New insights into second and fourth-order direction finding for NonCircular sources

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***Abstract*—These last three decades, many second order (SO) and higher order (HO) high resolution (HR) direction finding (DF) methods, such as $2q$-MUSIC ($q \geq 1$), exploiting the information contained in the SO or HO circular (C) cumulants of the data, have been developed. However, for $2q$th-order non-circular (NC) sources such as M-PSK sources with $M \leq 2q$, strong gains in performance may be obtained by taking into account the information contained in both $2q$th-order C and NC cumulants of the data, giving rise to NC $2q$th-order DF methods. Numerous NC DF methods have been developed these last fifteen years but mainly at the SO and under restrictive assumptions on the sources. The purpose of this paper is to give new insights into NC $2q$-MUSIC methods for $1 \leq q \leq 2$ and for arbitrary NC sources.**

I. Introduction

From the beginning of the 1980s, many SO, HR, DF methods have been developed among which the MUSIC method is the most popular [1]. To improve the performance of SO methods, in terms of resolution, robustness to modeling errors and number of sources to be processed in particular, HO HR DF methods have been developed for non-Gaussian sources from the end of the 1980s. Among these methods, extensions of MUSIC to both fourth-order (FO) and $2q$th-order ($q > 1$), called respectively $4-$MUSIC [2] and $2q-$MUSIC [3], are the most popular. These $2q-$MUSIC methods ($q \geq 1$) exploit the information contained in the $2q$th-order circular cumulants of the data. However, for $2q$th-order NC sources such as $M-$PSK sources with $M \leq 2q$, omnipresent in radio-communications, the information contained in the $2q$th-order circular cumulants of the data is not exhaustive and some information is also contained in the $2q$th-order NC cumulants of the data. In such conditions, strong gains in performance may be obtained by taking into account the information contained in all the $2q$th-order cumulants of the data, circular or not, giving rise to $2q$th-order NC DF methods.

Numerous SO NC DF methods, and NC extensions of MUSIC in particular, have been developed these last fifteen years, but under restrictive assumptions about the sources such as the assumptions of rectilinear sources [4][5] or of mixtures of rectilinear and circular sources [6]. Let us recall that a rectilinear source has a real-valued complex envelope to within a constant phase term. Only two papers [7][8] propose NC extensions of MUSIC for arbitrary sources, among which only [7] proposes a generic algorithm able to process all kind of sources with the capability to process up to $2N-2$ rectilinear sources from N antennas. On the other hand, HO NC DF methods are very scarce among which [9] and [10] propose, for rectilinear sources only, a bi-quaternion NC extension of $4-$MUSIC and a NC extension of $2q-$MUSIC ($q \geq 1$) respectively. In [11] the concept of $k-rectilinear$ source has been defined as a source which can be decomposed as the sum of k statistically independent rectilinear sources and the results of [10] have been extended for mixtures of $k-rectilinear$ and $2q$th-order circular sources. More precisely two NC extensions of 2q-MUSIC, called NC1-2q-MUSIC and NC2-2q-MUSIC respectively, have been proposed in [11] for such mixtures although the latter may be used in all contexts. The first one implements a search procedure in both the phase and the direction of the sources. The second one, based on the application of [12] to the first one, limits the search to the direction only but is very costly due to the need to compute a $(q+1) \times (q+1)$ determinant for each point of the pseudo-spectrum.

The purpose of this paper is to give new insights into the full (C + NC) $2q$th-order statistics of the data for $1 \leq q \leq 2$ and for arbitrary sources potentially $2q$th-order NC. These new insights allow to show in particular that the $NC1$-$2q$-MUSIC method developed in [11] for $k-rectilinear$ sources only is in fact powerful for most of the sources (C and NC) encountered in practice.

II. Model and problem formulation

A. Model and extended model

We consider an array of N narrow-band (NB) sensors and we call $\mathbf{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean statistically independent NB sources corrupted by a noise. Under these assumptions, the observation vector can be written as follows

$$\mathbf{x}(t) = \sum_{i=1}^{P} \mathbf{a}(\Theta_i) m_i(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{n}(t)$ is the noise vector, assumed to be zero-mean, spatially white, circular and Gaussian, $\mathbf{a}(\Theta)$ is the steering vector, $m_i(t)$ and Θ_i are the complex envelope and the direction of the source i .

