# On Coding Efficiency for Flash Memories 

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#### Abstract

Recently, flash memories have become a competitive solution for mass storage. The flash memories have rather different properties compared with the rotary hard drives. That is, the writing of flash memories is constrained, and flash memories can endure only limited numbers of erases. Therefore, the design goals for the flash memory systems are quite different from these for other memory systems. In this paper, we consider the problem of coding efficiency. We define the "coding-efficiency" as the amount of information that one flash memory cell can be used to record per cost. Because each flash memory cell can endure a roughly fixed number of erases, the cost of data recording can be well-defined. We define "payload" as the amount of information that one flash memory cell can represent at a particular moment. By using information-theoretic arguments, we prove a coding theorem for achievable coding rates. We prove an upper and lower bound for coding efficiency. We show in this paper that there exists a fundamental trade-off between "payload" and "coding efficiency". The results in this paper may provide useful insights on the design of future flash memory systems.


## I. Introduction

Recently, flash memories have become a competitive solution for mass storage. Compared with the conventional rotary hard drives, flash memories have high random access read speed, because there is no mechanical seek time. Flash memory storage devices are also more lightweight, power efficient, and kinetic shock resistant. Therefore, they are becoming desirable choices for many applications ranging from highspeed servers in data centers to portable devices.

Flash memories are one type of solid state memories. Each piece of flash memory usually contains multiple arrays of flash memory cells. Each memory cell is a transistor with a floating gate. Information is recorded using one memory cell by injecting and removing electrons into and from the floating gate. The process of injecting electrons is called programming and the process of removing electrons is called erase. Programming increases the threshold voltage level of the memory cell, while erase decreases the threshold voltage level. The threshold voltage level of the memory cell is the voltage level at the control gate that the transistor becomes conducting. In the reading process for the memory cell, the threshold voltage level is detected, thus the recorded information can be recovered.

The memory cells are organized into pages and then into blocks. The programming is page-wise and erase is blockwise. Usually, one memory block is first erased, so that all memory cells within the block return to an initial threshold voltage level. After the erase operation, the pages in the block are programmed (possibly multiple times), until normal
threshold voltage level ranges are used up. Then, the memory block is erased again for further use.
One challenge for flash memories is that the number of erase operations that one memory cell can withstand is quite limited. For current commercial flash memories, such maximal numbers of block erase operations range from 5,000 to 100,000 . After such a limited number of erase operations, the flash memory cell would become broken or unreliable. Therefore, data encoding methods must be carefully designed to address such an issue.

In fact, flash memories can be considered as one type of write-once-memories. The write-once-memories were first discussed in the seminal work by Rivest and Shamir [1]. Previous examples of write-once-memories include digital optical disks, punched paper tapes, punched cords, and programmable readonly memories etc. Rivest and Shamir show that by using advanced data encoding methods, the write-once-memories can be rewritten. In [1], one theorem for the achievable data recording rates of binary write-once-memories has been proven using combinatorial arguments. During the passed research, many data encoding methods for rewriting the write-once-memories have been proposed, see for example, [2] [3] etc.

In this paper, we consider a coding efficiency problem for data encoding on flash memories. Unlike other type of computer memories, the cost of data encoding can be welldefined for flash memories. That is, the cost for each erase operation can be defined based on the cost of the flash memory block and the total number of erase operations that the memory block can have. The coding efficiency problem is therefore the problem of recording more information using fewer erase operations. To our best knowledge, such a design problem for flash memories has never been discussed before.

We assume that one flash memory block has $N$ cells, and each cell can take $K$ voltage levels. We assume that the data encoding scheme uses the memory block for $T$ rounds between two consecutive erase operations. That is, in the first round, a message $M[1]$ is recorded using the block, and in the second round, a new message $M[2]$ is recorded, and so on. Suppose that $N l_{t}$ bits are recorded during the $t$-th round. We define the payload $p$ and coding efficiency $c$ as

$$
\begin{equation*}
p=\frac{1}{T} \sum_{t=1}^{T} l_{t}, \quad c=\frac{\alpha}{K} \sum_{t=1}^{T} l_{t} \tag{1}
\end{equation*}
$$

where, $\alpha$ is a constant depending on the type of the memory block, e.g., NOR type, NAND type, single-level-cell, multi-level-cell etc. The constant $\alpha$ may be used to reflect the cost
for the flash memory block. It should be clear that the coding efficiency measures the amount of recording information per voltage level cost. We may also define the voltage level cost per recorded bit, which is exactly $1 / c$.

In this paper, we first prove a coding theorem for achievable rates of data encoding on flash memories using informationtheoretic arguments. Using the coding theorem in this paper, we prove an upper bound for the optimal coding efficiency. We also show a lower bound of optimal coding efficiency using a specific coding scheme. Surprisingly, we find that there exists a tradeoff between the optimal coding efficiency and payload. These results may provide useful insights and tools for designing future flash memory storage systems.

The rest of this paper is organized as follows. In Section $\Pi$, we present the coding theorem for achievable coding rates. In Section III, we show the upper bound of the optimal coding efficiency. In Section IV we present the lower bound for optimal coding efficiency using a specific coding scheme. The coding efficiency to payload tradeoff is discussed in Section V. Some concluding remarks are presented at Section VI.

## II. Coding Theorem

We consider a memory block with $N$ memory cells. Each memory cell can take $K$ threshold voltage levels, that is, each memory cell can be at one of the states $0,1, \ldots, K-1$. After one erase operation, all memory cells are at the state $K-1$. During each programming process, the state of each cell can be decreased but never increased. Assume that the memory block can be reliably used for $T$ rounds of information recording, where messages $M(1), M(2), \ldots, M(t), \ldots, M(T)$ are recorded. We define the corresponding data rate in the $t$ th round $l(t)=\log _{2}(|M(t)|) / N$, where $|M(t)|$ denote the alphabet size of the message $M(t)$. In this case, we say that the sequence of data rates $l(t), t=1, \ldots, T$ is achievable. We assume that all the $T$ messages are statistical independent. We denote the state of the $n$-th cell in the block during time $t$ by $X_{n}(t)$. We use the notation $X_{1}^{N}(t)$ to denote the sequence $X_{1}(t), X_{2}(t), \ldots, X_{N}(t)$. Similarly, $X_{1}^{n}(t)$ denotes the sequence $X_{1}(t), X_{2}(t), \ldots, X_{n}(t)$, where $1 \leq n \leq N$. We use $H(\cdot)$ to denote the entropy and conditional entropy functions as in [4].

