# The Unique Games Conjecture, Integrality Gap for Cut Problems and Embeddability of Negative Type Metrics into $\ell_1^*$

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#### **Abstract**

In this paper, we disprove a conjecture of Goemans [23] and Linial [36] (also see [6, 38]); namely, that every negative type metric embeds into  $\ell_1$  with constant distortion. We show that for an arbitrarily small constant  $\delta > 0$ , for all large enough n, there is an n-point negative type metric which requires distortion at least  $(\log \log n)^{1/6-\delta}$  to embed into  $\ell_1$ .

Surprisingly, our construction is inspired by the Unique Games Conjecture (UGC) of Khot [28], establishing a previously unsuspected connection between probabilistically checkable proof systems (PCPs) and the theory of metric embeddings. We first prove that the UGC implies a super-constant hardness result for the (non-uniform) SPARSESTCUT problem. Though this hardness result relies on the UGC, we demonstrate, nevertheless, that the corresponding PCP reduction can be used to construct an "integrality gap instance" for SPARSESTCUT. Towards this, we first construct an integrality gap instance for a natural SDP relaxation of UNIQUEGAMES. Then we "simulate" the PCP reduction and "translate" the integrality gap instance of UNIQUEGAMES to an integrality gap instance of SPARSESTCUT. This enables us to prove a  $(\log\log n)^{1/6-\delta}$  integrality gap for SPARSESTCUT, which is known to be equivalent to the metric embedding lower bound.

<sup>\*</sup>A preliminary version of this paper appeared in FOCS 2005, see [32].

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## **Contents**

1	Introduction		3
	1.1	Metric Embeddings and their Algorithmic Applications	3
	1.2	Our Contribution	4
	1.3	Subsequent Works	5
	1.4	Difficulty in Proving $\ell_2^2$ vs. $\ell_1$ Lower Bound	6
	1.5	Overview of Our $\ell_2^2$ vs. $\ell_1$ Lower Bound	7
	1.6	Organization of the Main Body of the Paper	10
2	Preliminaries 13		
	2.1	The $(\ell_2^2, \ell_1, O(1))$ -Conjecture	11
	2.2	Balanced Edge-Separator, Sparsest Cut and their SDP Relaxations	12
	2.3	Relation Between $(\ell_2^2, \ell_1, O(1))$ -Conjecture, Sparsest Cut and Balanced Edge-Separator	13
	2.4	Our Integrality Gap Instance for Balanced Edge-Separator	15
	2.5	Fourier Analysis	16
3	The Integrality Gap Instance for Unique Games		17
	3.1	The Unique Games Problem, its SDP Relaxation and the UGC	18
	3.2	The Integrality Gap Instance	21
4	A P	CP Reduction from Unique Games to Balanced Edge-Separator	23
	4.1	The PCP Verifier	24
	4.2	From the PCP Verifier to a Balanced Edge-Separator Instance	25
5	The Integrality Gap Instance for Balanced Edge-Separator		26
	5.1	The Graph	28
	5.2	The SDP Solution	29
A	Proc	of of Soundness of the PCP Reduction	36
В	Proc	of of Lemma 5.8 (Triangle Inequality Constraint)	38

#### 1 Introduction

#### 1.1 Metric Embeddings and their Algorithmic Applications

In recent years, the theory of metric embeddings has played an increasing role in algorithm design. The best approximation algorithms for several NP-hard problems rely on techniques (and theorems) used to embed one metric space into another while preserving all pairwise distances up to a certain *not too large* factor, known as the *distortion* of the embedding.

Perhaps, the most well-known application of this paradigm is the SPARSESTCUT problem. Given an n-vertex graph along with a set of  $demand\ pairs$ , one seeks to find a non-trivial partition of the graph that minimizes the sparsity, i.e., the ratio of the number of edges cut to the number of demand pairs cut. Strictly speaking, the problem thus defined is the non-uniform version of SPARSESTCUT and in the absence of a qualification, we always mean the non-uniform version. In contrast, the uniform version refers to the special case when the set of demand pairs consists of all possible  $\binom{n}{2}$  vertex pairs. In the uniform version, the sparsity is the same (up to a factor 2 and a normalization factor of n) as the ratio of the number of edges cut to the size of the smaller side of the partition. A closely related problem is the BALANCEDEDGE-SEPARATOR problem where one desires a partition that cuts a constant fraction of demand pairs and minimizes the number of the edges cut. In its uniform version, one desires a balanced partition, say a (1/3, 2/3)-partition, 1 that minimizes the number of the edges cut.