NC DF methods exploit the information contained in the extended observation vector $\tilde{\mathbf{x}}(t) = [\ \mathbf{x}^T(t) \quad \mathbf{x}^H(t)\ ]^T$, where

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T and H denote transposition and conjugation-transposition respectively. From (1) we deduce that $\tilde{\mathbf{x}}(t)$ can be written as

$$\tilde{\mathbf{x}}(t) = \sum_{i=1}^{P} \mathbf{A}(\Theta_i)\,\mathbf{m}_i(t) + \tilde{\mathbf{n}}(t) = \sum_{i=1}^{P} \tilde{\mathbf{A}}(\Theta_i)\,\tilde{\mathbf{m}}_i(t) + \tilde{\mathbf{n}}(t) \tag{2}$$

where $\mathbf{m}_i(t) = [\ \Re(m_i(t)) \quad \Im(m_i(t))\ ]^T$, $\tilde{\mathbf{m}}_i(t) = [\ m_i(t) \quad m_i^*(t)\ ]^T$, $*$ means complex conjugate, $\tilde{\mathbf{n}}(t) = [\ \mathbf{n}^T(t) \quad \mathbf{n}^H(t)\ ]^T$, $\tilde{\mathbf{A}}(\Theta) = [\ \mathbf{a}_1(\Theta) \quad \mathbf{a}_2(\Theta)\ ]$ where $\mathbf{a}_1(\Theta) = [\ \mathbf{a}^T(\Theta) \quad \mathbf{0}_N^T\ ]^T$, $\mathbf{a}_2(\Theta) = [\ \mathbf{0}_N^T \quad \mathbf{a}^H(\Theta)\ ]^T$ and $\mathbf{0}_N$ is the null vector of size N, $\mathbf{A}(\Theta) = [\ \mathbf{a}_3(\Theta) \quad \mathbf{a}_4(\Theta)\ ]$ where $\mathbf{a}_3(\Theta) = [\ \mathbf{a}^T(\Theta) \quad \mathbf{a}^H(\Theta)\ ]^T$ and $\mathbf{a}_4(\Theta) = [\ j\mathbf{a}^T(\Theta) \quad -j\mathbf{a}^H(\Theta)\ ]^T$.

B. Particular case of $k-rectilinear$ sources

If the source i is a $k_i - rectilinear$ source, $m_i(t)$ can be written as [11]

$$m_i(t) = \sum\nolimits_{j=1}^{k_i} v_{ij}(t)\exp(j\Phi_{ij}) \tag{3}$$

where the signals $v_{ij}(t)$ $(1 \le j \le k_i)$ are real-valued and statistically independent whereas Φ_{ij} is a phase term. A rectilinear source (BPSK or ASK source) is a $1-rectilinear$ source, whereas a QPSK source, a square or a rectangular QAM source are three examples of $2-rectilinear$ sources. From (3) we deduce that $\mathbf{m}_i(t)$ and $\tilde{\mathbf{m}}_i(t)$ can be written as

$$\mathbf{m}_i(t) = \sum_{j=1}^{k_i} v_{ij}(t)\,\mathbf{e}(\Phi_{ij})\ ,\quad \tilde{\mathbf{m}}_i(t) = \sum_{j=1}^{k_i} v_{ij}(t)\,\tilde{\mathbf{e}}(\Phi_{ij}) \tag{4}$$

$$\mathbf{e}(\Phi) = \begin{bmatrix} \cos(\Phi) \\ \sin(\Phi) \end{bmatrix} \quad \tilde{\mathbf{e}}(\Phi) = \begin{bmatrix} \exp(j\Phi) \\ \exp(-j\Phi) \end{bmatrix} \tag{5}$$

