Proceedings of the 1996 International Conference on Robotics and Automation, Minneapolis, EUA, pp. 3714-3719.

OPTIMAL CONTROL SEQUENCE FOR UNDERACTUATED MANIPULATORS

Marcel Bergerman Yangsheng Xu

The Robotics Institute
Carnegie Mellon University
Pittsburgh PA 15213 USA

Abstract

In this paper, we consider the problem of controlling an underactuated manipulator with less actuators than passive joints. The control methodology consists of dividing the passive joints in several groups, and of controlling one group at a time via its dynamic coupling with the actuators. Among the many possible control sequences for a given robot, we choose the optimal one based on the dynamic programming method. The optimization is based on a control cost defined as the reciprocal of the coupling index, a measure of the dynamic coupling available between the active and the passive joints of the manipulator. The detailed theory, computational procedures, simulation results, and experimental results are presented.

1 Introduction

Underactuated robot manipulators are fixed-base serial chain mechanisms with both passive and active joints. The study of such mechanisms is important for fault-tolerant robot design, control of hyper-redundant robots, analysis and control of space robots, and for the analysis of sport mechanics. These robots are inherently nonlinear and nonholonomic, and some researchers have been working on their control considering the nonholonomic constraints [5], [6]. We, however, intend to study the control problem by assuming that all passive joints are equipped with brakes [1]. This allows us to constrain the robot to be holonomic at every instant, and to use classical robot control techniques in this new class of mechanisms. Control consists of controlling the passive joints to their set-points via their dynamic coupling with the active ones. Once the passive joints converge, they are locked and the active ones can be controlled until the entire robot reaches a desired configuration. When the number of passive joints is less than or equal to the number of active ones, all passive joints can be controlled simultaneously. Because this specific case has been studied by several authors—see [1], [3], [5], [7]—we restrict ourselves to the study of manipulators with more passive joints than actuators. In this case, the passive joints have to be divided in several groups and each group has to be controlled individually. The problem discussed in this paper is how to select the passive joints in each group, and how each group can be asymptotically controlled to its desired position.

More specifically, let n be the number of joints in the manipulator, r be the number of actuators installed at the active joints, and p be the number of passive joints equipped with brakes, where $n = r + p$. The passive joints are divided in groups of r joints each, except maybe for the last group, which may contain less than r joints. The groups are numbered from 1 to s; the control scheme consists of controlling the joints in group i via their dynamic coupling with the active joints, while the joints in all other groups are kept locked. When all passive joints in group i reach their set-points, they are locked and control of the joints in group i+1 can be performed. After all s groups of passive joints are controlled, the active ones are brought to their desired set-points. The control problem is then solved in $f = s + 1$ phases.

An apparent problem with the above control scheme is that there are many different ways of choosing the joints to be classified into each group. For example, consider a 4-link manipulator with one actuator. In this case $s = 3$ and there are 6 possible sequences for control of the 3 passive joints: 1→2→3, 1→3→2, 2→1→3, 2→3→1, 3→1→2, 3→2→1. We take advantage of this redundancy to minimize a control cost that is directly related to the torques applied at the actuators. In other words, we use the redundancy present in the system to minimize the energy necessary for the control of the passive joints.

We use as control cost the reciprocal of a measure of the dynamic coupling between the active and the passive joints, which we call the coupling index [2]. The greater the coupling index, the easier it is for the active joints to drive the passive ones. Based on dynamic programming results, we ensure that the problem of dividing the passive joints in groups is globally optimum with respect to the coupling index.

2 Problem definition

Consider an n-link underactuated manipulator, composed of r actuators located at the active joints, and p passive joints equipped with brakes. Assuming that there is sufficient dynamic coupling between the active and passive joints at every configuration of the mechanism, the manipulator can be driven indirectly by applying torques at the actuators. The following is the control problem we are investigating:

Given an initial and a final desired configurations of the manipulator, denoted by q_0 and q_f respectively, drive all the joints so that after some time $T > 0$, $q(T) = q_f$.

