On Force and Form Closure For Multiple Finger Grasps

Elon Rimon
Dept. of Mechanical Engineering
Technion, Israel Inst. of Technology
Haifa, ISRAEL

Joel Burdick
Dept. of Mechanical Engineering
California Inst. of Technology
Pasadena, CA 91125

Abstract. *This paper considers the relationship between force and form closure. Based on a recently developed mobility theory, we give precise definitions for 1st and 2nd order form closure for frictionless grasps. We also introduce the new concept of 2nd order force closure. We show for the case of frictionless contacts, a grasp is 1st order force closure if and only if it is 1st order form closure. We further show that a grasp is 2nd order force closure if and only if it is also 2nd order form closure.*

1 Introduction

This paper considers *force closure, form closure*, and the relationship between the two. In the first part of this paper, we consider these issues in the case of frictionless contacts. Force closure is a well established theory, and researchers have converged on a consistent set of definitions and mathematics for its use in the analysis of grasping and fixturing. On the other hand, the concept of form closure is less well defined in the robotics literature, as different authors have used this term to describe similar, but slightly different, concepts. One of the goals of this paper is to introduce precise notions of form closure and relate them to force closure. We give precise definitions of 1st and 2nd order form closure. Further, we show that 1st order form closure implies force closure (which we term 1st order force closure), and vice versa. We also introduce a novel 2nd *order force closure* definition, and show that our 2nd-order form closure definition is dual to 2nd-order force closure.

Our analysis and discussion is based on a recent mobility theory which has been developed by the authors [10, 11, 12, 13]. This theory determines the mobility of a smooth object, $\mathcal{B}$, held in frictionless contact by smooth rigid and stationary finger bodies or fixtures $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Our mobility analysis (which is reviewed in Section 3) is formulated in configuration space (c-space).

2 Relation to Previous Work

The history of force/form closure analysis dates back to the work of Reuleaux (1876) [9] and Somoff (1900)

[15]. More recent analysis dates to the work of Lakshminarayana [3]. Numerous investigators have provided various equivalent definitions of force closure. However, all of these definitions are based on what we term 1st order notions of velocity and force. Our definition of 2nd *order force closure* appears to be new.

The concept of form closure has not been precisely defined in the literature. Mirtich and Canny [5] consider a grasp to be form closure when $\mathcal{B}$'s configuration, which is represented by a point in configuration space, is surrounded by c-space obstacles. However, they give no analytical or computational procedure whereby one could determine if a grasp is form closure. Their notion coincides with our concept of immobility. However, we give more precise definitions of 1st and 2nd order immobility (or form closure) which can be determined by calculating 1st and 2nd order mobility indices. Others (for example [16]) use a notion of form closure which is similar to our notion of 1st order form closure. Other investigators have used Screw Theory (a first order theory) to study the relative motions of bodies in contact [8]. Our notion of 1st order mobility analysis gives results that are equivalent to Screw Theory.

However, traditional 1st-order theories do not account for higher order geometrical effects. For example, in Fig. 1 a smooth triangular object is grasped by 3 point "fingers" or fixtures. First-order theories would indicate that the object can instantaneously rotate about the point of concurrency of the contact normals. However, if the fingers are rigidly immobile, the triangular object is in fact completely immobilized. First-order theories are inadequate because the relative mobility of an object in contact with finger bodies is *not an infinitesimal notion, but a local one*. This deficiency motivated our development of a novel 2nd order mobility theory, which includes the curvatures of the contacting surfaces in the analysis of mobility. While examples show that 1st order effects alone are insufficient to correctly characterize mobility, the combination of 1st and 2nd effects does suffice to completely characterize the mobility of generic grasps.

Our 2nd order mobility theory includes the effects of

curvature, or "higher order" kinematics in mobility analysis. Prior researchers have also included curvature or 2nd order effects in the analysis of grasp stability. Howard and Kumar [2] have included finger and object curvatures in the calculation of the stiffness matrix for compliant grasps. Their compliance model, which is based on a simple linear spring idealization, can be used to assess the stability of a given grasp. In comparison, Rimon and Burdick [11] have shown that kinematic immobility automatically implies dynamic stability for a large class of nonlinear compliance models. Trinkle et. al [18] have also considered 2nd terms in the stability analysis of frictionless grasps of polygonal objects. While Ref.s [2] and [18] both give means to determine the stability of a given grasp (under the assumptions of their models), neither work addresses the more general issue of how to develop a unifying viewpoint on the issues of force and form closure.
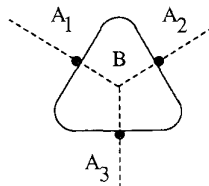