Theorem 2.1: A sequence of data rates $l(t), t=1, \ldots, T$ is achievable, if and only if, there exist random variables $U(1), \ldots, U(T)$ jointly distributed with a probability distribution $\mathbb{P}(U(1), \ldots, U(T))$, such that,

$$
\begin{align*}
& \mathbb{P}(U(t)=j \mid U(t-1)=i)=0, \text { if } j>i, \text { for } t=2, \ldots, T \\
& l(t) \leq H(U(t) \mid U(t-1)), \text { for } t=2, \ldots, T \\
& l(1) \leq H(U(1)) \tag{2}
\end{align*}
$$

By convention, $U(0)=K-1$ with probability 1 .
Proof: The achievable part is proven by random binning. For the $t$-th round of data recording, we construct a random code by throwing typical sequences of $U(t)$ into $\exp \{N l(t)\}$ bins uniformly in random. The message $m(t)$ is encoded by finding a sequence $X_{1}^{N}(t)$ in the $m(t)$-th bin, such that the
sequence $X_{1}^{N}(t)$ is jointly typical with $X_{1}^{N}(t-1)$. If such a sequence can not be found, then one encoding error is declared.

Suppose that $l(t) \leq H(U(t) \mid U(t-1))-2 \epsilon$, where $\epsilon$ is an arbitrarily small positive number. Then, the probability of encoding error can be upper bounded as follows.

$$
\begin{align*}
\mathbb{P}(\text { error }) & =\left(1-\frac{1}{\exp (N l(t))}\right)^{N_{1}} \\
& \stackrel{(a)}{\leq} \exp \left(-\frac{N_{1}}{\exp (N l(t))}\right) \\
& \stackrel{(b)}{\leq} \exp \left(-\frac{\exp (N(H(U(t) \mid U(t-1))-\epsilon))}{\exp \{N(H(U(t) \mid U(t-1))-2 \epsilon)\}}\right) \\
& \leq \exp (-\exp (\epsilon N)) \tag{3}
\end{align*}
$$

where, $N_{1}$ denotes the number of typical sequences $X_{1}^{N}(t)$ that are jointly typical with $X_{1}^{N}(t-1)$, (a) follows from the inequality, $(1-x) \leq \exp (-x)$, for $0 \leq x<1$, (b) follows from the fact that $\bar{N}_{1} \geq \exp \{N(H(\bar{U}(t) \mid U(t-1))-\epsilon)\}$. The achievable part of the proof then follows from the fact that $\epsilon$ can be taken arbitrarily small.

We prove the converse part by constructing some random variables $U(1), \ldots, U(T)$, which satisfy the conditions in the theorem. Assume that there exists at least one coding scheme, which satisfies the conditions in the theorem.

In the first step, we wish to show

$$
\begin{equation*}
H(M(t)) \leq H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \tag{4}
\end{equation*}
$$

This is because, on the one hand,

$$
\begin{align*}
& H\left(M(t), X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \\
& =H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right)+H\left(M(t) \mid X_{1}^{N}(t), X_{1}^{N}(t-1)\right) \\
& \stackrel{(a)}{=} H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \tag{5}
\end{align*}
$$

where, (a) follows from the fact that $M(t)$ can be completely determined by observing $X_{1}^{N}(t)$. On the other hand,

$$
\begin{align*}
& H\left(M(t), X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \\
& =H\left(M(t) \mid X_{1}^{N}(t-1)\right)+H\left(X_{1}^{N}(t) \mid M(t), X_{1}^{N}(t-1)\right) \\
& \stackrel{(a)}{=} H(M(t))+H\left(X_{1}^{N}(t) \mid M(t), X_{1}^{N}(t-1)\right) \tag{6}
\end{align*}
$$

where, (a) follows from the fact that $M(t)$ is independent of $X_{1}^{N}(t-1)$.

In the second step, we can show that

$$
\begin{equation*}
H(M(t)) \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \tag{7}
\end{equation*}
$$

This is because,

$$
\begin{align*}
H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) & =\sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{1}^{n-1}(t), X_{1}^{N}(t-1)\right) \\
& \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \tag{8}
\end{align*}
$$

where the last inequality follows from the fact that conditions do not increase entropy.

Let us define random variables $Z, U(1), U(2), \ldots, U(T)$ as follows. The random variable $Z$ takes values in $\{1,2, \ldots, N\}$ uniformly in random.

$$
\begin{equation*}
U(t)=X_{n}(t), \text { if } Z=n \tag{9}
\end{equation*}
$$

The probability distribution of the random variables $Z, U(1), U(2),, \ldots, U(T)$ can be factored as follows.

$$
\begin{equation*}
\mathbb{P}(Z) \prod_{t=1}^{T} \mathbb{P}(U(t) \mid U(1), \ldots, U(t-1), Z) \tag{10}
\end{equation*}
$$

It can be checked that

$$
\begin{equation*}
\mathbb{P}(U(t)=j \mid U(t-1)=i)=0, \text { if } j>i \tag{11}
\end{equation*}
$$

Finally, we wish to show that

$$
\begin{equation*}
N l(t)=H(M(t)) \leq N H(U(t) \mid U(t-1)) \tag{12}
\end{equation*}
$$

This is because

$$
\begin{align*}
& H(M(t)) \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \\
& \stackrel{(a)}{=} N H(U(t) \mid U(t-1), Z) \\
& \stackrel{(b)}{\leq} N H(U(t) \mid U(t-1)) \tag{13}
\end{align*}
$$

where, (a) follows from the definition of $Z$, (b) follows from the fact that conditions do not increase entropy.

Therefore, we have constructed the random variables $U(1), \ldots, U(T)$, which satisfy the conditions in the theorem. The theorem is proven.

## III. Upper Bound

In this section, we prove an upper bound for the achievable coding efficiency. It is clear that the coding efficiency can be calculated based on the Theorem 2.1 by forming an optimization problem. Let us define a random variable $V(t)=$ $U(t-1)-U(t)$ with an alphabet $\{0,1, \ldots, K-1\}$. With a given payload $p$, the optimization problem is as follows.

$$
\begin{gather*}
\min _{\mathbb{P}(V(1), \ldots, V(t), \ldots V(T))} \mathbb{E}\left[\sum_{t} V(t)\right]  \tag{14}\\
\text { Subject to: } \sum_{t} H(V(t) \mid U(t-1)) \geq T p  \tag{15}\\
\mathbb{P}\left(\sum_{t} V(t) \geq K\right)=0 \tag{16}
\end{gather*}
$$

By convention, $U(0)=K-1$ with probability 1 . It should be clear that the coding efficiency

$$
\begin{equation*}
c \leq \frac{\alpha T p}{\sum_{t} \mathbb{E}\left(V(t)^{*}\right)} \tag{17}
\end{equation*}
$$

where $V(t)^{*}$ denotes the minimizer of the optimization problem.