The case $r \geq p$ has been studied in our earlier work and by other researchers as well (see [1], [3], [5], and [7]). The solution consists of first controlling only the passive joints to their desired set-points via dynamic coupling with the active ones. Once this objective is reached, the passive joints can be locked in place. The active joints can then be easily controlled as if the mechanism were a regular fully-actuated manipulator. If the number of actuators is less than the number of passive joints, this scheme breaks down, since it has been shown that only r joints can be controlled at any instant ([1], [2]).

The above control scheme can be extended to the case where $r < p$. We start by dividing the passive joints into groups of r joints each. Label these groups $g_1, \ldots, g_s$. We propose the following solution to the problem posed above:

Step 1: set $i = 1$
Step 2: lock all joints in groups g_k, $k = 1, \ldots, s$, $k \neq i$; unlock all joints in group g_i
Step 3: control all joints in group g_i to their desired set-points, and as the joints reach their desired position, lock them
Step 4: increment i; if $i \leq s$, go back to step 2
Step 5: control all active joints to their desired positions

This scheme guarantees the convergence of all joints in the mechanism to a desired final configuration after a finite number of control loops.

3 Optimal control sequence

Given the groups of passive joints to control, the algorithm presented in Section 2 guarantees convergence of all joints to a desired configuration. In this section we consider the problem of selecting the s groups among all possible ones. This is done off-line before control is executed.

Consider first the case $r = 1$, $p = n - 1$. To facilitate the comprehension of the following paragraphs, consider Figure 1, where a 4-link planar manipulator with one actuator is to be controlled from $q_0 = [0, 0, 0, 0]$ to $q_f = [90^o, 90^o, 90^o, 90^o]$. Starting from the top, that figure shows all possible ways of attaining the desired objective. For example, if one follows the left-most path, then the passive joints are controlled in this order: {4, 3, 2}. (For the sake of clarity, we drew the manipulator with the angle of the active joint at zero degrees after each phase. In practice, this is not always true, as will be seen later in the simulations and experimental results.) Note that control from q_0 to q_f requires $f = 4$ phases, three of them for the control of the passive joints. Note also that there is a total of 9 different configurations of the manipulator, and 6 possible ways of reaching q_f from q_0. More generally, the case $r = 1$ requires n control phases, for a total of 2^p+1 different configurations, and $p!$ possible ways of reaching q_f from q_0.

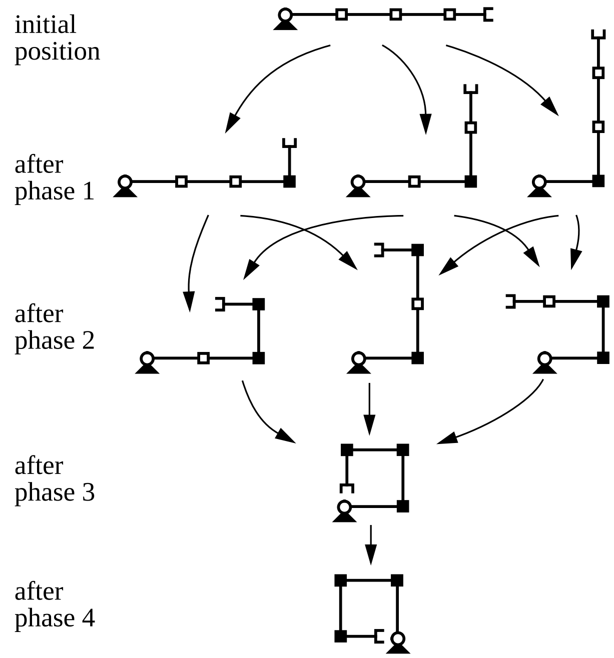

Figure 1: Possible control sequences of a 4-link manipulator with one actuator.