Figure 1: A 3-finger immobilizing grasp

Although the main goal of this paper is to consider the more fundamental issue of the definitions of force/form closure, it is worth noting that the concepts of force and form closure have been used to analyze grasping [7, 16], fixturing [6], and whole arm manipulation [17]. One fundamental application of the force/form closure concept is the issue of the number of frictionless contacts that are required to immobilize an object. Reuleaux [9] found that *at least* 4 *frictionless point contacts* are required to immobilize 2-dimensional (2D) objects. Somoff [15] found that at least 7 frictionless contact points are required to immobilize 3-dimensional (3D) objects. Much later, Markenscoff et. al (1990) [4] established that 4 contact points *suffice* to immobilize generic 2D objects, and 7 suffice to immobilize generic 3D objects. Czyzowicz et. al (1991) [1] have shown that generic 2D and 3D *polygonal objects* could be immobilized respectively by 3 and 4 frictionless point contacts. More recently, Rimon and Burdick [14] have shown that any 2D polygonal or smooth object can immobilized with 3 convex fingers (some fingers might have to be flat). They were also able to overcome the limitation of the analysis in [1] to polygons without parallel edges. Further, they have shown that if the finger curvature can be chosen, then it is possible to immobilize planar smooth and polygonal objects with 2
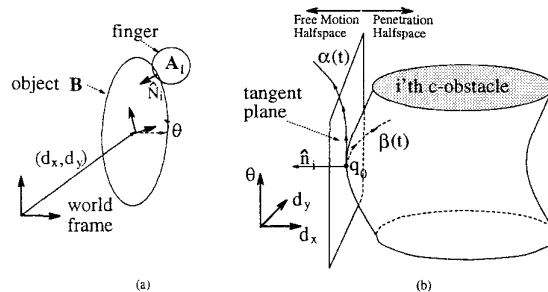

Figure 2: Schematic view of c-space obstacle.

frictionless finger contacts.

3 Rigid Body Mobility Analysis

The essential components of our mobility theory are now reviewed, as these concepts are the basis for our discussions. We study the mobility of an object $\mathcal{B}$ held in point contact by k stationary and frictionless finger bodies $\mathcal{A}_1, \ldots, \mathcal{A}_k$. The analysis is formulated in $\mathcal{B}$'s configuration space. The object $\mathcal{B}$ is described in its own nominal space $\mathcal{R} = I\!R^n$ ($n = 2$ or 3), in terms of a body fixed reference frame in $\mathcal{R}$, $F_\mathcal{B}$. Points in $\mathcal{R}$ are denoted r. Points in the physical space, $\mathcal{E}$, are denoted x. Every rigid placement of $\mathcal{B}$ in $\mathcal{E}$ is described by the following rigid body transformation:

$$ x = X\big(r, (d, R)\big) \triangleq Rr + d \quad r \in \mathcal{R}, x \in \mathcal{E}. \quad (1) $$

The c-space of $\mathcal{B}$ is parametrized in terms of *hybrid coordinates* by $q = (d, \theta)$, where $d \in I\!R^3$ directly parametrizes translation and $\theta \in I\!R^3$ parametrizes $SO(3)$ via the usual exponential map, where $\hat{\theta} = \theta/\|\theta\|$ is the axis of rotation and $\|\theta\|$ is the angle of rotation. Thus $\mathcal{B}$'s c-space is parametrized by $I\!R^m$, where $m = \frac{1}{2}n(n+1)$ ($m = 3$ or 6).

The fingers are represented as c-space obstacles (or *c-obstacles*). For example, in Fig. 2(a) $\mathcal{B}$ is contacted by a single finger $\mathcal{A}_i$. $\mathcal{B}$'s c-space is $q = (d_x, d_y, \theta)$, and the c-obstacle due to $\mathcal{A}_i$ (shown schematically in Fig. 2(b)) is the set of all configurations where $\mathcal{B}$ intersects the stationary "obstacle" $\mathcal{A}_i$. Thus, if q_0 is $\mathcal{B}$'s contact configuration, q_0 lies on the ith c-obstacle boundary, which is denoted $\mathcal{S}_i$. When $\mathcal{B}$ is contacted by k fingers, q_0 lies on the intersection of $\mathcal{S}_i$ for $i = 1, ..., k$.

