the desired control performance. In case (iii), the joint stiffness and link stiffness are of the same order, then the overall full-oredr system can formulated as a usually seen two time-scale singular perturbation system. Furthermore, if the fast subsystem is naturally asymptotically stable, the control performance of the full-order system can still be made.

References

[1] W. J. Book,"Recursive Largranigan Dynamics of Flexible Manipulator arm", *Int. J. Robotics Research, Vol. 3, No. 3, 1984.*

[2] B. Siciliano and W. J. Book,"A Singular Perturbation Approach to Control of Lightweight Flexible Manipulators", *Int. J. of Robotic Research, Vol. 7, No. 4, 1988*

[3] S. Cetinkunt and W. J. Book,"Symbolic Modeling of Flexible Manipulators", *IEEE, Conf. on Robotics and Automation, 1987*

[4] B. Siciliano, B. S. Yuan, and W. J. Book,"Model Reference Adaptive Control for A One-Link Flexible Arm", *IEEE, Conf. on Decision and Control, 1986*

[5] P. Lucibello and F. Bellezza,"Nonlinear Adaptive Control of a Two-Link Flexible Arm", *IEEE, Conf. on Decision and Control, 1990*

[6] S. N. Singh,"Control and Stabilization of a Nonlinear Uncertain Elastic Robotic Arm", *IEEE, Trans. on Aerospace and Electronic System, Vol. 24, No. 2, 1988*

[7] A. D. Luca and B. Siciliano,"Trajectory Control of a Nonlinear One-Link Flexible Arm", *Int. J. Control, Vol. 50, No. 5, 1989*

[8] J. H. Yang and L. C. Fu,"Nonlinear Adaptive Control for Flexible Manipulators", *Japan, U.S.A, Conf. on Flexible Automation, 1992*

[9] M. Corless,"Controllers Which Guarantee Robustness With Respect to Unmodelled Flexibilities for a Class of Uncertain Mechanical Systems", *Int. J. Adaptive Control and Signal Processing, Vol. 4, 1990*

[10] F. Ghorbel and M. W. spong,"Stability Analysis of Adaptively Controlled Flexible Joint Robots", *IEEE, Conf. on Decision and Control, 1990*

[11] S. B. Lee and H. S. Cho,"Dynamic Characteristics of Balanced Robotic Manipulators With Joints Flexibility", *Robotica, Vol. 10, 1992*

[12] F. Ghorbel and M. W. Spong,"Adaptive Integral Manifold Control of Flexible Joint Robots With Configuration Invariant Inernia", *American Control Conference, 1992*

[13] F. T. Mrad and S. Ahmad,"Adaptive Control of Flexible Joint Robots Using Position And Velocity Feedback", *Int. J. Control, Vol. 55, No. 5, 1992*

[14] T. E. Alberts, A. Kelkar, and B. Siciliano,"Two Time-Scale Control for an Arm with Joint And Link Compliance" *Proc. of the Int. Symposium on Intelligent Robotics, 1991, India.*

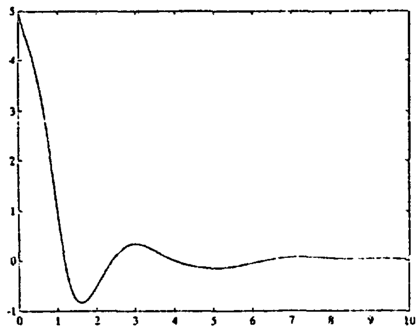

Fig 1 Position Error in Case(ii)

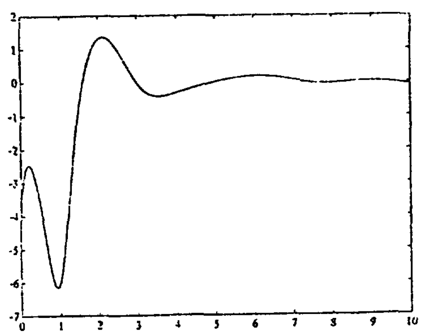

Fig 2 Velocity Error in Case(ii)

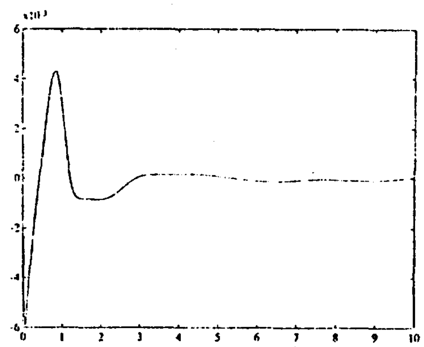

Fig 3 First Mode Trajectory in Case(ii)

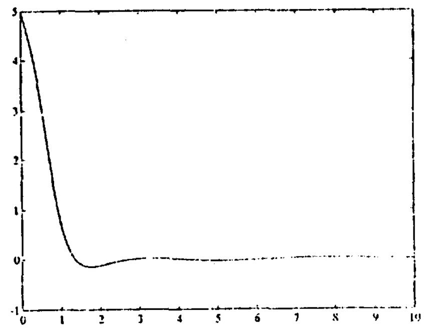

Fig 4 Position Error in Case(iii)

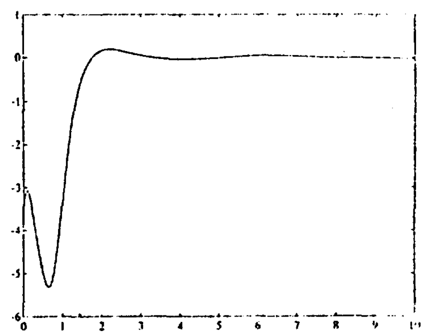

Fig 5 Velocity Error in Case(iii)

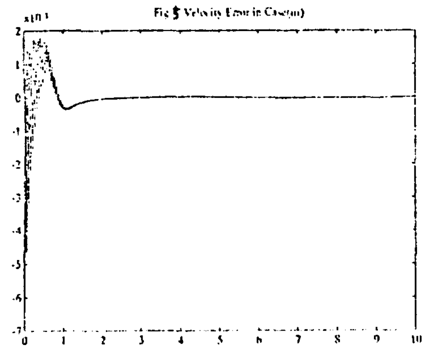

Fig 6 First Mode Trajectory in Case(iii)