However, the above optimization problem is difficult to solve in closed-form. We will consider instead a relaxed optimization problem. First, we remove the constraint in Eqn 16. Second, we relax the constraint $\sum_{t} H(V(t) \mid U(t-1)) \geq$ $T p$ to $\sum_{t} H(V(t)) \geq T p$, due to the fact that conditions do not increase entropy. Thus, the original optimization problem becomes

$$
\begin{align*}
& \min _{\mathbb{P}(V(1), \ldots, V(t), \ldots V(T))} \mathbb{E}\left[\sum_{t} V(t)\right] \\
& \text { Subject to: } \sum_{t} H(V(t)) \geq T p \tag{18}
\end{align*}
$$

In a final step, because all the constraint and objective functions only depend on marginal distributions of $V(t)$, we may further relax the above optimization problem by replacing the joint distribution

$$
\begin{equation*}
\mathbb{P}(V(1), \ldots, V(t), \ldots, V(T)) \tag{19}
\end{equation*}
$$

with a set of pseudo marginal distributions

$$
\begin{equation*}
\mathbb{P}(V(1)), \ldots, \mathbb{P}(V(t)), \ldots, \mathbb{P}(V(T)) \tag{20}
\end{equation*}
$$

The pseudo marginal distributions may or may not correspond to a joint distribution. The final relax optimization problem is thus as follows.

$$
\begin{align*}
& \min _{\mathbb{P}(V(1)), \ldots, \mathbb{P}(V(t)), \ldots, \mathbb{P}(V(T)))} \mathbb{E}\left[\sum_{t} V(t)\right] \\
& \text { Subject to: } \sum_{t} H(V(t)) \geq T p \tag{21}
\end{align*}
$$

Using the Lagrangian method, we can find that the optimal distribution for $V(t)$ takes the following form

$$
\begin{equation*}
\mathbb{P}(V(t)=j)=\frac{\exp \left(-\beta_{t} j\right)}{\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)} \tag{22}
\end{equation*}
$$

for a certain parameter $\beta_{t}>0$. Let us define the cost function $\operatorname{cost}\left(\beta_{t}\right)$ and rate function $\operatorname{rate}\left(\beta_{t}\right)$ at the $t$-th data encoding round as follows.

$$
\begin{equation*}
\operatorname{cost}\left(\beta_{t}\right)=\mathbb{E}[V(t)], \quad \operatorname{rate}\left(\beta_{t}\right)=H(V(t)) \tag{23}
\end{equation*}
$$

where $V(t)$ has a probability distribution in Eqn. 22. Both the two functions have closed-form formula,

$$
\begin{align*}
& \operatorname{cost}\left(\beta_{t}\right)=\frac{\sum_{j=0}^{K-1} j \exp \left(-\beta_{t} j\right)}{\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)} \\
& \operatorname{rate}\left(\beta_{t}\right)=\beta_{t} \operatorname{cost}\left(\beta_{t}\right)+\log \left(\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)\right) \tag{24}
\end{align*}
$$

Theorem 3.1: The coding efficiency $c$ is upper bounded by

$$
\begin{equation*}
c \leq \frac{\alpha \sum_{t} \operatorname{rate}\left(\left(\beta_{t}\right)\right)}{\sum_{t} \operatorname{cost}\left(\beta_{t}\right)} \tag{25}
\end{equation*}
$$

where, $\beta_{t}$ corresponds to the solution to the relaxed optimization problem in Eqn. 21

Proof: The optimal value of a relaxed maximization optimization problem is greater than or equal to the optimal value of the original optimization problem.

In our further discussion, we need to define a stage coding efficiency function

$$
\begin{equation*}
f(\beta)=\frac{\operatorname{rate}(\beta)}{\operatorname{cost}(\beta)} \tag{26}
\end{equation*}
$$

Lemma 3.2:

$$
\begin{equation*}
\frac{\mathrm{d}(\operatorname{rate}(\beta))}{\mathrm{d}(\operatorname{cost}(\beta))}=\beta \tag{27}
\end{equation*}
$$

Proof:

$$
\begin{align*}
& \frac{\mathrm{d} \operatorname{rate}(\beta)}{\mathrm{d} \operatorname{cost}(\beta)} \\
& =\frac{\operatorname{cost}(\beta)+\beta \operatorname{cost}^{\prime}(\beta)+\sum_{k}-k \exp (-\beta k) / \sum_{s} \exp (-\beta s)}{\operatorname{cost}^{\prime}(\beta)} \\
& =\frac{\operatorname{cost}(\beta)+\beta \operatorname{cost}^{\prime}(\beta)-\operatorname{cost}(\beta)}{\operatorname{cost}^{\prime}(\beta)}=\beta
\end{align*}
$$

where, the derivatives at the right hand sides are with respect to $\beta$.

Lemma 3.3: The function $\operatorname{cost}(\beta)$ is a decreasing function with respect to $\beta$.

Proof: In order to show that $\operatorname{cost}(\beta)$ is a decreasing function, it is sufficient to show that $\log (\operatorname{cost}(\beta))$ is a decreasing function. The derivative of $\log (\operatorname{cost}(\beta))$ is

$$
\begin{equation*}
\frac{\sum_{k=0}^{K-1} k \exp (-k \beta)}{\sum_{k=0}^{K-1} \exp (-k \beta)}-\frac{\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)}{\sum_{k=0}^{K-1} k \exp (-k \beta)} \tag{29}
\end{equation*}
$$

By using the Cuachy-Schwarz inequality, we have

$$
\begin{equation*}
\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2} \leq \sum_{k=0}^{K-1} \exp (-k \beta) \sum_{k=0}^{K-1} k^{2} \exp (-k \beta) \tag{30}
\end{equation*}
$$

and the equality holds only when $\beta$ goes to infinity. It thus follows that the derivative of $\log (\operatorname{cost}(\beta))$ is strictly negative for any finite $\beta$. The lemma follows.

Lemma 3.4: The function $f(\beta)$ is an increasing function with respect to $\beta$.

Proof: The derivative of $f(\beta)$ is as in Eqn. 31
The lemma is proven if we can show that

$$
\begin{equation*}
\frac{\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right]}{\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2}} \geq 1 \tag{32}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2} \leq\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right] \tag{33}
\end{equation*}
$$

We can show that this is indeed the case by using the CuachySchwarz inequality,

$$
\begin{equation*}
\left(\sum_{k} \sqrt{x_{k} y_{k}}\right)^{2} \leq\left(\sum_{k} x_{k}\right)\left(\sum_{k} y_{k}\right) \tag{34}
\end{equation*}
$$

Theorem 3.5: In the solution to the optimization problem in Eqn. 21

$$
\begin{equation*}
\beta_{1}=\beta_{2}=\ldots=\beta_{t}=\ldots=\beta_{T}=\beta \tag{35}
\end{equation*}
$$

Therefore, the coding efficiency

$$
\begin{equation*}
c \leq \frac{\alpha \operatorname{rate}((\beta))}{\operatorname{cost}(\beta)} \tag{36}
\end{equation*}
$$