3.1 1st and 2nd order free motions

The *free motions* of $\mathcal{B}$ are those local motions of $\mathcal{B}$ along which it either breaks away from or roll-slides[1]

---

[1] By a "roll-slide" motion we mean a general displacement between two bodies which maintains surface contact.

on the surface of the finger bodies. In c-space, the free motions of $\mathcal{B}$ at q_0 are the c-space paths that emanate from q_0 and locally lie in the *freespace*, which is the complement of the c-obstacle interiors. The first-order properties of the free motion paths and the c-obstacle boundaries (i.e. tangents and tangent hyperplanes) determine what we call the 1st *order mobility* of $\mathcal{B}$. This notion can be equated to other well known first-order theories, such as Screw Theory [8]. We cast these notions in a c-space framework, as this interpretation is the basis for the consideration of higher order aspects of mobility. In the following, $\hat{n}_i(q_0)$ denotes the outward pointing unit normal to $\mathcal{S}_i$ at q_0 (Fig. 2(b)).

Definition 1 : *The 1st order free motions of $\mathcal{B}$ at q_0 is the halfspace of tangent vectors $\dot{q} \in T_{q_0}I\!R^m$ satisfying $M_i^1(q_0) \triangleq \{\dot{q} \in T_{q_0}I\!R^m : \hat{n}_i(q_0) \cdot \dot{q} \geq 0\}$. The halfspace's boundary, $T_{q_0}\mathcal{S}_i = \{\dot{q}: \hat{n}_i(q_0) \cdot \dot{q} = 0\}$, is called the set of 1st order roll-slide motions. Its interior, $\{\dot{q}: \hat{n}_i(q_0) \cdot \dot{q} > 0\}$, is termed the set of 1st order escape motions. For k fingers, the set of 1st order free motions is: $M_{1,\dots,k}^1(q_0) \triangleq \bigcap_{i=1}^k M_i^1(q_0)$.*

In other words, $\mathcal{B}$ is increasing its distance from $\mathcal{A}_i$ to first order along 1st order escape motions, which implies that it locally breaks away from $\mathcal{A}_i$. To first order, $\mathcal{B}$ maintains zero distance from $\mathcal{A}_i$ along 1st order roll-slide motions, and *it is not possible* to determine from 1st order considerations if $\mathcal{B}$ locally breaks away or penetrates $\mathcal{A}_i$. For example, the c-space curves $\alpha(t)$ and $\beta(t)$ in Fig. 2 are equivalent to first order, yet $\alpha(t)$ locally lies in the freespace, while $\beta(t)$ does not. It can be shown that *all the free motions of $\mathcal{B}$ at an equilibrium grasp are necessarily roll-slide to 1st order* [12]. Hence, $\mathcal{B}$'s mobility depends on the second order properties of its local motions.

The second-order geometry of the free-motion curves and the c-obstacle boundaries is determined by their curvature and curvature form, respectively. The *curvature form* of $\mathcal{S}_i$ at $q_0 \in \mathcal{S}_i$ is denoted $\kappa_i(q_0, \dot{q})$, where $\kappa_i(q_0, \dot{q}) = \dot{q}^T[D\hat{n}_i(q_0)]\dot{q}$ for $\dot{q} \in T_{q_0}\mathcal{S}_i$. The matrix $D\hat{n}_i(q_0) = D^2 d_i(q_0)$ encodes the curvature of the c-obstacle surface at q_0. In Ref. [13] we give a closed-form formula for the curvature form in terms of the surface curvatures and location of the contacting bodies. We show in Ref. [12] that the free motion curves are determined to 2nd-order by their *velocity and acceleration* at q_0.

Definition 2 : *The 2nd order free motions of $\mathcal{B}$ at q_0 is the subset of $(\dot{q}, \ddot{q})$ satisfying*

$M_i^2(q_0) \triangleq \{(\dot{q}, \ddot{q}): \hat{n}_i(q_0) \cdot \dot{q} = 0 \text{ and } \dot{q}^T[D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0) \cdot \ddot{q} \geq 0\}$. *Pairs $(\dot{q}, \ddot{q})$ that satisfy $\hat{n}_i(q_0) \cdot \dot{q} = 0$ and $\dot{q}^T[D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0) \cdot \ddot{q} = 0$ are called $\mathbf{2^{nd}}$ order roll-slide motions, and the other pairs in $M_i^2(q_0)$ are termed $\mathbf{2^{nd}}$ order escape motions. For k fingers, $M_{1,\dots,k}^2(q_0) \triangleq \bigcap_{i=1}^k M_i^2(q_0)$.*

We show in Ref. [12] that if $(\dot{q}, \ddot{q}) \in M_i^2(q_0)$ is a 2nd order escape motion, its corresponding c-space path, $\alpha(t)$ with $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, $\ddot{\alpha}(0) = \ddot{q}$, *locally lies in the freespace* for $t \geq 0$.