We propose to formulate the problem of selecting the optimal control sequence as a dynamic programming problem. For this purpose, it is necessary to assign a cost to the transitions between all possible joint configurations. This cost must be a function of the magnitude of the actuator's torque so that energy can be rationally utilized. We propose to use the reciprocal of the global coupling index, which will be defined in Section 4, after the transition is completed, as a measure of the cost for each transition. More specifically, let the state transition diagram corresponding to Figure 1 be that given in Figure 2. (The numbers inside the circles represent the number of passive joints already controlled.) Then the cost for, say, transition 1a to 2b, is equal to the inverse of the value of the global coupling index at 2b. The rationale behind this choice is the following: after control phase k is completed, we want the coupling between the active and the passive joints to be as high as possible. If the coupling is low, then a large amount of energy will be spent on phase k+1 (according to the definition of the coupling index). The DP algorithm will then choose the solution with minimum cost, i.e., the one which maximizes the dynamic coupling among all possible solutions for the control problem. Note that we can assign

Proceedings of the 1996 International Conference on Robotics and Automation, Minneapolis, EUA, pp. 3714-3719.

the cost of the transitions from phase $s-1$ to phase s to zero, since after phase s is reached there will be no more passive joints to control.

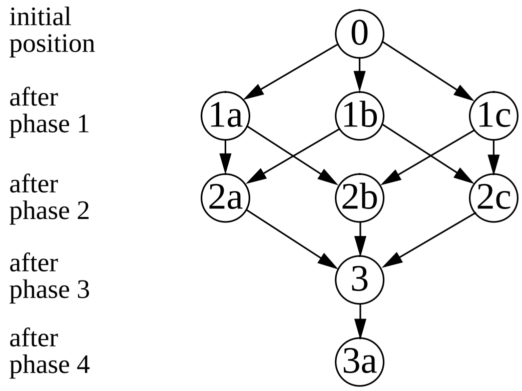

Figure 2: State transition diagram corresponding to Fig. 1.

Finally, given the state transition diagram and the cost of each transition, all one has to do is apply the DP algorithm described in [4]. As explained in that reference, the DP solution is guaranteed to be globally optimum and to reach this optimum solution in less time than an exhaustive search. Note that other tree-search algorithms, such as A*, can also be used instead of DP.

We close this section commenting briefly on the case $r \neq 1$, which is an extension of the previous case. Two sub-cases can be distinguished:

(i) p is a multiple of r ($p = kr$, with $k \geq 2$ an integer number): in this case the passive joints are controlled in groups of r joints each; k phases are necessary for their control, before the active ones can be controlled. The total number of phases, f, different configurations, c, and possible ways of reaching the desired configuration from the initial position, d, are respectively equal to:

$$f = k + 1$$

$$c = \binom{p}{0} + \binom{p}{r} + \binom{p}{2r} + \ldots + \binom{p}{kr} + 1$$

$$d = \binom{p}{r} \times \binom{p-r}{r} \times \binom{p-2r}{r} \times \ldots \times \binom{p-(k-1)r}{r}$$
$$= \binom{kr}{r} \times \binom{(k-1)r}{r} \times \binom{(k-2)r}{r} \times \ldots \times \binom{r}{r}$$

Note that these numbers reduce respectively to n, 2^p+1, and $p!$ when $r = 1$.

(ii) p is not a multiple of r ($p = k_1 r + k_2$, $k_1 \geq 2$, $k_2 < r$, with k_1 and k_2 integer numbers): in this case we first control $k_1 r$ passive joints in groups of r joints each, using k_1 control phases; then the remaining k_2 passive joints are controlled in one phase, and finally the active ones are also controlled in one phase. The parameters f, c, and d in this case are equal to:

$$f = k_1 + 2$$

$$c = \binom{p}{0} + \binom{p}{r} + \binom{p}{2r} + \ldots + \binom{p}{k_1 r} + 2$$

$$d = \binom{p}{r} \times \binom{p-r}{r} \times \ldots \times \binom{p-(k_1-1)r}{r}$$
$$= \binom{k_1 r + k_2}{r} \times \binom{(k_1-1)r + k_2}{r} \times \ldots \times \binom{r + k_2}{r}$$

4 Dynamic coupling measure

The dynamic equation of an underactuated manipulator is similar to that of a regular fully-actuated one, the difference being that, in the present case, the torque vector has p components equal to zero. Lumping these p components at the bottom part of the torque vector, the following partitioned dynamic equation is obtained:

$$\begin{bmatrix} \tau_a \\ 0 \end{bmatrix} = \begin{array}{c} r \\ p \\ \end{array} \begin{array}{c} \begin{bmatrix} M_{aa} & M_{ap} \\ M_{pa} & M_{pp} \end{bmatrix} \\ \begin{array}{cc} r & \;\;p \end{array} \end{array} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_p \end{bmatrix} + \begin{bmatrix} b_a \\ b_p \end{bmatrix} \tag{1}$$