Bourgain [11] showed that every n-point metric embeds into  $\ell_2$  (and, hence, into  $\ell_1$  since every n-point subset of  $\ell_2$  isometrically embeds into  $\ell_1$ ) with distortion  $O(\log n)$ . Aumann and Rabani [7] and Linial, London and Rabinovich [37] independently gave a striking application of Bourgain's theorem: An  $O(\log n)$  approximation algorithm for SPARSESTCUT. The approximation ratio is exactly the distortion incurred in Bourgain's theorem. This gave an alternate approach to the seminal work of Leighton and Rao [35], who obtained an  $O(\log n)$  approximation algorithm for SPARSESTCUT via a linear programming (LP) relaxation based on multi-commodity flows.<sup>2</sup> It is well-known that an f(n) factor algorithm for SPARSESTCUT can be used iteratively to design an O(f(n)) factor algorithm for BALANCEDEDGE-SEPARATOR. In particular, in the uniform case, given a graph that has a (1/2,1/2)-partition cutting an  $\alpha$  fraction of the edges, the algorithm produces a (1/3,2/3)-partition that cuts at most  $O(f(n)\alpha)$  fraction of the edges. Such partitioning algorithms are very useful as sub-routines in the design of graph theoretic algorithms via the divide-and-conquer paradigm.

The results of [7, 37] are based on the *metric LP relaxation* of SPARSESTCUT. Given an instance G(V,E) of SPARSESTCUT, let  $d_G$  be the *n*-point metric obtained as a solution to this LP. The metric  $d_G$  is then embedded into  $\ell_1$  via Bourgain's theorem. Since  $\ell_1$  metrics are non-negative linear combinations of *cut metrics*, an embedding into  $\ell_1$  essentially gives the desired sparse cut (up to an  $O(\log n)$  approximation factor). Subsequent to this result, it was realized that one could write a semi-definite programming (SDP) relaxation of SPARSESTCUT with the so-called *triangle inequality constraints* and enforce an additional condition that the metric  $d_G$  belongs to a special subclass of metrics called the *negative type metrics* (denoted by  $\ell_2^2$ ). Clearly, if  $\ell_2^2$  embeds into  $\ell_1$  with distortion g(n), then one gets a g(n) approximation to SPARSESTCUT via this SDP (and in particular the same upper bound on the *integrality gap* of the SDP).

The results of [7, 37] led to the conjecture that  $\ell_2^2$  embeds into  $\ell_1$  with distortion C, where C is an absolute constant. This conjecture has been attributed to Goemans [23] and Linial [36], see [6, 38]. This conjecture, which we henceforth refer to as the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture, if true, would have had tremendous algorithmic applications (apart from being an important mathematical result). Several problems, specifically cut

<sup>&</sup>lt;sup>1</sup>In the uniform case, for a parameter  $b \in (0, 1/2]$ , a partition of the vertex set is said to be a (b, 1-b) partition if each side of the partition contains at least b fraction of the vertices.

<sup>&</sup>lt;sup>2</sup>In fact, algorithms based on metric embeddings work for the more general *non-uniform* version of SPARSESTCUT. The Leighton-Rao algorithm worked only for the uniform version.

problems (see [18]), can be formulated as optimization problems over the class of  $\ell_1$  metrics, and optimization over  $\ell_1$  is an NP-hard problem in general. However, one can optimize over  $\ell_2^2$  metrics in polynomial time via SDPs (and since  $\ell_1 \subseteq \ell_2^2$ , this is indeed a relaxation). Hence, if  $\ell_2^2$  metrics were embeddable into  $\ell_1$  with constant distortion, one would get a computationally efficient constant factor approximation to  $\ell_1$  metrics.