3.2 1st and 2nd Order Mobility Indices

Mobility indices are coordinate invariant integer-valued functions that measure the mobility, or effective number of degrees of freedom, of $\mathcal{B}$ when it is held in an equilibrium-grasp configuration q_0. At an *equilibrium grasp* the net wrench on $\mathcal{B}$ must be zero. In c-space, the wrench due to a normal contact force applied by $\mathcal{A}_i$ on $\mathcal{B}$ is represented as a positive multiple of the outward pointing finger c-obstacle normals $\hat{n}_i(q_0)$ [12]. The equilibrium condition in c-space is thus characterized by the requirement that the origin of $T_{q_0}I\!R^m$ be in the **convex hull** of the finger c-obstacles' normals. That is, there must exist scalars $\lambda_1, \dots, \lambda_k$ such that

$$0 = \lambda_1 \hat{n}_1(q_0) + \cdots + \lambda_k \hat{n}_k(q_0),$$
$$\lambda_i \geq 0 \text{ and } \sum_{i=1}^k \lambda_i = 1. \qquad (2)$$

We assume that each $\hat{n}_i(q_0)$ in (2) is *essential* for the grasp, meaning that the origin cannot be positively spanned with any subcollection of $\{\hat{n}_1(q_0), \dots, \hat{n}_k(q_0)\}$ [10]. Grasps are generically essential in the planar case for $k = 2, 3, 4$ and in the spatial case for $k = 2, \dots, 7$. It can be shown that at a k-fingered equilibrium grasp, $M_{1,\dots,k}^1(q_0)$ forms a subspace [12]. This subspace consists of 1st order motions which are roll-slide with respect to each of the fingers. For essential grasps, the dimension of $M_{1,\dots,k}^1(q_0)$ is $(m - k + 1)$ ($m = 3$ or 6). The 1st **order mobility index** of an equilibrium grasp, denoted $m_{q_0}^1$, is defined as the dimension of this subspace. Thus $m_{q_0}^1 = (m - k + 1)$ in the essential case. $m_{q_0}^1$ is shown in Ref. [12] to be coordinate invariant. A key fact is that $m_{q_0}^1$ *is identical for all k-fingered grasps*. Thus, any first order theory, such as screw theory, will be unable to distinguish between alternative equilibrium grasps which have the same number of fingers. This lack of discriminating power is remedied with our 2nd order index.

Consider the λ_i's in the equilibrium condition (2). While the individual c-obstacle curvature forms are in general not coordinate invariant, it is shown in Ref.

[10] that their weighted sum, called the *relative c-space curvature*, has a coordinate invariant structure which characterizes the 2nd order mobility of $\mathcal{B}$.

Definition 3 [10]: *Let $\lambda_1, ..., \lambda_k$ be the coefficients of the equilibrium equation (2). The* **c-space relative curvature form** *for the equilibrium grasp is the quadratic form:* $\kappa_{\text{rel}}(q_0, \dot{q}) \stackrel{\triangle}{=} \sum_{i=1}^{k} \lambda_i \kappa_i(q_0, \dot{q})$, *such that* $\dot{q} \in M_{1,...,k}^{1}(q_0)$.

The 2nd **order mobility index** *of an equilibrium grasp configuration, denoted* $m_{q_0}^2$, *is the* **number of non-negative eigenvalues** *of the* **c-space relative curvature,** $\sum_{i=1}^{k} \lambda_i [D\hat{n}_i(q_0)]$.

