A New Line Tracking Method for Nonholonomic Vehicles

Yutaka J. Kanayama
Department of Computer Science
Naval Postgraduate School
Monterey, CA 93943
kanayama@cs.nps.navy.mil

Fariba Fahroo
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943
fahroo@math.nps.navy.mil

Abstract

In this paper, we investigate the problem of finding an algorithm for the movement of a vehicle under the nonholonomic constraint to track a given directed straight line without allowing any spinning motion. We propose a new principle of computing the derivative of path curvature as a linear combination of the current vehicle path curvature, vehicle orientation error, and positional error. We call this function the steering function. By linearization we find an optimal selection of parameters for critically damped motions and obtain a single parameter, σ, for tracking, which we call smoothness. The uniform asymptotic stability of the feedback rule is proved through a Lyapunov function. Numerous simulation results as well as experimental results obtained on the autonomous robot Yamabico at the Naval Postgraduate School are included to show the effectiveness of this method.

1 Introduction

The motion planning/control problems for autonomous vehicles with the nonholonomic constraint have given rise to a vast body of literature in recent years. Due to a celebrated result by Brocket [1], it has been known that nonholonomic wheeled mobile robots can not be stabilized to a given configuration by a smooth feedback control. Therefore, to handle the "configuration tracking problem", some researchers such as Canudas and Sørdalen in [2] have proposed piecewise smooth feedback laws for exponentially stabilizing a mobile robot. Samson in [3, 4] has proposed a smooth time varying feedback control law for stabilizing a robot, or even a chained system of wheeled robots to a given final configuration.

While these approaches are mathematically elegant and general enough to be applied to a variety of situations, for the specific problem of "straight-line tracking" the feedback laws given by these procedures do not necessarily yield the simplest and most efficient feedback controllers. The problem considered in this paper is to find an algorithm for the movement of the vehicle to track a given directed line without allowing any spinning motion. Our approach to this problem is based on a geometrical notion leading to a feedback design that is easy to use and implement, and also guarantees the exponential convergence of the mobile vehicle's path from an arbitrary initial configuration to a given directed straight line.

This problem was first studied by one of the authors. In [5] a motion planning algorithm using a sequence of straight lines was proposed. After this preliminary investigation, he found a stable tracking rule using a Lyapunov function [6]. The problem in this paper was to find a suitable pair (v, ω), the linear and angular velocity of the vehicle so that the vehicle would track a target moving on a straight directed line. A condition for the parameter in the feedback control rule for a critically damped response was obtained through linearization. The results from both papers were successfully implemented and tested on the Yamabico autonomous vehicle at the University of Tsukuba and at the University of California at Santa Barbara.

Recently, DeSantis, [7] has developed a control rule based on the geometric path tracking principle for a tractor-trailer-like robots to track a straight line or a circular arc. In a related effort, Thuilot and others, [8] have developed a theory based on linearized dynamic feedback as well as time-varying feedbacks for the moving reference trajectory tracking problem for vehicles with several steering wheels.

The new method proposed in this paper is based on the fundamental observation on the role of path curvature in vehicle motion control. In this method, the derivative $d\kappa/ds$ of curvature is computed rather than the curvature κ itself, where s is path length. A rule to compute the derivative of path curvature is to express it as a linear combination of the vehicle path curvature, vehicle heading errot, and signed positional error Δd. This function is called a *steering function* in this paper.

The main results of this theory are summarized as follows: (1) The uniform asymptotic stability of the feedback rule is proved through linearization and a Lyapunov function. (2) Assuming the critical damping condition, the steering function contains only one parameter σ that controls the smoothness (or equivalently, sharpness) of vehicle motions. (3) As the condition $|d\kappa/ds| < \infty$ is satisfied, this scheme ensures the continuity of curvature κ that is needed for rigid body vehicle motion [9]. (4) This theory is independent of vehicle architecture.

To demonstrate the effectiveness of this method numerous simulation results are included. This smooth motion control algorithm has been successfully working on the Yamabico autonomous mobile vehicle at Naval Postgraduate School. Because of the simplicity of the steering function, the implementation is easy and the processing time is short. The real trajectories of Yamabico generated by this algorithm are also shown. From a comparison of these results with the simulation results, we observe a good agreement between the two types of results.