Inserting (4) into (2), it is straightforward to show that $\tilde{\mathbf{x}}(t)$ takes the form

$$\tilde{\mathbf{x}}(t) = \sum\nolimits_{i=1}^{P}\sum\nolimits_{j=1}^{k_i} \tilde{\mathbf{b}}(\Theta_i, \Phi_{ij})\, v_{ij}(t) + \tilde{\mathbf{n}}(t) \tag{6}$$

$$\tilde{\mathbf{b}}(\Theta, \Phi) = \tilde{\mathbf{A}}(\Theta)\,\tilde{\mathbf{e}}(\Phi) = \mathbf{A}(\Theta)\,\mathbf{e}(\Phi) \tag{7}$$

where $\tilde{\mathbf{b}}(\Theta, \Phi)$ is the generic extended steering vector of a $rectilinear$ source. This shows that without noise, $\tilde{\mathbf{x}}(t)$ is spanned by the extended steering vectors $\tilde{\mathbf{b}}(\Theta_i, \Phi_{ij})$ $(1 \le j \le k_i)$ $(1 \le i \le P)$ and this allows straightforward NC extensions of $2q-$MUSIC methods $(q \ge 1)$ from the generic extended steering vector $\tilde{\mathbf{b}}(\Theta, \Phi)$ as done in [11].

C. Problem formulation

The first purpose of this paper is to show that whatever the kind and the non-circularity properties of the sources, the signal subspace of the SO statistical matrix of $\tilde{\mathbf{x}}(t)$ has the same algebraic structure as the one obtained for k-rectilinear sources. The second purpose of this paper is to show that this result remains valid at the FO for most of the sources of practical interest. These results, completely unknown by the scientific community, allow to use, whatever the kind of sources $(q = 1)$ and for most of sources of practical interest $(q = 2)$, NC extensions of $2q$-MUSIC $(1 \le q \le 2)$ initially developed for k-rectilinear sources such as those presented in [11].

III. NON-CIRCULAR SECOND-ORDER DF METHODS

A. Extended Second-Order Statistics

Most of SO NC DF methods exploit the information contained in the time-averaged correlation matrix of $\tilde{\mathbf{x}}(t)$, defined by $\mathbf{R}_{\tilde{x}} = \left\langle \mathbb{E}\left[\tilde{\mathbf{x}}(t)\,\tilde{\mathbf{x}}^H(t)\right]\right\rangle$, where $\langle . \rangle$ is the time averaging operation on a given observation window and $\mathbb{E}[.]$ is the expected value operation. Under the assumptions of section II-A, we deduce from (2) that $\mathbf{R}_{\tilde{x}}$ can be written as

$$\mathbf{R}_{\tilde{x}} = \sum\nolimits_{i=1}^{P} \mathbf{A}(\Theta_i)\,\mathbf{R}_{m_i}\mathbf{A}^H(\Theta_i) + \sigma^2\mathbf{I}_N \tag{8}$$

where σ^2 is the noise power per sensor and $\mathbf{R}_{m_i}$ is the time-averaged correlation matrix of $\mathbf{m}_i(t)$. As $\mathbf{R}_{m_i}$ is a real-valued (2×2) symmetric matrix, its eigen decomposition can be written as

$$\mathbf{R}_{m_i} = \sum\nolimits_{k=1}^{2} \mu_{ik}\,\mathbf{e}(\Phi_{ik})\,\mathbf{e}^T(\Phi_{ik}) \tag{9}$$

where the orthonormal eigenvectors $\mathbf{e}(\Phi_{ik})$ $(1 \le k \le 2)$, such that $\mathbf{e}^T(\Phi_{ik})\,\mathbf{e}(\Phi_{ik'}) = \delta(k-k')$, are associated with the real eigenvalues μ_{ik} where $\delta(.)$ is the Kronecker symbol. For this reason, $\Phi_{i2} = \Phi_{i1} \pm \pi/2$. Using (9) into (8), we obtain

$$\mathbf{R}_{\tilde{x}} = \sum\nolimits_{i=1}^{P}\sum\nolimits_{k=1}^{2} \mu_{ik}\,\tilde{\mathbf{b}}(\Theta_i, \Phi_{ik})\,\tilde{\mathbf{b}}^H(\Theta_i, \Phi_{ik}) + \sigma^2\mathbf{I}_N \tag{10}$$