Since the relative curvature form is evaluated on the set of 1st order free motions, $m_{q_0}^1$ is an upper bound on the values of $m_{q_0}^2$, i.e. $0 \leq m_{q_0}^2 \leq m_{q_0}^1$. This inequality has an important practical interpretation: 2nd *order effects can reduce the mobility of the grasped object*, as predicted by 1st order effects. In particular, if $m_{q_0}^1 = 0$, $\mathcal{B}$ is completely immobilized, and the 2nd order index carries no immediately useful information. The 2nd order index is always useful for determining the mobility of planar grasps involving $k = 2, 3$ fingers, and in 3D grasps involving $k = 2, ..., 6$ fingers. In these cases $\mathcal{B}$ is not immobilized to first order ($m_{q_0}^1 > 0$), but may be immobilized to 2nd order ($m_{q_0}^2 = 0$). A key interpretation of the 2nd order index is provided in Proposition 5.6 of Ref. [10]: if $m_{q_0}^2 = 0$, any local motion of $\mathcal{B}$ is either 1st order penetration, or it is 1st order roll-slide which is necessarily a 2nd order penetration motion. Thus $m_{q_0}^2 = 0$ implies that $\mathcal{B}$ is completely immobilized, and a test of $m_{q_0}^2$ is sufficient to determine immobility.

4 Form and Force Closure

The dual concepts of *force closure* and *form closure* have often been discussed in the literature on multi-fingered grasping, fixture planning, and the kinematics of bodies in contact. Unfortunately, the distinction between these terms has at times been unclear. We revisit these issues here to point out how our work fits in the historical development of grasping and fixturing analysis and how it can be used as the basis for developing a more unified terminology.

4.1 1st and 2nd Order Form Closure

One of the significant differences between form and force closure lies in the way in which the contact between a fixture and an object is modeled. In form

closure the contact between $\mathcal{B}$ and $\mathcal{A}_i$ *is modeled as a constraint on $\mathcal{B}$'s allowable motions*. That is, one might consider that the sole purpose of a finger or fixture is to limit the possible motions of $\mathcal{B}$. If enough constraints are placed on $\mathcal{B}$'s motion, then $\mathcal{B}$ will be immobilized. Thus, from a practical perspective, *form closure* is equivalent to our notion of *immobilization*. From a c-space point of view, a grasp is form closure or immobilized if $\mathcal{B}$'s configuration, q_0, is completely surrounded by c-obstacles. The concepts of ith order mobility and i^{th} order mobility index give us a means to make the following precise characterization of form closure, which has heretofore been missing from the literature.

Definition 4 *Let $\mathcal{B}$ be at configuration q_0, in contact with k finger bodies. $\mathcal{B}$ is in 1st* **order form closure** *if it is* **immobilized** *to 1st order by the fingers i.e., if the intersection of the 1st order free halfspaces, $M_i^1(q_0)$ for $i = 1, ..., k$, contains only the origin, $\dot{q} = 0$, of $T_{q_0}\mathbb{R}^m$.*

$\mathcal{B}$ is held in 2nd **order form closure** *if it is* **immobilized** *to 2nd order i.e., if the intersection of the 2nd order free motion sets, $M_i^2(q_0)$ for $i = 1, ..., k$, contains only the zero velocity and acceleration vectors, $\dot{q} = 0$ and $\ddot{q} = 0$.*

For grasps whose fingers are **essential**, *1st order form closure is equivalent to $m_{q_0}^1 = 0$, and 2nd order form closure is equivalent to $m_{q_0}^2 = 0$.*

Hence, the mobility indices give precise and computable tests for 1st and 2nd order form closure.

4.2 1st and 2nd Order Force Closure

In the dual modeling approach of force closure, the effect of the contacting finger or fixture on $\mathcal{B}$ is modeled as a *contact force*. In a general frictional contact, $\mathcal{A}_i$ can apply a set of contact forces on $\mathcal{B}$ at a given contact point. This set reduces to a force along the surface normal for frictionless contact. Let W_i be the collection of wrenches (i.e. force and torque) generated by applying all possible contact forces at the ith contact point, and let $W = W_1 + \cdots + W_k$ be the collection of all wrenches which can possibly be generated by the k fingers. If W contains a neighborhood about the zero wrench, the grasp is said to be *force closure*—though the term *wrench closure* is more appropriate. In practice it means that any external wrench can be resisted by a proper combination of the finger forces.

We now show that frictionless 1st order form closure is equivalent to frictionless force closure.

Lemma 4.1 *Let $\mathcal{B}$ be held in a k-finger* **frictionless** *equilibrium grasp, such that all the fingers apply nonzero force. Then the grasp is* **force closure** *if and only if it is* 1st **order form closure**.