Proof: The theorem is proven by contradiction. Suppose that in the optimization solution for Eqn. 21 there exist $\beta_{s}$ and $\beta_{t}$ such that $\beta_{s}>\beta_{t}$. According to Lemma 3.3, $\operatorname{cost}\left(\beta_{s}\right)<$ $\operatorname{cost}\left(\beta_{t}\right)$. We may modifity $\beta_{s}$ and $\beta_{t}$ slightly into $\beta_{s}-\Delta \beta_{s}$ and $\beta_{t}+\Delta \beta_{t}$, such that

$$
\begin{align*}
& \operatorname{cost}\left(\beta_{s}-\Delta \beta_{s}\right)=\operatorname{cost}\left(\beta_{s}\right)+\Delta \operatorname{cost}  \tag{37}\\
& \operatorname{cost}\left(\beta_{t}+\Delta \beta_{t}\right)=\operatorname{cost}\left(\beta_{t}\right)-\Delta \operatorname{cost} \tag{38}
\end{align*}
$$

where $\Delta$ cost $>0$. Therefore, the total sum of cost functions remains the same. On the other hand, the rate function corresponding to $\beta_{s}$ increases with derivative $\beta_{s}$, and the rate function corresponding to $\beta_{t}$ decreases with derivative $\beta_{t}$. The total sum of rate functions increases. Therefore, $\beta_{s}$ and $\beta_{t}$ can not be a part of the optimization solution. This results in a contradiction. The theorem is proven.

## IV. Achievable Lower Bound using Random Coding Arguments

In this section, we prove a lower bound for the coding efficiency by using a specific random coding scheme. The data encoding scheme consists of multiple stages. During all the stages, the cells in the block are restricted to take one of two states, $k$ or $k-1$, where $k=1, \ldots, K-1$. Assume in a certain stage, there are $l$ cells that take the state $k-1$, and the rest $N-l$ cells take state $k$. Then, during this stage, the state of only one memory cell is changed from $k$ to $k-1$ and $l(t)=\log _{2}\lfloor(1-\epsilon)(N-l)\rfloor$ bits can be recorded, where $\lfloor\cdot\rfloor$ denotes the floor function, and $\epsilon$ is a small real number, $0<\epsilon<1$.

The data encoding process is as follows. Let us throw all the sequences of symbols with length $N$ and alphabet $\{0,1,2, \ldots, K-1\}$ into $2^{(l(t))}$ bins uniformly in random. If the to-be-recorded message is $m[t]$, then we check the $m[t]$ th bin. We try to find one sequence in the bin, such that the current configuration of the memory cells can be modified to be equal to the sequence by turning the state of one memory cell $X_{n}$ from $k$ to $k-1$. If such a sequence can be found, then we turn the state of the memory cell $X_{n}$ from $k$ to $k-1$. If such a sequence can not be found in the bin, then a decoding error is declared and we randomly turn one memory cell from $k$ to $k-1$ and go to the next coding stage.

$$
\begin{equation*}
f^{\prime}(\beta)=\log \left(\sum_{k=0}^{K-1} \exp (-k \beta)\right)\left\{-1+\frac{\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right]}{\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2}}\right\} \tag{31}
\end{equation*}
$$

We assume that the data decoding process knows the random coding schemes, for example, by sharing the same random source with the encoder, or using a pseudo random source. In the first step of data decoding, the decoder can determine the stage of data encoding by looking at the states of the memory cells and the number $l$ of cells being at the state $k-1$. The recorded message $m(t)$ can then be recovered by looking at the bin index of the current configuration of the memory cells.

The encoding error probability can be bounded as follows,

$$
\begin{align*}
& \mathbb{P}(\text { error }) \leq\left[1-\frac{1}{\lfloor(1-\epsilon)(N-l)\rfloor}\right]^{N-l} \\
& \stackrel{(a)}{\leq} \exp \left(-\frac{N-l}{\lfloor(1-\epsilon)(N-l)\rfloor}\right) \leq \exp \left(\frac{-1}{1-\epsilon}\right) \tag{39}
\end{align*}
$$

where, (a) follows from the inequality, $(1-x)^{y} \leq$ $\exp (-x y)$, for $x \in(0,1), y \geq 0$.

The expected total amount of recoded information between two erase operations can be bounded as

$$
\begin{align*}
& \mathbb{E}(\text { rate }) \geq(K-1) \\
& \times \sum_{l=0}^{N}\left[1-\exp \left(\frac{-1}{1-\epsilon}\right)\right] \log _{2}(\lfloor(1-\epsilon)(N-l)\rfloor) \tag{40}
\end{align*}
$$

For sufficiently large $N$ and $\epsilon=0.5$, the total expected recorded information is lower bounded as

$$
\begin{equation*}
\mathbb{E}(\text { rate }) \geq \frac{(K-1) N}{2}[1-\exp (-2)] \log (N / 2) \tag{41}
\end{equation*}
$$

Therefore, the coding efficiency is bounded as follows.

$$
\begin{equation*}
c \geq \frac{\alpha}{2}[1-\exp (-2)] \log (N / 2) \tag{42}
\end{equation*}
$$

The payload can be calculated as

$$
\begin{equation*}
p=\frac{1}{N^{2}} \sum_{l=0}^{N-1} \log _{2}(\lfloor(1-\epsilon)(N-l)\rfloor) \tag{43}
\end{equation*}
$$

Based on the above discussions in this section, we arrive at the following theorem.

Theorem 4.1: The optimal coding efficiency for $K$ level $N$ cell flash memories can go to infinity as $N$ goes to infinity.

## V. The Coding-Efficiency-to-Payload Tradeoff

Some important insights can be gained from the upper and lower bounds for coding efficiency proved in the previous sections. From the upper bound, it can be seen clearly that the coding efficiency decreases as the payload increases. From the lower bound, it can be seen that the coding efficiency may go to infinity as the payload decreases to zero. Therefore, we can conclude that there exists a tradeoff between the coding
efficiency and payload. The tradeoff is illustrated in Fig. 1 In the figure, the upper and lower bound for coding efficiency are shown, where the x -axis shows the payload. We assume $\alpha=1$, and the flash memories are 8 -level (3bit) TLC type flash memories.


Fig. 1. Upper and lower bounds for coding efficiency of 3-bit flash memory cells.

## VI. Conclusion

In this paper, we study the coding efficiency problem for flash memories. A coding theorem for achievable rates is proven. We prove an upper and lower bounds for the coding efficiency. We show that there exists a tradeoff between the coding efficiency and payload. Our discussions in this paper provide useful insights on the design of future flash memory systems.