2 Motion Control by Path Curvature

A two dimensional nonholonomic vehicle's status is described by a *vehicle configuration*

$$q = (p, \theta, \kappa) = \Big((x, y), \theta, \kappa\Big), \qquad (1)$$

where p and θ denote its position and heading orientation respectively, and κ is the path curvature in a global Cartesian coordinate system. All these variables are functions of time or path length. We include the path curvature κ in the configuration, partially because a vehicle under the nonholonomic constraint can execute only a path that has curvature continuity [9, 10, 11].

Although a two dimensional vehicle motion has generally three degrees of freedom [9], a nonholonomic motion has only two degrees of freedom. Such a motion can be described as follows: There is one and only one line $\mathcal{L}$ fixed on the vehicle so that there is a rotational center c on $\mathcal{L}$ with a rotational speed of ω (Fig. 1). We call this line $\mathcal{L}$ the *major* axis of the vehicle. An arbitrary point on the major axis can be taken as a *vehicle reference point* O_V. Since the position of c is specified by the distance r from O_V, an instantaneous motion $\mathcal{Q}$ is described by only two variables (r, ω) or (r, v), where v is the linear speed at O_V. A better representation is

$$\mathcal{Q} = (\kappa, v), \qquad (2)$$

where $\kappa = 1/r$ is curvature. If the center c of rotation is at infinity, the vehicle is moving on a straight line and $\kappa = 0$. This representation clearly shows the two degrees of freedom possessed by nonholonomic vehicle motions. The path curvature κ is the best variable as a tool to describe vehicle's motions, because (1) κ is more directly related to vehicle control, and (2) the curvature is independent of how a global coordinate system is placed. As we notice every day, the path curvature of an automobile is positively controlled through its steering wheel.
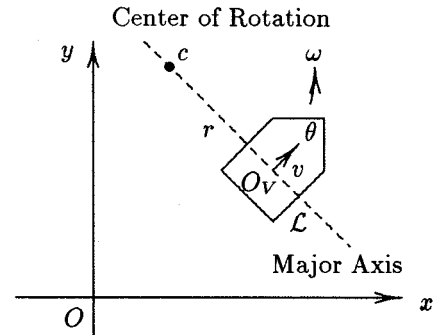

Figure 1: Constraint in Vehicle Motion

In this paper, we will mainly discuss designing the optimal curvature κ, but not of speed v. With this background, the configuration q and motion $\mathcal{Q}$ are represented as functions of length s as the independent variable rather than time t. By the definition of curvature, and by the fact that the tangential direction of the trajectory is equal to the vehicle heading $\theta(s)$ (Fig. 1), we obtain

$$\theta(s) = \theta(0) + \int_0^s \kappa(u)du, \qquad (3)$$

$$x(s) = x(0) + \int_0^s \cos\theta(u)du, \qquad (4)$$

$$y(s) = y(0) + \int_0^s \sin\theta(u)du. \qquad (5)$$

3 Line Tracking

3.1 Steering Function

Now our original problem becomes the one of finding the "optimal" curvature κ by feedback at each moment in order to track a given line L. Actually, rather than computing the curvature itself, we propose a method which computes the derivative $d\kappa/ds$ of curvature κ,

$$\lambda = \frac{d\kappa}{ds} = f(L, q), \qquad (6)$$

because of the constraint that the curvature must be continuous. Since the magnitude $|\lambda|$ is finite, the curvature is always continuous. Let a vehicle $q = (p, \theta, \kappa)$
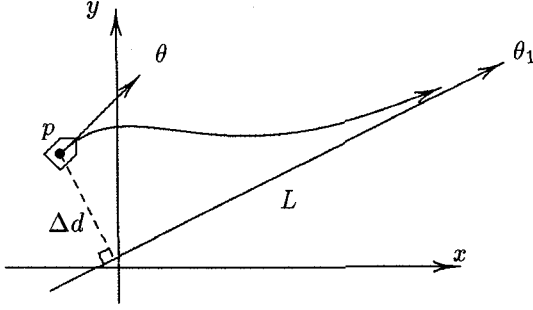
Figure 2: Principle of Path Tracking

be supposed to track a directed line L (Fig. 2). We propose a specific form of a control rule

$$\frac{d\kappa}{ds} = -a\kappa - b(\theta - \theta_1) - c\Delta d, \qquad (7)$$

where, a, b, and c are positive constants. This negative feedback rule is called a *steering function*. The first term, $-a\kappa$ is a feedback term (a damping factor) for the curvature, the second term $-b(\theta - \theta_1)$ is a feedback term for the angle error, and the third term $-c\Delta d$ is a feedback term for the *positional error*. The *positional error* Δd is the signed distance from p to L, where $\Delta d > 0$ if the vehicle is on the left side of the directed line, $\Delta d < 0$ if it is on the right side, and $\Delta d = 0$ if it is on the line.