B. Non-circular second order MUSIC methods

We deduce from (10) that the signal space of $\mathbf{R}_{\tilde{x}}$ is spanned by the vectors $\tilde{\mathbf{b}}(\Theta_i, \Phi_{ik})$ associated with the non-zero μ_{ik} $(1 \le i \le P)$ $(1 \le k \le 2)$. For each i at least one value of μ_{ik} is not zero and thus at least one $\tilde{\mathbf{b}}(\Theta_i, \Phi_{ik})$ is in the signal subspace of $\mathbf{R}_{\tilde{x}}$. According to [11][1], the directions Θ_i $(1 \le i \le P)$ can then be estimated by searching for the zeros, over (Θ, Φ), of the NC1-MUSIC criterion

$$J_{1,2}(\Theta, \Phi) = \left(\tilde{\mathbf{b}}^H(\Theta, \Phi)\,\mathbf{\Pi}_2\tilde{\mathbf{b}}(\Theta, \Phi)\right) / \left\|\tilde{\mathbf{b}}(\Theta, \Phi)\right\|^2$$

where $\|\mathbf{u}\|^2 = \mathbf{u}^H\mathbf{u}$ and $\mathbf{\Pi}_2$ is the orthogonal projector on the noise subspace of $\mathbf{R}_{\tilde{x}}$. Thus the NC1-MUSIC algorithm developed for rectilinear or $k-rectilinear$ sources [11] can also be used for arbitrary SO NC sources and is able to process up to $2N-1$ rectilinear sources from a 2D search process with respect to (Θ, Φ) when Θ is a scalar, i.e for 1D DF estimation.

IV. NON-CIRCULAR FOURTH-ORDER DF METHODS

A. Extended Fourth-Order Statistics

FO NC DF methods exploit the information contained in the time-averaged circular FO cumulants of $\tilde{\mathbf{x}}(t)$, defined by $c_{\tilde{\mathbf{x}},ijkl} =< cum(\tilde{x}_i(t), \tilde{x}_j(t), \tilde{x}_k^*(t), \tilde{x}_l^*(t)) >$ for $1 \le i,j,k,l \le 2N$, where $\tilde{x}_i(t)$ is the component i of $\tilde{\mathbf{x}}(t)$. These latter entries can be arranged in the $(2N)^2 \times (2N)^2$ matrix $\mathbf{C}_{\tilde{\mathbf{x}}}$ in different ways as done in [3] or [11] but it is easy to verify [11] that all these arrangements are equivalent. We then choose the natural arrangement defined by $\mathbf{C}_{\tilde{\mathbf{x}}}(I, J) = c_{\tilde{\mathbf{x}},ijkl}$ with $I = 2N(i-1)+j$ and $J = 2N(k-1)+l$. Under the assumptions of section II-A, we deduce from (2) that $\mathbf{C}_{\tilde{\mathbf{x}}}$ can be written as

$$\mathbf{C}_{\tilde{\mathbf{x}}} = \sum_{i=1}^{P} \tilde{\mathbf{A}}^{\otimes 2}(\Theta_i)\, \mathbf{C}_{\tilde{\mathbf{m}}_i} \tilde{\mathbf{A}}^{\otimes 2H}(\Theta_i) \qquad (11)$$

where $\tilde{\mathbf{A}}^{\otimes 2}(\Theta) = \tilde{\mathbf{A}}(\Theta) \otimes \tilde{\mathbf{A}}(\Theta)$, $\otimes$ is the kronecker product and $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ is the (4×4) matrix of the time-averaged circular FO cumulants of $\tilde{\mathbf{m}}_i(t)$. Denoting by β_i and γ_i the parameters: $\beta_i = <cum(m_i(t), m_i(t), m_i(t), m_i^*(t))>/c_i$, $\gamma_i = <cum(m_i(t), m_i(t), m_i(t), m_i(t))>/c_i$ and $c_i = c_{\tilde{\mathbf{m}}_i,1111}$, the matrix $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ can be written as $\mathbf{C}_{\tilde{\mathbf{m}}_i} = c_i \mathbf{C}_i$ where $\mathbf{C}_i$ is defined by