Proof: We show that 1st order form-closure implies force closure. For a frictionless contact $W_i = \sigma_i \hat{n}_i(q_0)$ for $\sigma_i \geq 0$. Hence it suffices to show that the normals $\hat{n}_i(q_0)$ for $i = 1, ..., k$ positively span a neighborhood about the origin. By definition of 1st order form-closure, for every $\dot{q} \in T_{q_0} I\!R^m$ there exists a normal $\hat{n}_i(q_0)$ such that $\hat{n}_i(q_0) \cdot \dot{q} < 0$. It follows that the $k \times m$ matrix M, whose rows are $\hat{n}_i(q_0)^T$ for $i = 1, ..., k$, has rank m (necessarily $k \geq m$). Otherwise there would exist a vector $\dot{q}$ such that $\hat{n}_i(q_0) \cdot \dot{q} = 0$ for $i = 1, ..., k$. But the rank of M^T is also m. Hence M^T maps open neighborhoods in its domain, $I\!R^k$, *onto* open neighborhoods in its image space, $I\!R^m$. From the equilibrium condition we know that $\lambda_1 \hat{n}_1(q_0) + \cdots + \lambda_k \hat{n}_k(q_0) = 0$. Hence M^T maps an open neighborhood about the point $(\lambda_1, ..., \lambda_k)$ in $I\!R^k$ onto an open neighborhood about the origin in $I\!R^m$. Since by assumption each λ_i is positive, a sufficiently small neighborhood about $(\lambda_1, ..., \lambda_k)$ consists of points of $I\!R^k$ whose coordinates are positive. Thus the normals $\hat{n}_i(q_0)$ for $i = 1, ..., k$ positively span a neighborhood about the origin. The arguments are all reversible, hence it also follows that force closure implies 1st order form-closure. $\square$

We have just seen that for frictionless equilibrium grasps, 1st order form-closure is equivalent to force closure. It is therefore natural to regard force closure as 1st *order force closure*, and to seek a definition of 2nd order force closure. We propose such a definition for the case of frictionless contact, where $W_i = \sigma_i \hat{n}_i(q_0)$ for $\sigma_i \geq 0$. In the following definition, U is the space spanned by all possible finger wrenches. We know that U is a subspace since $\mathcal{B}$ is in an equilibrium grasp [12]. Let V be the subspace consisting of the orthogonal complement of U. We also use in the definition the *wrench derivatives* which are generated by a stationary finger $\mathcal{A}_i$, while $\mathcal{B}$ moves along a general roll-slide motion on the surface of $\mathcal{A}_i$. Since every finger wrench has the form $w_i = \sigma_i \hat{n}_i(q_0)$, its time-derivative along a c-space trajectory $q(t)$ of $\mathcal{B}$ is $\dot{w}_i = \dot{\sigma}_i \hat{n}_i(q_0) + \sigma_i D\hat{n}_i(q_0)\dot{q}$. Intuitively, one component of $\dot{w}_i$ is generated by varying the magnitude of the contact force, while the other is generated by changing the location of the contact point.

Definition 5 *Let $\mathcal{B}$ be held in a k-finger* **frictionless** *equilibrium grasp. The grasp is* 2nd **order force closure** *if, first, the feasible wrenches* $W = W_1 + \cdots +$

W_k *contain a neighborhood about the origin in the subspace U. And second, if at each $w = w_1 + \cdots + w_k$ in W, the wrench-derivatives $\dot{w}_i$ for $i = 1, ..., k$, span the orthogonal complement subspace V.*

Note that any 1st order force-closure grasp is automatically force-closure to 2nd order. Let us discuss the intuition behind the definition. Every external wrench w can be uniquely written as $w = u + v$, such that $u \in U$ and $v \in V$. From the definition it is clear that the u component can be directly resisted by proper choice of the finger forces. The definition also requires that the fingers be able to generate a wrench derivative whose integration yields a wrench that resists the v component of w. The following lemma gives further evidence that the definition of 2nd order force closure captures the notion dual to 2nd order form-closure.

Lemma 4.2 *Let $\mathcal{B}$ be held at a k-finger* **frictionless** *equilibrium grasp, such that all the fingers are essential for the grasp. Then the grasp is* 2nd **order force closure** *iff it is* 2nd **order form closure**.