3.2 Linearization

Choosing the "optimal" combination of the three constants (a, b, c) is the next important issue to tackle. If we select an inappropriate combination, the vehicle may not converge to the line L or may oscillate around the directed line and takes longer tracking for convergence. We first perform an appropriate coordinate transformation so that the given directed line L becomes the X-axis of the global coordinate system in the positive direction [12]. After this transformation,

$$\theta_1 = 0, \qquad \Delta d = y. \qquad (8)$$

Thus, the steering function (7) becomes

$$\frac{d\kappa}{ds} = -a\kappa - b\theta - cy. \qquad (9)$$

Notice that this is a nonlinear homogeneous differential equation and there is no closed form solution. We impose the condition, $|\theta| < \frac{\pi}{2}$, so that the trajectory is a single-valued function $y(x)$.

We can formulate the problem as achieving asymptotic stability of the zero equilibrium solution of the nonlinear dynamic equation (9) by a proper choice of a, b, c. In this case, *uniform asymptotic stability* of the zero solution means that the trajectories of the vehicle with certain initial configurations converge to the

X-axis in an exponential rate. In order to study the stability properties of the zero solution and their dependence on the choice of a, b, c, we refer to Lyapunov stability theories. First, we use the indirect method which gives us information about asymptotic stability of the nonlinear equation by studying the related linearized first order system of equations. So we write Equation (9) as a system of first order equations by using the following variables:

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \end{pmatrix}. \qquad (10)$$

We would like to have the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (11)$$

where the dot denotes differentiation with respect to x. We represent all the variables in Equation (9) (y, θ, κ, and $\frac{d\kappa}{ds}$) in terms of x_1 to x_3.

$$
\begin{aligned}
y &= x_1 \\
\theta &= \tan^{-1} y' = \tan^{-1} x_2, \\
\kappa &= \frac{d\theta}{ds} = \frac{d\theta/dx}{ds/dx} = \frac{y''}{(1 + y'^2)^{3/2}} = \frac{x_3}{(1 + x_2^2)^{3/2}}, \\
\frac{d\kappa}{ds} &= \frac{d\kappa/dx}{ds/dx} = \frac{\frac{d}{dx}\left(\frac{y''}{(1+y'^2)^{3/2}}\right)}{\sqrt{1 + y'^2}} \\
&= y'''(1 + y'^2)^{-2} - 3y'y''^2(1 + y'^2)^{-3} \\
&= \dot{x}_3(1 + x_2^2)^{-2} - 3x_2 x_3^2(1 + x_2^2)^{-3}.
\end{aligned}
$$

After substituting these results into (9), we can obtain an equation in the form of Equation (11):

$$
\begin{aligned}
\dot{x}_1 &= y' = x_2 = f_1, \\
\dot{x}_2 &= y'' = x_3 = f_2, \\
\dot{x}_3 &= y''' \\
&= 3x_2 x_3^2(1 + x_2^2)^{-1} - ax_3(1 + x_2^2)^{1/2} \\
&\quad - b(1 + x_2^2)^2 \tan^{-1} x_2 - cx_1(1 + x_2^2)^2 \\
&= f_3,
\end{aligned}
$$

where $\mathbf{f} = [f_1, f_2, f_3]^T$ clearly satisfies

$$\mathbf{f}(0) = 0. \qquad (12)$$

Now we linearize the system by using the Jacobian of $\mathbf{f}$

$$\mathcal{A} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}. \qquad (13)$$

The Lyapunov's indirect method informs us that the nature of the eigenvalues of matrix $\mathcal{A}$ determines the stability properties of the system. More precisely the method states that the equilibrium point 0 is uniformly asymptotically stable if all the eigenvalues of $\mathcal{A}$ have negative real parts [13].

2910

For our problem the eigenvalues λ of A satisfy the equation,

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0. \tag{14}$$

To obtain the proper conditions on the coefficients for achieving asymptotic stability, we refer to the Routh-Hurwitz criterion which says a, b, and c must be all positive and $ab - c > 0$. Therefore, we have the following statement.

Proposition 3.1 *The equilibrium point* $\mathbf{0}$ *of Equation (9) is uniformly asymptotically stable if* a, b, c *are positive constants and* $ab > c$.