The vector q_a represents the positions of all active joints, while q_p represents those of the passive ones. Note that the inertia matrix M in (1) is symmetric and positive definite, as demonstrated previously in [2]. This leads to the submatrices M_{aa} and M_{pp} also being positive definite. Factoring $\ddot{q}_a$ in the first line of (1), and substituting the result on its second line, the following relationship between the acceleration of the passive joints and the torques applied at the active ones is obtained:

$$\ddot{q}_p = -W_{pp} M_{pa} M_{aa}^{-1} \tau_a + W_{pp}(M_{pa} M_{aa}^{-1} b_a - b_p) \tag{2}$$

where the $p \times p$ matrix W_{pp} is the inverse of the Schur complement of M_{aa} in M, shown in [2] to be positive definite:

$$W_{pp} = (M_{pp} - M_{pa} M_{aa}^{-1} M_{ap})^{-1} \tag{3}$$

We focus on the relationship between the accelerations of the passive joints and the active joints' torques, and rewrite equation (2) as:

$$\ddot{\tilde{q}}_p = -W_{pp} M_{pa} M_{aa}^{-1} \tau_a \tag{4}$$

where:

$$\ddot{\tilde{q}}_p = \ddot{q}_p - W_{pp}(M_{pa} M_{aa}^{-1} b_a - b_p) \tag{5}$$

The vector $\ddot{\tilde{q}}_p$ can be considered as a virtual acceleration of the passive joints, generated by the active torques and the nonlinear torques due to velocity and gravitational effects.

Equation (4) illustrates how the passive joints in an underactuated manipulator can be driven. Namely, torques are applied at the active joints, and the structure of the $p \times r$ *coupling matrix*

$$\hat{M} = -W_{pp} M_{pa} M_{aa}^{-1} \tag{6}$$

dictates how these torques are transmitted to the passive joints. The greater the transmission, the easier the passive joints can be driven via their dynamic coupling with the active ones.

A convenient measure of the coupling between the active and passive joints is given by the norm of the coupling matrix, which is proportional to the product of its singular values. We thus define the *coupling index* as:

$$\rho = \prod_{k=1}^{q} \sigma_k \tag{7}$$

The coupling index thus defined is generally positive, and provides a quantitative indication of the possibility to control the passive joints given the available actuators' torques.

In our earlier work, we also defined a coupling index based on the relationship between the accelerations of the active and of the passive joints. That index can be considered as an acceleration transmission ratio, while the index defined above can be considered as a virtual inertia. Both are useful to characterize the movement of the passive joints; we use in this paper the one that is directly related to the actuators' torques because we are interested on minimizing energy spending.

The coupling index defined above is local in nature; it measures the amount of coupling available between the active and passive joints at a given configuration of the underactuated arm. In design and path planning problems, however, one is more interested in a global measure of the dynamic coupling all over the workspace of the mechanism. Such a *global coupling index* is defined as:

$$\rho^g = \frac{\int_\Theta \rho^2 d\Theta}{\int_\Theta d\Theta} \tag{8}$$

where the integrals are taken over the entire joint space $\Theta \in \Re^n$ of the manipulator. We use the reciprocal of the global coupling index as a cost measure for the transition between control phases, as explained in Section 3, with one minor variation. Namely, we compute the global coupling index only between the passive joints not yet controlled and the active ones. This is justified by the following rationale: after a passive joint reaches its set-point and is locked, it is of no importance to know the amount of coupling between this passive joint and the active ones.

5 Feedback control

Once the passive joints are grouped and the ordering of the groups for control is established, we have to design a feedback control law that guarantees that all joints will reach their set-points asymptotically. As shown in [3], robustness of the closed-loop system to parametric uncertainties and external disturbances is very important for underactuated manipulators. We present here a variable structure controller (VSC) which guarantees asymptotic stability and the necessary system robustness.