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# On Coding-Efficiency for Flash Memories 

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#### Abstract

Recently, flash memories have become a competitive solution for mass storage. The flash memories have rather different properties compared with the rotary hard drives. That is, the writing of flash memories is constrained, and flash memories can endure only limited numbers of erases. Therefore, the design goals for the flash memory systems are quite different from these for other memory systems. In this paper, we consider the problem of coding efficiency. We define the "coding-efficiency" as the amount of information that one flash memory cell can be used to record per cost. Because each flash memory cell can endure a roughly fixed number of erases, the cost of data recording can be well-defined. We define "payload" as the amount of information that one flash memory cell can represent at a particular moment. By using information-theoretic arguments, we prove a coding theorem for achievable coding rates. We prove an upper and lower bound for coding efficiency. We show in this paper that there exists a fundamental trade-off between "payload" and "coding efficiency". The results in this paper may provide useful insights on the design of future flash memory systems.


## I. Introduction

Recently, flash memories have become a competitive solution for mass storage. Compared with the conventional rotary hard drives, flash memories have high random access read speed, because there is no mechanical seek time. Flash memory storage devices are also more lightweight, power efficient, and kinetic shock resistant. Therefore, they are becoming desirable choices for many applications ranging from highspeed servers in data centers to portable devices.

Flash memories are one type of solid state memories. Each piece of flash memory usually contains multiple arrays of flash memory cells. Each memory cell is a transistor with a floating gate. Information is recorded using one memory cell by injecting and removing electrons into and from the floating gate. The process of injecting electrons is called programming and the process of removing electrons is called erase. Programming increases the threshold voltage level of the memory cell, while erase decreases the threshold voltage level. The threshold voltage level of the memory cell is the voltage level at the control gate that the transistor becomes conducting. In the reading process for the memory cell, the threshold voltage level is detected, thus the recorded information can be recovered.

The memory cells are organized into pages and then into blocks. The programming is page-wise and erase is blockwise. Usually, one memory block is first erased, so that all memory cells within the block return to an initial threshold voltage level. After the erase operation, the pages in the block are programmed (possibly multiple times), until normal
threshold voltage level ranges are used up. Then, the memory block is erased again for further use.
One challenge for flash memories is that the number of erase operations that one memory cell can withstand is quite limited. For current commercial flash memories, such maximal numbers of block erase operations range from 5,000 to 100,000 . After such a limited number of erase operations, the flash memory cell would become broken or unreliable. Therefore, data encoding methods must be carefully designed to address such an issue.

In fact, flash memories can be considered as one type of write-once-memories. The write-once-memories were first discussed in the seminal work by Rivest and Shamir [?]. Previous examples of write-once-memories include digital optical disks, punched paper tapes, punched cords, and programmable readonly memories etc. Rivest and Shamir show that by using advanced data encoding methods, the write-once-memories can be rewritten. In [?], one theorem for the achievable data recording rates of binary write-once-memories has been proven using combinatorial arguments. During the passed research, many data encoding methods for rewriting the write-once-memories have been proposed, see for example, [?] [?] etc.

In this paper, we consider a coding efficiency problem for data encoding on flash memories. Unlike other type of computer memories, the cost of data encoding can be welldefined for flash memories. That is, the cost for each erase operation can be defined based on the cost of the flash memory block and the total number of erase operations that the memory block can have. The coding efficiency problem is therefore the problem of recording more information using fewer erase operations. To our best knowledge, such a design problem for flash memories has never been discussed before.

We assume that one flash memory block has $N$ cells, and each cell can take $K$ voltage levels. We assume that the data encoding scheme uses the memory block for $T$ rounds between two consecutive erase operations. That is, in the first round, a message $M[1]$ is recorded using the block, and in the second round, a new message $M[2]$ is recorded, and so on. Suppose that $N l_{t}$ bits are recorded during the $t$-th round. We define the payload $p$ and coding efficiency $c$ as

$$
\begin{equation*}
p=\frac{1}{T} \sum_{t=1}^{T} l_{t}, \quad c=\frac{\alpha}{K} \sum_{t=1}^{T} l_{t} \tag{1}
\end{equation*}
$$

where, $\alpha$ is a constant depending on the type of the memory block, e.g., NOR type, NAND type, single-level-cell, multi-level-cell etc. The constant $\alpha$ may be used to reflect the cost
for the flash memory block. It should be clear that the coding efficiency measures the amount of recording information per voltage level cost. We may also define the voltage level cost per recorded bit, which is exactly $1 / c$.

In this paper, we first prove a coding theorem for achievable rates of data encoding on flash memories using informationtheoretic arguments. Using the coding theorem in this paper, we prove an upper bound for the optimal coding efficiency. We also show a lower bound of optimal coding efficiency using a specific coding scheme. Surprisingly, we find that there exists a tradeoff between the optimal coding efficiency and payload. These results may provide useful insights and tools for designing future flash memory storage systems.

The rest of this paper is organized as follows. In Section $\Pi$, we present the coding theorem for achievable coding rates. In Section III, we show the upper bound of the optimal coding efficiency. In Section IV we present the lower bound for optimal coding efficiency using a specific coding scheme. The coding efficiency to payload tradeoff is discussed in Section V. Some concluding remarks are presented at Section VI.

## II. Coding Theorem

We consider a memory block with $N$ memory cells. Each memory cell can take $K$ threshold voltage levels, that is, each memory cell can be at one of the states $0,1, \ldots, K-1$. After one erase operation, all memory cells are at the state $K-1$. During each programming process, the state of each cell can be decreased but never increased. Assume that the memory block can be reliably used for $T$ rounds of information recording, where messages $M(1), M(2), \ldots, M(t), \ldots, M(T)$ are recorded. We define the corresponding data rate in the $t$ th round $l(t)=\log _{2}(|M(t)|) / N$, where $|M(t)|$ denote the alphabet size of the message $M(t)$. In this case, we say that the sequence of data rates $l(t), t=1, \ldots, T$ is achievable. We assume that all the $T$ messages are statistical independent. We denote the state of the $n$-th cell in the block during time $t$ by $X_{n}(t)$. We use the notation $X_{1}^{N}(t)$ to denote the sequence $X_{1}(t), X_{2}(t), \ldots, X_{N}(t)$. Similarly, $X_{1}^{n}(t)$ denotes the sequence $X_{1}(t), X_{2}(t), \ldots, X_{n}(t)$, where $1 \leq n \leq N$. We use $H(\cdot)$ to denote the entropy and conditional entropy functions as in [?].