$$\mathbf{C}_i = \begin{bmatrix} 1 & \beta_i & \beta_i & \gamma_i \\ \beta_i^* & 1 & 1 & \beta_i \\ \beta_i^* & 1 & 1 & \beta_i \\ \gamma_i^* & \beta_i^* & \beta_i^* & 1 \end{bmatrix} = \mathbf{\Gamma} \bar{\mathbf{C}}_i \mathbf{\Gamma}^H \qquad (12)$$

where $\mathbf{\Gamma}$ and $\bar{\mathbf{C}}_i$ are defined by

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and } \bar{\mathbf{C}}_i = \begin{bmatrix} 1 & \beta_i & \gamma_i \\ \beta_i^* & 1 & \beta_i \\ \gamma_i^* & \beta_i^* & 1 \end{bmatrix} \qquad (13)$$

It becomes obvious from (13) that the rank of $\mathbf{C}_i$ is at most equal to 3 and the eigen-decomposition of $\mathbf{C}_{\tilde{\mathbf{m}}_i}$, Hermitian matrix, can be written as

$$\mathbf{C}_{\tilde{\mathbf{m}}_i} = c_i \sum_{j=1}^{3} \mu_{ij} \mathbf{u}_{ij} \mathbf{u}_{ij}^H \qquad (14)$$

where the μ_{ij}'s $(1 \le j \le 3)$ are the three real eigenvalues of $\mathbf{C}_i$ with the greatest modulus, whereas the $\mathbf{u}_{ij}$'s are the associated orthonormal eigenvectors. Depending on the source i, one, two or three of the μ_{ij}'s may not be zero. We define a $r_i - rank$ source i $(1 \le r_i \le 3)$, a source i for which only r_i eigenvalues μ_{ij}'s $(1 \le j \le r_i)$ are not zero.

The purpose of what follows is to show that the space spanned by the vectors $\mathbf{u}_{ij} (1 \le j \le r_i)$ is also spanned by vectors of the form $\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{ij})$ $(1 \le j \le r_i)$. In other words, we will show that $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ can also be written as

$$\mathbf{C}_{\tilde{\mathbf{m}}_i} = \sum_{j=1}^{r_i} \sum_{l=1}^{r_i} \tilde{\mathbf{e}}^{\otimes 2}(\Psi_{ij})\, \tilde{\mathbf{e}}^{\otimes 2H}(\Psi_{il})\, q_{jl}^i \qquad (15)$$

where the q_{jl}^i's and Ψ_{il}'s $(1 \le j, l \le r_i)$ are scalar quantities.

B. Algebraic structure of $\mathbf{C}_{\tilde{\mathbf{m}}_i}$

We analyse in this section the algebraic structure of $\mathbf{C}_i = \mathbf{C}_{\tilde{\mathbf{m}}_i}/c_i$ for each possible value of its rank r_i, i.e. for $1 \le r_i \le 3$.

1) Rank one source i: The matrix $\mathbf{C}_i$ (12) has a rank equal to 1 if and only if the determinants of all the (2×2) sub-matrices of $\mathbf{C}_i$ are equal to zero. This is obtained if **C1**, defined by (16), is verified

$$\mathbf{C1}: \quad |\beta_i| = 1 \text{ and } \gamma_i = \beta_i^2 \qquad (16)$$

In this case, $\exists \Psi_{i1}$ such that $\beta_i = \exp(2j\Psi_{i1})$ and $\gamma_i = \exp(4j\Psi_{i1})$. It is then easy to verify that in this case $\mathbf{C}_{\tilde{\mathbf{m}}_i} = c_i \mathbf{C}_i$ takes the form

$$\mathbf{C}_{\tilde{\mathbf{m}}_i} = c_i\, \tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i1})\, \tilde{\mathbf{e}}^{\otimes 2H}(\Psi_{i1})$$

which is a particular case of (15) with $q_{11}^i = c_i$ and $r_i = 1$.