Proof: We show that 2nd order form-closure implies 2nd order force closure. For a frictionless contact $W_i = \sigma_i \hat{n}_i(q_0)$ for $\sigma_i \geq 0$. Hence we may regard U as the subspace spanned by the normals $\hat{n}_i(q_0)$ for $i = 1, ..., k$. By definition of 2nd order form-closure, every $\dot{q} \in T_{q_0} I\!R^m$ which lies in U satisfies $\hat{n}_i(q_0) \cdot \dot{q} < 0$ for some $1 \leq i \leq k$. Using the same arguments given in the previous lemma, this implies that the vectors $\hat{n}_1(q_0), ..., \hat{n}_k(q_0)$ positively span a neighborhood about the origin in U. Next consider tangent vectors $\dot{q}$ in the orthogonal subspace V. By definition of 2nd order form closure, the c-space relative curvature is negative definite on V. Hence every $\dot{q} \in V$ satisfies $\lambda_1 \dot{q}^T D\hat{n}_1(q_0)\dot{q} + \cdots + \lambda_k \dot{q}^T D\hat{n}_k(q_0)\dot{q} < 0$. Since the λ_i's are non-negative, every $\dot{q} \in V$ satisfies $\dot{q}^T D\hat{n}_i(q_0)\dot{q} < 0$ for some $1 \leq i \leq k$. Consider now the collection of vectors $D\hat{n}_i(q_0)\dot{q}$, for all $\dot{q} \in V$ and for $i = 1, ..., k$. We have just shown that these vectors span the subspace V. Otherwise there would exist a vector $\dot{q} \in V$ such that $\dot{q}^T D\hat{n}_i(q_0)\dot{q} = 0$ for $i = 1, ..., k$. Each $\dot{w}_i$ can be realized as $\dot{w}_i = \sigma_i D\hat{n}_i(q_0)\dot{q}$ by instantaneously varying the contact point location while the magnitude σ_i is held fixed. Hence the collection of all wrench-derivatives, $\dot{w}_i$ for $i = 1, ..., k$, spans the subspace V. The arguments given are all reversible and hence the lemma is proved. $\square$

5   Frictional Force and Form Closure

In this section we briefly consider how to incorporate friction into our mobility theory. We assume the stan-

dard Coulomb friction model: $|F_1^t| \leq \mu |F_i^n|$, where F_i^t and F_i^n are the tangent and normal components of the ith finger force, F_i, and μ is the coefficient of Coulomb friction.

Let FC_i denote the friction cone associated with the ith finger. Let the ith finger contact $\mathcal{B}$ at point x_i on its boundary. The c-space wrench (c-wrench) due to a single-finger contact force, $F_i \in FC_i$, on $\mathcal{B}(q)$ is:

$$w_i(q) = [DX_{r_i}(q)]^T F_i, \qquad (3)$$

where $DX_{r_i}(q) = \frac{d}{dq} X_{r_i}(q)$, and X_{r_i} is the rigid body transformation of Eq. (1). The set of forces inside the friction cone maps to a cone of wrenches in c-space. Let $\mathcal{W}_i$ represent the c-wrench cone due to the ith finger: $\mathcal{W}_i = \{w: w = [DX_{r_i}]^T F_i, \forall F_i \in FC_i\}$.

In the frictionless case, the set of 1st order free motions was defined using the virtual work principle. That is, 1st order free motions must satisfy $F_i \cdot \dot{X}_{r_i} = F_i \cdot [DX_{r_i}]\dot{q} = ([DX_{r_i}]^T F_i) \cdot \dot{q} = w_i \cdot \dot{q} > 0$, where F_i is a force that can be generated by $\mathcal{A}_i$.. We can apply this concept here to derive the following.

Definition 6 : *Let $\mathcal{B}$ be in frictional contact with stationary rigid finger $\mathcal{A}_i$ at configuration q_0. The set of 1st order unimpeded motions is:*

$$FM_i^1(q_0) = \{\dot{q} \in T_{q_0}\mathbb{R}^m: \dot{q} \cdot w_\alpha \geq 0 \;\; \forall w_\alpha \in \mathcal{W}_i\}$$

For k fingers, $FM_{1,\cdots,k}^1(q_0) = \cap_i FM_i^1(q_0)$. The set of 1st order impeded motions is:

$$FI_i^1 = M_i^1 - FM_i^1$$

The unimpeded motions are those 1st order motions which are 1st order free, and which can not be affected by frictional forces. For a single contact, the set of unimpeded motions is the cone in $T_{q_0}\mathbb{R}^m$ that is polar to the negated wrench cone, $-\mathcal{W}_i$. The impeded motions are those motions which are 1st order free, but which can be *instantaneously* impeded by frictional forces. If $\mathcal{B}$ is contacted by k fingers, then we say it is in *frictional equilibrium* if there exists $w_i \in \mathcal{W}_i$ for $i = 1, \cdots, k$ such that $\sum_{i=1}^k w_i = 0$. Frictional force and form closure can be now by defined.