We begin by partitioning the joint vector q as:

$$q = \begin{bmatrix} q_a^T & q_c^T & q_w^T \end{bmatrix}^T \tag{9}$$

The r-vector q_a represents the active joints, the r-vector q_c represents the passive joints currently being controlled, the w-vector q_w represents both the passive joints waiting to be controlled and those that already converged, and $w = p - r$. Equation (1) can be further partitioned as:

$$\begin{bmatrix} \tau_a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} M_{aa} & M_{ac} & M_{aw} \\ M_{ca} & M_{cc} & M_{cw} \\ M_{wa} & M_{wc} & M_{ww} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_c \\ \ddot{q}_w \end{bmatrix} + \begin{bmatrix} b_a \\ b_c \\ b_w \end{bmatrix} \tag{10}$$

Factoring $\ddot{q}_a$ in the second line, substituting the obtained expression in the first line, and noting that $\ddot{q}_w \equiv 0$, we obtain:

$$\tau_a = (M_{ac} - M_{aa} M_{ca}^{-1} M_{cc}) \ddot{q}_c - M_{aa} M_{ca}^{-1} b_c + b_a \tag{11}$$

Expression (11) determines the open-loop relationship between the accelerations of the controlled joints and the torques applied at the actuators. The VSC proposed in [3] can now be used for control of the joint variables q_c. Define the following sliding surface:

$$s_c = \Gamma_c \tilde{q}_c + \dot{\tilde{q}}_c \tag{12}$$

where $\tilde{q}_c$ represents the error of the variable q_c and Γ_c is an $r \times r$ diagonal gain matrix. The sliding surface is reached in finite time if the acceleration $\ddot{q}_c$ in (11) is equal to:

$$\ddot{q}_c = \Gamma_c \dot{\tilde{q}}_c + \ddot{q}_{c,d} + P_c \operatorname{sgn}(s_c) \tag{13}$$

where $\ddot{q}_{c,d}$ is the desired acceleration of the controlled joints.

Control law (11)-(13) guarantees asymptotic convergence of q_c to $q_{c,d}$. As each phase is completed, the joints in the vector q_c are substituted for those in q_w corresponding to the next control phase. This guarantees asymptotic convergence of all passive joints to their set-

Proceedings of the 1996 International Conference on Robotics and Automation, Minneapolis, EUA, pp. 3714-3719.

points. During the last phase, the active joints are controlled; the dynamic equation of the system becomes:

$$\tau_a = M_{aa}\ddot{q}_a + b_a \tag{14}$$

As before, a sliding surface is defined in the phase plane of $(q_a, \dot{q}_a)$ and an expression similar to (13) is used instead of the acceleration $\ddot{q}_a$ in (14). The active joints then converge to $q_{a,d}$ and the joint control problem is solved.

6 Simulation results

In this section we present simulation results to demonstrate the feasibility of the method proposed for control of an underactuated manipulator with less actuators than passive joints. We consider a 4-link manipulator with one actuator located at the first joint, as depicted in Figure 1. The dynamic parameters adopted are given in Table 1.

Table 1: Dynamic parameters adopted (simulation).

link	m_i (Kg)	I_i (Kg m^2)	l_i (m)	l_{ci} (m)
1	2.0	0.2	0.30	0.15
2	1.0	0.1	0.30	0.15
3	1.0	0.1	0.30	0.15
4	1.0	0.1	0.30	0.15

In order to make use of the DP method, we compute the cost of each transition in the diagram shown in Figure 2. These costs, found with the use of the software package Matlab, are given in Table 2.

Table 2: Cost of each transition of the state diagram in Figure 2.