2) Rank 2 source i: The matrix $\mathbf{C}_i$ (12) has a rank equal to 2 if and only if the determinant of $\bar{\mathbf{C}}_i$ (13) is equal to zero while **C1** is not verified. After some elementaries computations, the rank 2 condition is obtained if **C2**, defined by (17), is verified

$$\mathbf{C2}: \exists \Omega_i / \ \gamma_i = (\beta_i)^2 + e^{j\Omega_i}\left(|\beta_i|^2 - 1\right) \text{ and } |\beta_i| \neq 1 \qquad (17)$$

Condition **C2** implies that the matrix $\mathbf{Q}_i$ composed of the two first columns of $\mathbf{C}_i$ has a rank equal to 2. Let us analyze the conditions under which there exist Ψ_{i1} and Ψ_{i2} such that $Span(\mathbf{Q}_i) = Span(\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i1}), \tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i2}))$. This last property is verified if and only if $\exists (\Psi_{i1}, \Psi_{i2}) \in \mathbb{R}^2$ and $\exists$ $\mathbf{T}$, a full rank (2×2) matrix, such that

$$\mathbf{Q}_i = \begin{bmatrix} 1 & \beta_i \\ \beta_i^* & 1 \\ \beta_i^* & 1 \\ \gamma_i^* & \beta_i^* \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^i \\ \mathbf{Q}_2^i \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{12}^i \\ \mathbf{E}_{12}^i \Omega_{12}^i \end{bmatrix} \mathbf{T} \qquad (18)$$

where the (2×2) matrices $\mathbf{Q}_j^i$, Ω_{12}^i and $\mathbf{E}_{12}^i$ are defined by

$$\mathbf{Q}_1^i = \begin{bmatrix} 1 & \beta_i \\ \beta_i^* & 1 \end{bmatrix}; \quad \mathbf{Q}_2^i = \begin{bmatrix} \beta_i^* & 1 \\ \gamma_i^* & \beta_i^* \end{bmatrix} \qquad (19)$$

$$\mathbf{E}_{12}^i = \begin{bmatrix} e^{j2\Psi_{i1}} & e^{j2\Psi_{i2}} \\ 1 & 1 \end{bmatrix}; \quad \Omega_{12}^i = \begin{bmatrix} e^{-j2\Psi_{i1}} & 0 \\ 0 & e^{-j2\Psi_{i2}} \end{bmatrix}$$

After straightforward manipulations, it is easy to verify that property (18) is equivalent to $\mathbf{Q}_1^i = \mathbf{E}_{12}^i \mathbf{T}$ and $\mathbf{T}(\mathbf{Q}_1^i)^{-1}$ $\mathbf{Q}_2^i \mathbf{T}^{-1} = \Omega_{12}^i$ which requires that $e^{-j2\Psi_{i1}}$ and $e^{-j2\Psi_{i2}}$ are eigenvalues of $(\mathbf{Q}_1^i)^{-1}\, \mathbf{Q}_2^i$ and that the associated eigenvectors correspond to the columns of $\mathbf{T}^{-1}$. From (17) and (19), we obtain, after some elementary computations

$$\left(\mathbf{Q}_1^i\right)^{-1} \mathbf{Q}_2^i = \begin{bmatrix} \alpha_i & 1 \\ -e^{-j\Omega_i} & 0 \end{bmatrix} \qquad (20)$$

$$\text{where } \alpha_i = 2\Re\left(\beta_i e^{-j\Omega_i/2}\right) e^{-j\Omega_i/2} \qquad (21)$$

We then deduce that the eigenvalues, λ_{ik} $(1 \le k \le 2)$, of (20) are given by

$$\lambda_{ik} = e^{-j\Omega_i/2}\left(\Re\left(\beta_i e^{-j\Omega_i/2}\right) \pm \sqrt{\Re\left(\beta_i e^{-j\Omega_i/2}\right)^2 - 1}\right) \qquad (22)$$