Theorem 2.1: A sequence of data rates $l(t), t=1, \ldots, T$ is achievable, if and only if, there exist random variables $U(1), \ldots, U(T)$ jointly distributed with a probability distribution $\mathbb{P}(U(1), \ldots, U(T))$, such that,

$$
\begin{align*}
& \mathbb{P}(U(t)=j \mid U(t-1)=i)=0, \text { if } j>i, \text { for } t=2, \ldots, T \\
& l(t) \leq H(U(t) \mid U(t-1)), \text { for } t=2, \ldots, T \\
& l(1) \leq H(U(1)) \tag{2}
\end{align*}
$$

By convention, $U(0)=K-1$ with probability 1 .
Proof: The achievable part is proven by random binning. For the $t$-th round of data recording, we construct a random code by throwing typical sequences of $U(t)$ into $\exp \{N l(t)\}$ bins uniformly in random. The message $m(t)$ is encoded by finding a sequence $X_{1}^{N}(t)$ in the $m(t)$-th bin, such that the
sequence $X_{1}^{N}(t)$ is jointly typical with $X_{1}^{N}(t-1)$. If such a sequence can not be found, then one encoding error is declared.

Suppose that $l(t) \leq H(U(t) \mid U(t-1))-2 \epsilon$, where $\epsilon$ is an arbitrarily small positive number. Then, the probability of encoding error can be upper bounded as follows.

$$
\begin{align*}
\mathbb{P}(\text { error }) & =\left(1-\frac{1}{\exp (N l(t))}\right)^{N_{1}} \\
& \stackrel{(a)}{\leq} \exp \left(-\frac{N_{1}}{\exp (N l(t))}\right) \\
& \stackrel{(b)}{\leq} \exp \left(-\frac{\exp (N(H(U(t) \mid U(t-1))-\epsilon))}{\exp \{N(H(U(t) \mid U(t-1))-2 \epsilon)\}}\right) \\
& \leq \exp (-\exp (\epsilon N)) \tag{3}
\end{align*}
$$

where, $N_{1}$ denotes the number of typical sequences $X_{1}^{N}(t)$ that are jointly typical with $X_{1}^{N}(t-1)$, (a) follows from the inequality, $(1-x) \leq \exp (-x)$, for $0 \leq x<1$, (b) follows from the fact that $\bar{N}_{1} \geq \exp \{N(H(\bar{U}(t) \mid U(t-1))-\epsilon)\}$. The achievable part of the proof then follows from the fact that $\epsilon$ can be taken arbitrarily small.

We prove the converse part by constructing some random variables $U(1), \ldots, U(T)$, which satisfy the conditions in the theorem. Assume that there exists at least one coding scheme, which satisfies the conditions in the theorem.

In the first step, we wish to show

$$
\begin{equation*}
H(M(t)) \leq H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \tag{4}
\end{equation*}
$$

This is because, on the one hand,

$$
\begin{align*}
& H\left(M(t), X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \\
& =H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right)+H\left(M(t) \mid X_{1}^{N}(t), X_{1}^{N}(t-1)\right) \\
& \stackrel{(a)}{=} H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \tag{5}
\end{align*}
$$

where, (a) follows from the fact that $M(t)$ can be completely determined by observing $X_{1}^{N}(t)$. On the other hand,

$$
\begin{align*}
& H\left(M(t), X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) \\
& =H\left(M(t) \mid X_{1}^{N}(t-1)\right)+H\left(X_{1}^{N}(t) \mid M(t), X_{1}^{N}(t-1)\right) \\
& \stackrel{(a)}{=} H(M(t))+H\left(X_{1}^{N}(t) \mid M(t), X_{1}^{N}(t-1)\right) \tag{6}
\end{align*}
$$

where, (a) follows from the fact that $M(t)$ is independent of $X_{1}^{N}(t-1)$.

In the second step, we can show that

$$
\begin{equation*}
H(M(t)) \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \tag{7}
\end{equation*}
$$

This is because,

$$
\begin{align*}
H\left(X_{1}^{N}(t) \mid X_{1}^{N}(t-1)\right) & =\sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{1}^{n-1}(t), X_{1}^{N}(t-1)\right) \\
& \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \tag{8}
\end{align*}
$$

where the last inequality follows from the fact that conditions do not increase entropy.

Let us define random variables $Z, U(1), U(2), \ldots, U(T)$ as follows. The random variable $Z$ takes values in $\{1,2, \ldots, N\}$ uniformly in random.

$$
\begin{equation*}
U(t)=X_{n}(t), \text { if } Z=n \tag{9}
\end{equation*}
$$

The probability distribution of the random variables $Z, U(1), U(2),, \ldots, U(T)$ can be factored as follows.

$$
\begin{equation*}
\mathbb{P}(Z) \prod_{t=1}^{T} \mathbb{P}(U(t) \mid U(1), \ldots, U(t-1), Z) \tag{10}
\end{equation*}
$$

It can be checked that

$$
\begin{equation*}
\mathbb{P}(U(t)=j \mid U(t-1)=i)=0, \text { if } j>i \tag{11}
\end{equation*}
$$

Finally, we wish to show that

$$
\begin{equation*}
N l(t)=H(M(t)) \leq N H(U(t) \mid U(t-1)) \tag{12}
\end{equation*}
$$

This is because

$$
\begin{align*}
& H(M(t)) \leq \sum_{n=1}^{N} H\left(X_{n}(t) \mid X_{n}(t-1)\right) \\
& \stackrel{(a)}{=} N H(U(t) \mid U(t-1), Z) \\
& \stackrel{(b)}{\leq} N H(U(t) \mid U(t-1)) \tag{13}
\end{align*}
$$

where, (a) follows from the definition of $Z$, (b) follows from the fact that conditions do not increase entropy.

Therefore, we have constructed the random variables $U(1), \ldots, U(T)$, which satisfy the conditions in the theorem. The theorem is proven.

## III. Upper Bound

In this section, we prove an upper bound for the achievable coding efficiency. It is clear that the coding efficiency can be calculated based on the Theorem 2.1 by forming an optimization problem. Let us define a random variable $V(t)=$ $U(t-1)-U(t)$ with an alphabet $\{0,1, \ldots, K-1\}$. With a given payload $p$, the optimization problem is as follows.

$$
\begin{gather*}
\min _{\mathbb{P}(V(1), \ldots, V(t), \ldots V(T))} \mathbb{E}\left[\sum_{t} V(t)\right]  \tag{14}\\
\text { Subject to: } \sum_{t} H(V(t) \mid U(t-1)) \geq T p  \tag{15}\\
\mathbb{P}\left(\sum_{t} V(t) \geq K\right)=0 \tag{16}
\end{gather*}
$$

By convention, $U(0)=K-1$ with probability 1 . It should be clear that the coding efficiency

$$
\begin{equation*}
c \leq \frac{\alpha T p}{\sum_{t} \mathbb{E}\left(V(t)^{*}\right)} \tag{17}
\end{equation*}
$$

where $V(t)^{*}$ denotes the minimizer of the optimization problem.