If the eigenvalues are three negative numbers, $-k_1$, $-k_2$, $-k_3$, the parameters a, b, c must satisfy

$$a = k_1 + k_2 + k_3, \tag{15}$$
$$b = k_2 k_3 + k_3 k_1 + k_1 k_2, \tag{16}$$
$$c = k_1 k_2 k_3. \tag{17}$$

Furthermore, if we take all the negative real eigenvalues equal,

$$k_1 = k_2 = k_3 = k \text{ for some } k, \tag{18}$$

then the relationship among the constants becomes simpler:

$$a = 3k, \quad b = 3k^2, \quad c = k^3. \tag{19}$$

We call k the *gain* of the steering function. Therefore, we can conclude

Corollary 3.1 *For the Equation (9), the equilibrium point* $\mathbf{0}$ *is uniformly asymptotically stable if* $a = 3k$, $b = 3k^2$, *and* $c = k^3$ *for some positive constant* k.

In the examples in Section 4, the parameters are supposed to satisfy the set of relations (19). We let

$$\sigma = \frac{1}{k} \tag{20}$$

that is called the *smoothness* of the steering function. A steering function has only one parameter σ.

3.3 Global Stability

The stability analysis presented in the previous section gives us asymptotic stability of the solutions in a local sense. This means that only for the initial configurations that are sufficiently close to the origin, the subsequent trajectories converge to the X-axis. To study the stability properties of the highly nonlinear system (9), we need to refer to the direct Lyapunov

method which gives us more information about the subset of of the real plane called the *domain of attraction* or the set of initial conditions that result in uniform asymptotic convergence to a desired path. Upon some numerical experiments, we have encountered examples where for some initial conditions the trajectories do not converge to the X-axis. Therefore, we already know that the system lacks global exponential stability. But we can indeed prove a stability result that is stronger than the one obtained by linearization. Under a restriction only on the magnitude of the direction θ, we will prove that all the resulting paths converge to the X-axis, regardless of the initial values of the other variables, y, and κ. This result is obtained by the direct method which relies on constructing a positive definite Lyapunov function whose path derivative along the trajectory is negative. We refer to the following theorem [14]:

Theorem 3.1 *Consider the autonomous equation*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(\mathbf{0}) = \mathbf{0}. \tag{21}$$

Let $V(\mathbf{x})$ *be a scalar function, and let* Ω *designate a region where* $V(\mathbf{x})$ *is positive-definite. Assume that* Ω *is bounded and that within* Ω,

- $V(\mathbf{x}) > 0$ *for* $\mathbf{x} \neq \mathbf{0}$, *and* $V(\mathbf{0}) = 0$,

- $\dot{V}(\mathbf{x}) \leq 0$,

- *if* $\dot{V}(\mathbf{x}) = 0$ *at points other than the origin, then these points are not a solution of the system.*

Then the origin is asymptotically stable and all motions starting in Ω *converge to the origin eventually.*

To use this result, we first choose Ω to be the set of configurations with its direction θ restricted in the range, $|\theta| < \frac{\pi}{2}$,

$$\Omega = \left\{ \Big((x,y),\theta,\kappa\Big) : |\theta| < \frac{\pi}{2} \right\}. \tag{22}$$

Second, we construct the following Lyapunov function:

$$V = \frac{b}{2c}\kappa^2 + \frac{b^2}{2c}\theta^2 + \frac{c}{2}y^2 + by\theta + \kappa\sin\theta + a(1 - \cos\theta), \tag{23}$$

where a, b, c are positive constants in Equation (9). It is easy to see that $V(0) = 0$. Now we need to show that V is a positive definite function for some values of a, b and c.

Lemma 3.1 *For positive constant* a, b *and* c *that satisfy the relationship* $ab > c$ *the Lyapunov function,* V *is positive definite in* Ω *if* $\mathbf{x} \neq \mathbf{0}$.

For the rest of the conditions in the Theorem, we need

Lemma 3.2 *For values a, b, c such that $ab > c$*

$$\dot{V} \leq 0 \ in \ \Omega$$

and the points at which $\dot{V} = 0$ are not a solution to the system of equations.

We can now summarize the results on the estimate of the region of asymptotic stability in the following proposition.