Proposition 5.1 *Let $\mathcal{B}$ be grasped in a frictional equilibrium by k fingers. Let $\mathcal{W}_i$ be the c-wrench cone due to the ith finger. The grasp is* **frictional 1st order force closure** *if there exists a neighborhood about 0 in $co(\mathcal{W}_i, \cdots, \mathcal{W}_k)$, where $co(\cdot)$ denotes the convex hull. The grasp is 1st order frictional* **form closure** *if $\cap_{i=1}^k FM_i^1 = \{0\}$.*

This definition of frictional force closure is equivalent to its standard definition in the literature. The new frictional form closure definition is dual to the force closure definition.

References

[1] J. Czyzowicz, I. Stojmenovic, and J. Urrutia. Immobilizing a polytope. In *Lecture Notes in Computer Science*, volume 519, pages 214–227. 1991.

[2] W. S. Howard and V. Kumar. Modeling and analysis of the compliance and stability of enveloping grasps. In *IEEE Int. Conference on Robotics and Automation*, pages 1367–1372, Nagoya, Japan, May 1995.

[3] K. Lakshminarayana. Mechanics of form closure. Technical Report 78-DET-32, ASME, 1978.

[4] X. Markenscoff, L. Ni, and C. H. Papadimitriou. The geometry of grasping. *Int. J. of Robotics Research*, 9(1):61–74, Feb 1990.

[5] B. Mirtich and J. F. Canny. Optimum force-closure grasps. Technical Report ESRC 93-11/RAMP 93-5, Dept. of EE-CS, U.C. Berkeley, July 1993.

[6] B. Mishra. Workholding. In *Proc. IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 53–57, 1991.

[7] V.-D. Nguyen. Constructing force-closure grasps. *Int. J. of Robotics Research*, 7(3):3–16, June 1988.

[8] M. S. Ohwovoriole and B. Roth. An extension of screw theory. *J. of Mechanical Design*, 103:725–735, 1981.

[9] F. Reuleaux. *The Kinematics of Machinery*. Macmillan 1876, republished by Dover, NY, 1963.

[10] E. Rimon and J. W. Burdick. Mobility of bodies in contact: A new 2nd order mobility index for multiple-finger grasps. In *Proc. IEEE Int. Conf. on Robotics and Automation*, pages 2329–2335, San Diego, CA, May 1994.

[11] E. Rimon and J. W. Burdick. Mobility of bodies in contact: How forces are generated by curvature effects. In *IEEE Int. Conf. on Robotics and Automation*, pages 2336–2341, San Diego, CA, May 1994.

[12] E. Rimon and J. W. Burdick. A configuration space analysis of bodies in contact—part i: 1st order mobility. *Mechanisms and Machine Theory*, 30(6):897–912, 1995.

[13] E. Rimon and J. W. Burdick. A configuration space analysis of bodies in contact—part ii: 2nd order mobility. *Mechanisms and Machine Theory*, 30(6):913–928, 1995.

[14] E. Rimon and J. W. Burdick. New bounds on the number of frictionless fingers required to immobilize planar objects. *J. Robotic Systems*, 12(6):433–451, June 1995.

[15] P. Somoff. Uber gebiete von schraubengeschwindigkeiten eines starren korpers bieverschiedener zahl von stuzflachen. *Zeitschrift fur Mathematic and Physik*, 45:245–306, 1900.

[16] J. C. Trinkle. A quantitative test for form closure grasps. In *Proc. IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, pages 1670–1677, Raleigh, NC, July 1992.

[17] J. C. Trinkle, A. O. Farahat, and P. F. Stiller. First-order stability cells of frictionless rigid body systems. Tech report. CS Dept., Texas A&M, Jan. 1993.

[18] J. C. Trinkle, A. O. Farahat, and P. F. Stiller. Second-order stability cells of frictionless rigid body systems. In *IEEE Int. Conference on Robotics and Automation*. San Diego, CA. May 1994.