However, the above optimization problem is difficult to solve in closed-form. We will consider instead a relaxed optimization problem. First, we remove the constraint in Eqn 16. Second, we relax the constraint $\sum_{t} H(V(t) \mid U(t-1)) \geq$ $T p$ to $\sum_{t} H(V(t)) \geq T p$, due to the fact that conditions do not increase entropy. Thus, the original optimization problem becomes

$$
\begin{align*}
& \min _{\mathbb{P}(V(1), \ldots, V(t), \ldots V(T))} \mathbb{E}\left[\sum_{t} V(t)\right] \\
& \text { Subject to: } \sum_{t} H(V(t)) \geq T p \tag{18}
\end{align*}
$$

In a final step, because all the constraint and objective functions only depend on marginal distributions of $V(t)$, we may further relax the above optimization problem by replacing the joint distribution

$$
\begin{equation*}
\mathbb{P}(V(1), \ldots, V(t), \ldots, V(T)) \tag{19}
\end{equation*}
$$

with a set of pseudo marginal distributions

$$
\begin{equation*}
\mathbb{P}(V(1)), \ldots, \mathbb{P}(V(t)), \ldots, \mathbb{P}(V(T)) \tag{20}
\end{equation*}
$$

The pseudo marginal distributions may or may not correspond to a joint distribution. The final relax optimization problem is thus as follows.

$$
\begin{align*}
& \min _{\mathbb{P}(V(1)), \ldots, \mathbb{P}(V(t)), \ldots, \mathbb{P}(V(T)))} \mathbb{E}\left[\sum_{t} V(t)\right] \\
& \text { Subject to: } \sum_{t} H(V(t)) \geq T p \tag{21}
\end{align*}
$$

Using the Lagrangian method, we can find that the optimal distribution for $V(t)$ takes the following form

$$
\begin{equation*}
\mathbb{P}(V(t)=j)=\frac{\exp \left(-\beta_{t} j\right)}{\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)} \tag{22}
\end{equation*}
$$

for a certain parameter $\beta_{t}>0$. Let us define the cost function $\operatorname{cost}\left(\beta_{t}\right)$ and rate function $\operatorname{rate}\left(\beta_{t}\right)$ at the $t$-th data encoding round as follows.

$$
\begin{equation*}
\operatorname{cost}\left(\beta_{t}\right)=\mathbb{E}[V(t)], \quad \operatorname{rate}\left(\beta_{t}\right)=H(V(t)) \tag{23}
\end{equation*}
$$

where $V(t)$ has a probability distribution in Eqn. 22. Both the two functions have closed-form formula,

$$
\begin{align*}
& \operatorname{cost}\left(\beta_{t}\right)=\frac{\sum_{j=0}^{K-1} j \exp \left(-\beta_{t} j\right)}{\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)} \\
& \operatorname{rate}\left(\beta_{t}\right)=\beta_{t} \operatorname{cost}\left(\beta_{t}\right)+\log \left(\sum_{s=0}^{K-1} \exp \left(-\beta_{t} s\right)\right) \tag{24}
\end{align*}
$$

Theorem 3.1: The coding efficiency $c$ is upper bounded by

$$
\begin{equation*}
c \leq \frac{\alpha \sum_{t} \operatorname{rate}\left(\left(\beta_{t}\right)\right)}{\sum_{t} \operatorname{cost}\left(\beta_{t}\right)} \tag{25}
\end{equation*}
$$

where, $\beta_{t}$ corresponds to the solution to the relaxed optimization problem in Eqn. 21

Proof: The optimal value of a relaxed maximization optimization problem is greater than or equal to the optimal value of the original optimization problem.

In our further discussion, we need to define a stage coding efficiency function

$$
\begin{equation*}
f(\beta)=\frac{\operatorname{rate}(\beta)}{\operatorname{cost}(\beta)} \tag{26}
\end{equation*}
$$

Lemma 3.2:

$$
\begin{equation*}
\frac{\mathrm{d}(\operatorname{rate}(\beta))}{\mathrm{d}(\operatorname{cost}(\beta))}=\beta \tag{27}
\end{equation*}
$$

Proof:

$$
\begin{align*}
& \frac{\mathrm{d} \operatorname{rate}(\beta)}{\mathrm{d} \operatorname{cost}(\beta)} \\
& =\frac{\operatorname{cost}(\beta)+\beta \operatorname{cost}^{\prime}(\beta)+\sum_{k}-k \exp (-\beta k) / \sum_{s} \exp (-\beta s)}{\operatorname{cost}^{\prime}(\beta)} \\
& =\frac{\operatorname{cost}(\beta)+\beta \operatorname{cost}^{\prime}(\beta)-\operatorname{cost}(\beta)}{\operatorname{cost}^{\prime}(\beta)}=\beta
\end{align*}
$$

where, the derivatives at the right hand sides are with respect to $\beta$.

Lemma 3.3: The function $\operatorname{cost}(\beta)$ is a decreasing function with respect to $\beta$.

Proof: In order to show that $\operatorname{cost}(\beta)$ is a decreasing function, it is sufficient to show that $\log (\operatorname{cost}(\beta))$ is a decreasing function. The derivative of $\log (\operatorname{cost}(\beta))$ is

$$
\begin{equation*}
\frac{\sum_{k=0}^{K-1} k \exp (-k \beta)}{\sum_{k=0}^{K-1} \exp (-k \beta)}-\frac{\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)}{\sum_{k=0}^{K-1} k \exp (-k \beta)} \tag{29}
\end{equation*}
$$

By using the Cuachy-Schwarz inequality, we have

$$
\begin{equation*}
\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2} \leq \sum_{k=0}^{K-1} \exp (-k \beta) \sum_{k=0}^{K-1} k^{2} \exp (-k \beta) \tag{30}
\end{equation*}
$$

and the equality holds only when $\beta$ goes to infinity. It thus follows that the derivative of $\log (\operatorname{cost}(\beta))$ is strictly negative for any finite $\beta$. The lemma follows.

Lemma 3.4: The function $f(\beta)$ is an increasing function with respect to $\beta$.

Proof: The derivative of $f(\beta)$ is as in Eqn. 31
The lemma is proven if we can show that

$$
\begin{equation*}
\frac{\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right]}{\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2}} \geq 1 \tag{32}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2} \leq\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right] \tag{33}
\end{equation*}
$$

We can show that this is indeed the case by using the CuachySchwarz inequality,

$$
\begin{equation*}
\left(\sum_{k} \sqrt{x_{k} y_{k}}\right)^{2} \leq\left(\sum_{k} x_{k}\right)\left(\sum_{k} y_{k}\right) \tag{34}
\end{equation*}
$$

Theorem 3.5: In the solution to the optimization problem in Eqn. 21

$$
\begin{equation*}
\beta_{1}=\beta_{2}=\ldots=\beta_{t}=\ldots=\beta_{T}=\beta \tag{35}
\end{equation*}
$$

Therefore, the coding efficiency

$$
\begin{equation*}
c \leq \frac{\alpha \operatorname{rate}((\beta))}{\operatorname{cost}(\beta)} \tag{36}
\end{equation*}
$$