It is then easy to verify that $|\lambda_{ik}|^2 = 1$ for $(1 \le k \le 2)$ and $\lambda_{i1} \neq \lambda_{i2}$ if condition **C3**, defined hereafter, is verified

$$\mathbf{C3}: \quad \left|\Re\left(\beta_i e^{-j\Omega_i/2}\right)\right| \le 1 \qquad (23)$$

In this case, it exist Ψ_{i1} and Ψ_{i2} such that $\lambda_{i1} = e^{-j2\Psi_{i1}}$ and $\lambda_{i2} = e^{-j2\Psi_{i2}}$. Moreover, it is also easy to verify that $\mathbf{Q}_1^i = \mathbf{E}_{12}^i \mathbf{T}$, which means that $\mathbf{T}(\mathbf{Q}_1^i)^{-1}\, \mathbf{Q}_2^i \mathbf{T}^{-1} = \Omega_{12}^i$ is verified and that $Span(\mathbf{Q}_i) = Span(\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i1}), \tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i2}))$. Matrix $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ then takes the form (15). However if condition **C4** is verified

$$\mathbf{C4}: \quad \left|\Re\left(\beta_i e^{-j\Omega_i/2}\right)\right| > 1 \qquad (24)$$

the previous results do no longer hold, it does not exist Ψ_{i1} and Ψ_{i2} such that $Span(\mathbf{Q}_i) = Span(\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i1}), \tilde{\mathbf{e}}^{\otimes 2}(\Psi_{i2}))$ and $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ has no longer the form (15). However, most of rank 2 sources encountered in practice, such as square QAM sources, verify (23) and not (24).

3) Rank 3 source i: In this case,

$$\mathbf{C5}: \quad \left|\gamma_i - (\beta_i)^2\right| \neq \left||\beta_i|^2 - 1\right| \tag{25}$$

the space spanned by the $\mathbf{u}_{ij}$'s ($1 \leq j \leq 3$) of (14) corresponds to the one spanned by the columns of $\mathbf{\Gamma}$, denoted by $Span(\mathbf{\Gamma})$. It is well-known that the orthogonal projector on the subspace orthogonal to the columns of $\mathbf{\Gamma}$ is defined by $\Pi_{\mathbf{\Gamma}}^{\perp} = \mathbf{I} - \mathbf{\Gamma}\left(\mathbf{\Gamma}^H\mathbf{\Gamma}\right)^{-1}\mathbf{\Gamma}^H$ where it is easy to verify that $\mathbf{\Gamma}^H\mathbf{\Gamma} = diag\left([1\ 2\ 1]\right)$. Then a vector $\mathbf{v}$ belongs to $Span(\mathbf{\Gamma})$ if and only if $\Pi_{\mathbf{\Gamma}}^{\perp}\mathbf{v} = 0$. It is then straightforward to verify that, whatever the value of Ψ, $\Pi_{\mathbf{\Gamma}}^{\perp}\tilde{\mathbf{e}}^{\otimes 2}(\Psi) = 0$, which means that all vectors $\tilde{\mathbf{e}}^{\otimes 2}(\Psi)$ for arbitrary values of Ψ belong to $Span(\mathbf{\Gamma})$. Moreover, it is easy to built three non-colinear vectors $\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{ij})$ ($1 \leq j \leq 3$) where $\Psi_{i1} \neq \Psi_{i2} \neq \Psi_{i3}$, which shows that there exists $\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{ij})$ ($1 \leq j \leq 3$) such that $Span(\mathbf{\Gamma}) = Span(\tilde{\mathbf{e}}^{\otimes 2}(\Psi_{ij}), 1 \leq j \leq 3)$ and then such that (15) holds.