Transition	*State after transition*	*Cost*
$0 \rightarrow 1a$	1a	0.3146
$0 \rightarrow 1b$	1b	0.3208
$0 \rightarrow 1c$	1c	0.4564
$1a \rightarrow 2a$ $1b \rightarrow 2a$	2a	0.3822
$1a \rightarrow 2b$ $1c \rightarrow 2b$	2b	0.6677
$1b \rightarrow 2c$ $1c \rightarrow 2c$	2c	0.7277
$2a \rightarrow 3$ $2b \rightarrow 3$ $2c \rightarrow 3$	3	0.0
$3 \rightarrow 3a$	3a	0.0

The DP algorithm generates the following order under which the passive joints should be controlled: $g_1 = \{4\}$, $g_2 = \{3\}$, $g_3 = \{2\}$. (The numbers between the braces represent the joints in each group.) Figure 3 presents the control of the mechanism from an initial position q_0 = [0, 0, 0, 0] to the desired configuration q_f = [60^o, 30^o, -40^o, 40^o]. Note that each passive joint is controlled independently, in the prescribed order, and that they reach their set-points respectively at t = 1.00 s, 2.05 s, and 2.87 s. After the passive joints are controlled, the active one is brought to its set-point at t = 4.26 s.

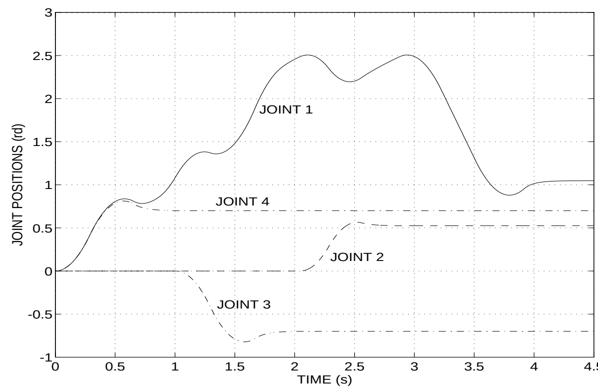

Figure 3: Simulated control of the 4-link manipulator.

7 Experimental results

To validate the simulation result presented, we built a 3-link planar horizontal manipulator equipped with one actuator at the first joint. The robot is shown in Figure 4, and its dynamic parameters are given in Table 3.

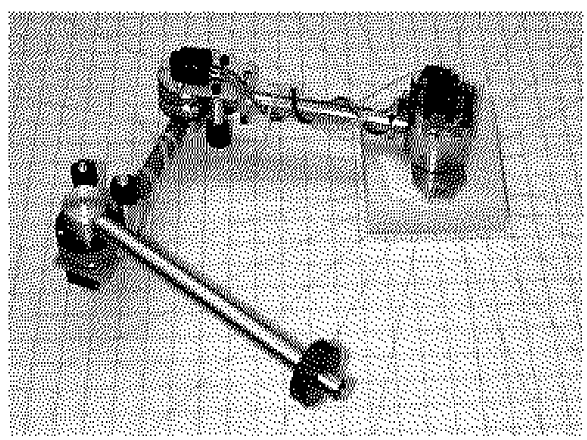

Figure 4: Experimental testbed.

Table 3: Dynamic parameters of the 3-link arm.

link	m_i (Kg)	I_i (Kg m^2)	l_i (m)	l_{ci} (m)
1	0.9938	0.027037	0.262	0.108
2	0.5639	0.016742	0.208	0.150
3	0.3683	0.015473	0.256	0.161

For the 3-link robot, the possible control sequences are shown in Figure 5 (for a generic set-point). The numbers beside the arrows indicate the cost of each transition. One can see that the optimal control sequence of the passive joints consists of first controlling the passive joint on joint 3, followed by that on joint 2; after these 2 joints converge, the active joint can be controlled. When joint 2 is controlled, the matrix M_{ca} is equal to:

$$M_{ca} = 0.0704 + 0.0422c_2 + 0.0248c_3 + 0.0156c_{23} \quad (15)$$

When the third joint is controlled, it is equal to:

$$M_{ca} = 0.0251 + 0.0124c_3 + 0.0156c_{23} \quad (16)$$

Using equation (7) one may conclude that the coupling index never reaches zero inside the manipulator's workspace ($-120^{o} \leq \theta_2, \theta_3 \leq 120^{o}$); accordingly, position control of both passive joints is always achievable.