Proof: The theorem is proven by contradiction. Suppose that in the optimization solution for Eqn. 21 there exist $\beta_{s}$ and $\beta_{t}$ such that $\beta_{s}>\beta_{t}$. According to Lemma 3.3, $\operatorname{cost}\left(\beta_{s}\right)<$ $\operatorname{cost}\left(\beta_{t}\right)$. We may modifity $\beta_{s}$ and $\beta_{t}$ slightly into $\beta_{s}-\Delta \beta_{s}$ and $\beta_{t}+\Delta \beta_{t}$, such that

$$
\begin{align*}
& \operatorname{cost}\left(\beta_{s}-\Delta \beta_{s}\right)=\operatorname{cost}\left(\beta_{s}\right)+\Delta \operatorname{cost}  \tag{37}\\
& \operatorname{cost}\left(\beta_{t}+\Delta \beta_{t}\right)=\operatorname{cost}\left(\beta_{t}\right)-\Delta \operatorname{cost} \tag{38}
\end{align*}
$$

where $\Delta$ cost $>0$. Therefore, the total sum of cost functions remains the same. On the other hand, the rate function corresponding to $\beta_{s}$ increases with derivative $\beta_{s}$, and the rate function corresponding to $\beta_{t}$ decreases with derivative $\beta_{t}$. The total sum of rate functions increases. Therefore, $\beta_{s}$ and $\beta_{t}$ can not be a part of the optimization solution. This results in a contradiction. The theorem is proven.

## IV. Achievable Lower Bound using Random Coding Arguments

In this section, we prove a lower bound for the coding efficiency by using a specific random coding scheme. The data encoding scheme consists of multiple stages. During all the stages, the cells in the block are restricted to take one of two states, $k$ or $k-1$, where $k=1, \ldots, K-1$. Assume in a certain stage, there are $l$ cells that take the state $k-1$, and the rest $N-l$ cells take state $k$. Then, during this stage, the state of only one memory cell is changed from $k$ to $k-1$ and $l(t)=\log _{2}\lfloor(1-\epsilon)(N-l)\rfloor$ bits can be recorded, where $\lfloor\cdot\rfloor$ denotes the floor function, and $\epsilon$ is a small real number, $0<\epsilon<1$.

The data encoding process is as follows. Let us throw all the sequences of symbols with length $N$ and alphabet $\{0,1,2, \ldots, K-1\}$ into $2^{(l(t))}$ bins uniformly in random. If the to-be-recorded message is $m[t]$, then we check the $m[t]$ th bin. We try to find one sequence in the bin, such that the current configuration of the memory cells can be modified to be equal to the sequence by turning the state of one memory cell $X_{n}$ from $k$ to $k-1$. If such a sequence can be found, then we turn the state of the memory cell $X_{n}$ from $k$ to $k-1$. If such a sequence can not be found in the bin, then a decoding error is declared and we randomly turn one memory cell from $k$ to $k-1$ and go to the next coding stage.

$$
\begin{equation*}
f^{\prime}(\beta)=\log \left(\sum_{k=0}^{K-1} \exp (-k \beta)\right)\left\{-1+\frac{\left[\sum_{k=0}^{K-1} \exp (-k \beta)\right]\left[\sum_{k=0}^{K-1} k^{2} \exp (-k \beta)\right]}{\left[\sum_{k=0}^{K-1} k \exp (-k \beta)\right]^{2}}\right\} \tag{31}
\end{equation*}
$$

We assume that the data decoding process knows the random coding schemes, for example, by sharing the same random source with the encoder, or using a pseudo random source. In the first step of data decoding, the decoder can determine the stage of data encoding by looking at the states of the memory cells and the number $l$ of cells being at the state $k-1$. The recorded message $m(t)$ can then be recovered by looking at the bin index of the current configuration of the memory cells.

The encoding error probability can be bounded as follows,

$$
\begin{align*}
& \mathbb{P}(\text { error }) \leq\left[1-\frac{1}{\lfloor(1-\epsilon)(N-l)\rfloor}\right]^{N-l} \\
& \stackrel{(a)}{\leq} \exp \left(-\frac{N-l}{\lfloor(1-\epsilon)(N-l)\rfloor}\right) \leq \exp \left(\frac{-1}{1-\epsilon}\right) \tag{39}
\end{align*}
$$

where, (a) follows from the inequality, $(1-x)^{y} \leq$ $\exp (-x y)$, for $x \in(0,1), y \geq 0$.

The expected total amount of recoded information between two erase operations can be bounded as

$$
\begin{align*}
& \mathbb{E}(\text { rate }) \geq(K-1) \\
& \times \sum_{l=0}^{N}\left[1-\exp \left(\frac{-1}{1-\epsilon}\right)\right] \log _{2}(\lfloor(1-\epsilon)(N-l)\rfloor) \tag{40}
\end{align*}
$$

For sufficiently large $N$ and $\epsilon=0.5$, the total expected recorded information is lower bounded as

$$
\begin{equation*}
\mathbb{E}(\text { rate }) \geq \frac{(K-1) N}{2}[1-\exp (-2)] \log (N / 2) \tag{41}
\end{equation*}
$$

Therefore, the coding efficiency is bounded as follows.

$$
\begin{equation*}
c \geq \frac{\alpha}{2}[1-\exp (-2)] \log (N / 2) \tag{42}
\end{equation*}
$$

The payload can be calculated as

$$
\begin{equation*}
p=\frac{1}{N^{2}} \sum_{l=0}^{N-1} \log _{2}(\lfloor(1-\epsilon)(N-l)\rfloor) \tag{43}
\end{equation*}
$$

Based on the above discussions in this section, we arrive at the following theorem.

Theorem 4.1: The optimal coding efficiency for $K$ level $N$ cell flash memories can go to infinity as $N$ goes to infinity.

## V. The Coding-Efficiency-to-Payload Tradeoff

Some important insights can be gained from the upper and lower bounds for coding efficiency proved in the previous sections. From the upper bound, it can be seen clearly that the coding efficiency decreases as the payload increases. From the lower bound, it can be seen that the coding efficiency may go to infinity as the payload decreases to zero. Therefore, we can conclude that there exists a tradeoff between the coding
efficiency and payload. The tradeoff is illustrated in Fig. 1 In the figure, the upper and lower bound for coding efficiency are shown, where the $x$-axis shows the payload. We assume $\alpha=1$, and the flash memories are 8 -level (3bit) TLC type flash memories.


Fig. 1. Upper and lower bounds for coding efficiency of 3-bit flash memory cells.

## VI. Conclusion

In this paper, we study the coding efficiency problem for flash memories. A coding theorem for achievable rates is proven. We prove an upper and lower bounds for the coding efficiency. We show that there exists a tradeoff between the coding efficiency and payload. Our discussions in this paper provide useful insights on the design of future flash memory systems.