Proposition 3.2 *If $ab > c$, all trajectories starting and remaining in Ω will eventually converge to the X-axis.*

4 Results and Applications

4.1 Fundamental Results

In this example, the vehicle is supposed to track a directed straight line L with the direction of $\theta_1 = \pi/4$, where its initial direction θ_0 is one of five distinct directions: $-\pi, -\pi/2, 0, \pi/2, \pi$.
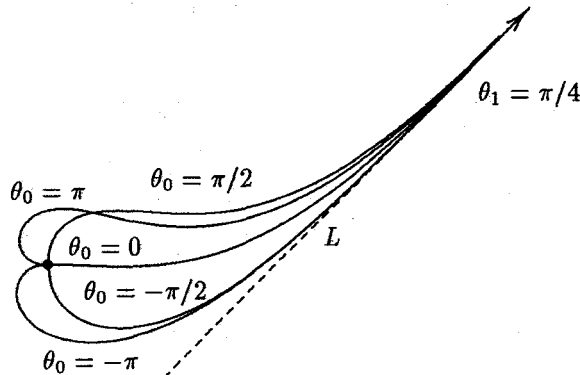

Figure 3: Line Tracking

4.2 Effect of Smoothness σ

The only parameter in the steering function is the smoothness σ. Selecting an appropriate smoothness in a given situation is fundamentally important. Generally speaking, using a larger smoothness is better for obtaining a smoother motion so that a faster motion is possible. However, in a tighter space, we need to use a smaller smoothness in order to make sharper turns. In this section we show the effect of the value of smoothness on the resultant vehicle motions.

Figure 4 shows how the vehicle tracks the X-axis after leaving an initial configuration, $q_0 =$

$((0, 100), 0, 0)$. The effect of using distinct values of smoothness with $\sigma = 20, 40, 80$, and 160 is clearly seen in this figure. The smoothness of a vehicle motion is perfectly controlled by σ.
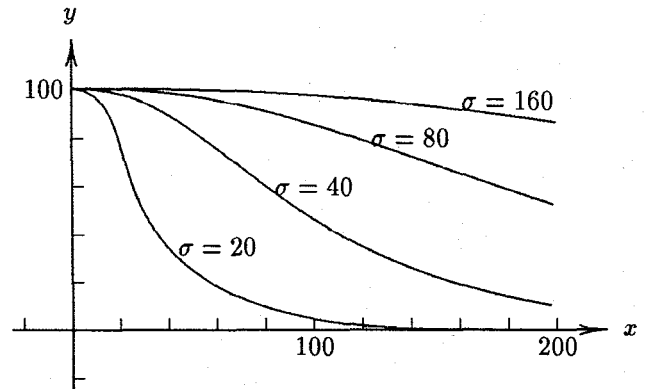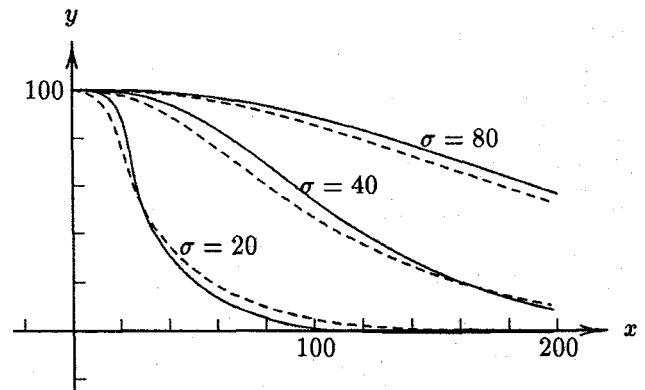

Figure 4: Effect of Smoothness

Figure 5: Results from Real Experiments (solid lines) and Simulation (broken lines)

This line tracking algorithm is already implemented as a part of the Yamabico autonomous vehicle software system. Figure 5 shows the difference between the results obtained by simulation and by real experiments. Both results are pretty close and the performance by Yamabico is satisfactory.

4.3 Wall Tracking

The *wall tracking* problem is one of planning a motion for a vehicle to track non-flat parallel walls with a given constant clearance of d_0, where the vehicle is equipped with side-looking range finders. The vehicle obtains the current clearance Δd to the wall by the sensor in real time in order to keep the clearance Δd equal to the given desired clearance d_0 by steering the vehicle. Unless the vehicle is in a transition status, the vehicle's orientation must be equal to the wall direction θ_1 and its curvature must be 0. This problem can be easily solved using the steering function

method. We use a term $c(\Delta d - d_0)$ instead of $c\Delta d$ in Equation (7). One simulation result is given in Figure 6, Part (I). This behavior shows how the vehicle faithfully follows the exact profile of the walls.