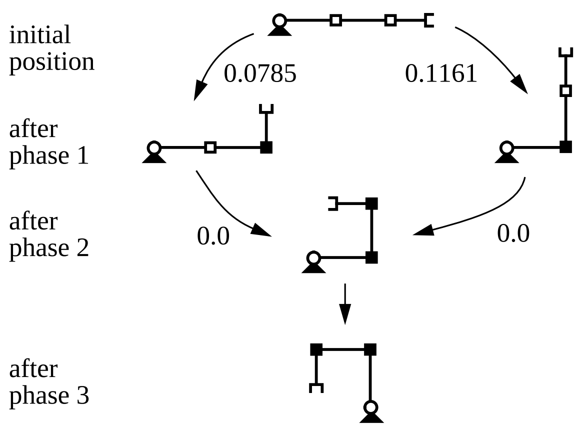

Figure 5: Possible control sequences for the 3-link arm.

Figure 6 presents the control of the manipulator for initial angles q_0 = [12.6°, -6.3°, -39.4°] and final desired angles q_f = [0°, -45°, 90°]. It can be seen that the controller successfully brings both passive joints to their set-points, in the optimal order prescribed by the dynamic programming result. The active joint is then controlled and the manipulator reaches its final desired configuration in joint space. The final steady-state error is $\tilde{q}$ = [-0.006°, 0.000°, 0.000°].

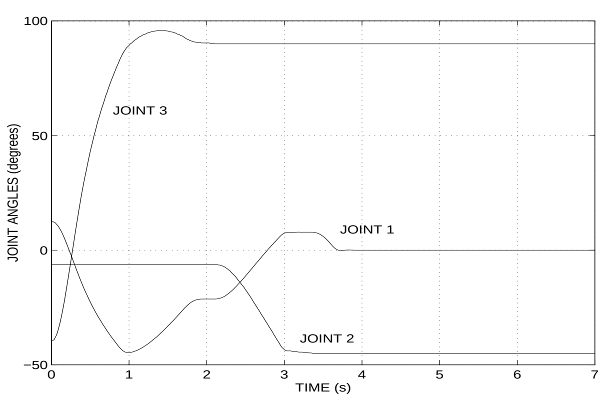

Figure 6: Control of the 3-link manipulator.

8 Conclusion

We present in this paper a method to control an underactuated manipulator with less actuators than passive joints. The methodology consists of dividing the passive joints in several groups, and of controlling each group sequentially. After all passive joints reach their set-points, the active ones can be controlled to their desired positions. We use a variable structure controller to guarantee robustness of the system to parameter uncertainty and external disturbances, and dynamic programming to ensure that the control sequence is optimal with respect to the coupling index.

We demonstrate through simulation and experimental studies the feasibility of the proposed method in controlling an actual 3-link manipulator in joint space with the use of only one actuator.

9 Acknowledgments

We would like to thank Christopher Lee for writing part of the control software for this project; and H. Ben Brown for helping with the design and assembly of the 3-link experimental mechanism. The first author is supported by a grant from the Brazilian National Council for Research and Development (CNPq).

10 References

[1] Arai, H.; Tachi, S. "Position control of a manipulator with passive joints using dynamic coupling." *IEEE Trans. Robotics and Automation*, vol. 7, no. 4, Aug. 1991, pp. 528-534.

[2] Bergerman, M.; Lee, C.; Xu, Y. "A dynamic coupling index for underactuated manipulators." *Journal of Robotic Systems*, vol. 12, no. 10, Oct. 1995, pp. 693-707.

[3] Bergerman, M.; Xu, Y. "Robust control of underactuated manipulators: analysis and implementation." *IEEE Conf. on Systems, Man and Cybernetics*, Oct. 1994, pp. 925-930.

[4] Kirk, D.E. *Optimal control theory: an introduction*. Prentice-Hall, Englewood-Cliffs, 1970, chapter 3.

[5] Mukherjee, R.; Chen, D. "Control of free-flying underactuated space manipulators to equilibrium manifolds." *IEEE Trans. Rob. Automation*, vol. 9, no. 5, Oct. 1993, pp. 561-570.

[6] Oriolo, G.; Nakamura, Y. "Free-joint manipulators: motion control under second-order nonholonomic constraints." *Intelligent Robots and Systems Conference*, 1991, pp. 1248-1253.

[7] Papadopoulos, E.; Dubowsky, S. "Failure recovery control for space robotic systems." *American Control Conference*, vol. 2, 1991, pp. 1485-1490.