C. Non-circular Fourth order MUSIC methods

It has been shown in section IV-B that in the presence of a mixture of P sources i ($1 \leq i \leq P$) with either rank 1, rank 2 verifying (23) or rank 3, $\mathbf{C}_{\tilde{\mathbf{m}}_i}$ has, for each i, the form (15). Inserting (15) into (11) and using (7) we finally obtain

$$\mathbf{C}_{\tilde{\mathbf{x}}} = \sum_{i=1}^{P}\sum_{j,l=1}^{r_i} q_{jl}^{i}\tilde{\mathbf{b}}^{\otimes 2}(\Theta_i, \Psi_{ij})\,\tilde{\mathbf{b}}^{\otimes 2H}(\Theta_i, \Psi_{il}) \tag{26}$$

We deduce from (26) that the signal space of $\mathbf{C}_{\tilde{\mathbf{x}}}$ is spanned by the vectors $\tilde{\mathbf{b}}^{\otimes 2}(\Theta_i, \Psi_{ij})$ for ($1 \leq i \leq P$) ($1 \leq j \leq r_i$). For each i at least one vector $\tilde{\mathbf{b}}^{\otimes 2}(\Theta_i, \Psi_{ij})$ is in the signal subspace of $\mathbf{C}_{\tilde{\mathbf{x}}}$ of rank $r = \sum_{i=1}^{P} r_i$. The directions Θ_i ($1 \leq i \leq P$) can then be estimated by searching for the zeros or the minima, over (Θ, Ψ), of the $NC1-4$–MUSIC [11] criterion.

$$J_{1,4}(\Theta, \Phi) = \left(\tilde{\mathbf{b}}^{\otimes 2H}(\Theta, \Phi)\,\mathbf{\Pi}_4\tilde{\mathbf{b}}^{\otimes 2}(\Theta, \Phi)\right) / \left\|\tilde{\mathbf{b}}^{\otimes 2}(\Theta, \Phi)\right\|^2$$

where $\mathbf{\Pi}_4$ is the orthogonal projector on the noise subspace of $\mathbf{C}_{\tilde{\mathbf{x}}}$. For rank 2 sources verifying (24), the $NC2-4$-MUSIC method presented in [11] must be used.

V. Computer Simulations

To illustrate the performance of the $NC1-2q$-MUSIC method for $q = 1, 2$, we consider a mixture of $P = 2$ statistically independent sources, having the same SNR equal to $10dB$, impinging on a uniform circular array of $N = 3$ antennas of radius $\lambda/2$, where λ is the wavelength. The first source is a $2-rectilinear$ source ($rank$ 2) whereas the second one is an ASK source ($rank$ 1). The angles of arrival of the 2 sources are $\Theta_1 = 100°$ and $\Theta_2 = 110°$, whereas their phase are $\Psi_{11} = 10°$, $\Psi_{12} = 80°$ and $\Psi_{21} = 45°$. Under these assumptions, Fig.1 shows the variations of the Root Mean Square Error (RMSE) of the direction estimate of the source 1 as a function of the number of snapshots L used to estimate the statistics for both $2q-$MUSIC and $NC1-2q-$MUSIC with $q = 1$ and 2. Note the best performance of $NC1-2q-$MUSIC with respect to $2q-$MUSIC for both $q = 1$ and 2 and the better performance of HO methods since the sources are poorly angularly separated.

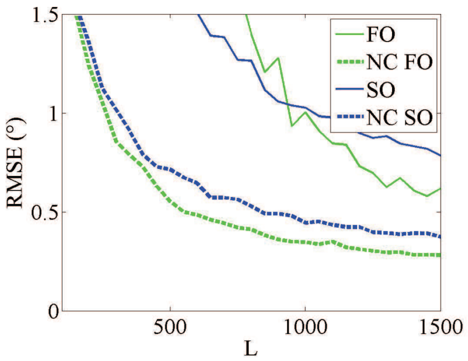

Fig. 1. RMSE of the source 1 as a function of L for $2q-$MUSIC and $NC1-2q-$MUSIC with $q = 1, 2$.

VI. Conclusion

It has been shown in this paper that the $NC1-2q$-MUSIC algorithm developed in [11] for $k-rectilinear$ sources is also powerful for arbitrary NC sources for $q = 1$ and for rank 1, rank 3 and most of rank 2 NC sources encountered in practice for $q = 2$.

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