(I) Wall Tracking with Exact Clearance
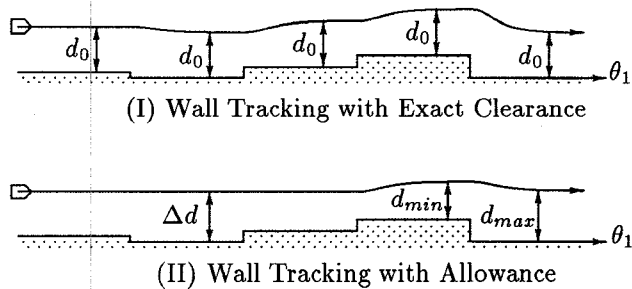
(II) Wall Tracking with Allowance

Figure 6: Wall Tracking

We can loosen the strict wall tracking requirement in order to reduce the frequency of lateral transitions. One method is that the vehicle keeps clearance Δd to try to satisfy the condition $d_{min} \leq \Delta d \leq d_{max}$, where d_{min} and d_{max} are positive constants with $d_{min} < d_{max}$. Namely, the clearance has some allowance. By this method, the vehicle executes steering so that slight wall-position changes do not cause vehicle's lateral motions (Fig. 6, Part (II)). Only when a significant change occurs, the vehicle steers.

5 Conclusion

In this paper, the new principle of controlling the derivative of path curvature for smooth nonholonomic motion planning is proposed and is specifically applied to line tracking. Through linearization and the Lyapunov stability theory, we have shown exponential convergence of the vehicle's path from an initial condition to a given straight line. This algorithm works well not only in simulation, but also in real vehicle navigation on the Yamabico autonomous mobile robot.

We will report on the application of this principle to other path tracking problems such as tracking a circle, or a more general C^1 curve in future work.

References

[1] Brockett, R.W., "Asymptotic Stability and Feedback Stabilization," in *Progress in Math*, vol.27, Birkhauser, pp. 181-208, 1983.

[2] Canudas de Witt, C. and Sørdalen, O.J., "Exponential Stabilization of Mobile Robots with Nonholonomic Constraints", IEEE Trans. on Automatic Control, vol. 37, pp. 1791-1797, 1992.

[3] Samson, D., "Velocity and Torque Feedback Control of a Nonholonomic Cart," in Advanced Robot Control, Lecture Notes in Control and Information Sciences, No. 162, pp. 125-151, 1991.

[4] Samson, C., "Time-varying Feedback Stabilization of Car-like Wheeled Mobile Robots," International Journal of Robotics Research, vol. 12, no. 1, pp. 55-64, 1995.

[5] Kanayama, Y, "Vehicle Path Specification by a Sequence of Straight Lines," IEEE Journal of Robotics and Automation, vol. 4, no. 3, pp. 265-276, 1988.

[6] Kanayama, Y., Kimura, Y., Miyazaki, F. and Noguchi, T., "A Stable Tracking Control Method for an Autonomous Mobile Robot," *Proc. IEEE International Conference on Robotics and Automation*, pp. 1315-1317, Cincinnati, Ohio, May 13-18, 1988.

[7] DeSantis, R.M., "Path-tracking for a Tractor-Trailer-like Robot," International Journal of Robotics Research, vol. 13, no. 6, pp. 533-543, 1994.

[8] Thuilot, B., d'Andréa-Novel, B., and Micaelli, A., "Modeling and Feedback Control of Mobile Robots Equipped with Several Steering Wheels," IEEE Transaction on Robotics and Automation, vol. 12, no. 3, pp. 375-390, 1996.

[9] Kanayama, Y., "Two Dimensional Wheeled Vehicle Kinematics," Proceedings of International Conference on Robotics and Automation, in San Diego, California, May 8-13, pp. 3079-3084, 1994.

[10] Kanayama, Y. and Miyake, N., "Trajectory Generation for Mobile Robots", *Robotics Research*, vol. 3, MIT Press, pp. 333-340, 1986.

[11] Kanayama, Y. and Hartman, B., "Smooth Local Path Planning for Autonomous Vehicles," IEEE International Conference on Robotics and Automation, (1989), pp. 1265-1270.

[12] Y. Kanayama, D. L. MacPherson, and G. W. Krahn, "Two Dimensional Transformations and Its Application to Vehicle Motion Control and Analysis," Proceedings of International Conference on Robotics and Automation, in Atlanta, Georgia, May 2-7, pp. 13-18, 1993.

[13] Vidyasagar, M., *Nonlinear Systems Analysis*, 2nd Edition, Prentice Hall, New Jersey, 1993.

[14] Pai, M.A., *Power System Stability*, North-Holland Publishing Co., 1981.