However, no better embedding of  $\ell_2^2$  into  $\ell_1$ , other than Bourgain's  $O(\log n)$  embedding (that works for all metrics), was known. A breakthrough result of Arora, Rao and Vazirani (ARV) [6] gave an  $O(\sqrt{\log n})$  approximation to (uniform) SPARSESTCUT by showing that the integrality gap of the SDP relaxation is  $O(\sqrt{\log n})$  (see also [39] for an alternate perspective on ARV). Subsequently, ARV techniques were used by Chawla, Gupta and Räcke [13] to give an  $O(\log^{3/4} n)$  distortion embedding of  $\ell_2^2$  metrics into  $\ell_2$  and, hence, into  $\ell_1$ . This result was further improved to  $O(\sqrt{\log n}\log\log n)$  by Arora, Lee and Naor [5].<sup>3</sup> Techniques from ARV have also been applied to obtain an  $O(\sqrt{\log n})$  approximation to MINUNCUT and related problems [1], to VertexSeparator [20], and to obtain a  $2 - O(1/\sqrt{\log n})$  approximation to VertexCover [27]. It was conjectured in the ARV paper that the integrality gap of the SDP relaxation of (uniform) SPARS-ESTCUT is bounded from above by an absolute constant.<sup>4</sup> Thus, if the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture and/or the ARV-Conjecture were true, one would potentially get a constant factor approximation to a host of problems, and perhaps, an algorithm for VertexCover with an approximation factor better than 2.

#### 1.2 Our Contribution

The main contribution of this paper is the disproval of the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture. This is an immediate corollary of the following theorem which proves the existence of an appropriate integrality gap instance for non-uniform Balancedede-Separator. See Section 2 for a formal description of the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture, the non-uniform Balancedede-Separator problem and its SDP relaxation, and how constructing an integrality gap for non-uniform Balancedede-Separator implies an integrality gap for non-uniform Sparsestout and, thus, disproves the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture.

**Theorem 1.1 (Integrality Gap Instance for Balanced Edge-Separator)** Non-uniform BALANCEDEDGE-SEPARATOR has an integrality gap of at least  $(\log\log n)^{1/6-\delta}$ , where  $\delta>0$  is an arbitrarily small constant. The integrality gap holds for a standard SDP relaxation with the triangle inequality constraints.

**Theorem 1.2**  $((\ell_2^2, \ell_1, O(1))$ -Conjecture is False) For an arbitrarily small constant  $\delta > 0$ , for all sufficiently large n, there is an n-point  $\ell_2^2$  metric which cannot be embedded into  $\ell_1$  with distortion less than  $(\log \log n)^{1/6-\delta}$ .

A surprising aspect of our integrality gap construction is that it proceeds via the Unique Games Conjecture (UGC) of Khot [28] (see Section 3 for the statement of the conjecture). We first prove that the UGC implies a super-constant hardness result for non-uniform BALANCEDEDGE-SEPARATOR.

**Theorem 1.3 (UG-Hardness for Balanced Edge-Separator)** Assuming the Unique Games Conjecture, non-uniform BalancedEdge-Separator is NP-hard to approximate within any constant factor.

This particular result was also proved independently by Chawla et al. [14]. Note that this result leads to the following implication: If the UGC is true and  $P \neq NP$ , then the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture must be false! This is a rather peculiar situation, because the UGC is still unproven, and may very well be false. Nevertheless,

<sup>&</sup>lt;sup>3</sup>This implies, in particular, that every *n*-point  $\ell_1$  metric embeds into  $\ell_2$  with distortion  $O(\sqrt{\log n} \log \log n)$ , almost matching decades old  $\Omega(\sqrt{\log n})$  lower bound due to Enflo [19].

<sup>&</sup>lt;sup>4</sup>The  $(\ell_2^2, \ell_1, O(1))$ -Conjecture implies the same also for the non-uniform version.

we are able to disprove the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture *unconditionally*. Indeed, the UGC plays a crucial role in our disproval. Let us outline the high-level approach we take. First, we build an integrality gap instance for a natural SDP relaxation of UNIQUEGAMES (see Figure 4). We then *translate* this integrality gap instance into an integrality gap instance of non-uniform BALANCEDEDGE-SEPARATOR. This translation *mimics* the PCP reduction from the UGC to this problem.

The integrality gap instance for the UNIQUEGAMES SDP relaxation (see Figure 4) is stated below and is one of our main contributions. Here, we choose to provide an informal description of this construction (the reader should be able to understand this construction without even looking at the SDP relaxation).

**Theorem 1.4 (Integrality Gap Instance for Unique Games-** *Informal Statement)* Let N be an integer and  $\eta > 0$  be a parameter (think of N as large and  $\eta$  as tiny). There is a graph G(V,E) of size  $2^N/N$  with the following properties: Every vertex  $u \in V$  is assigned a set of unit vectors  $B(u) \stackrel{\text{def}}{=} \{u_1, \dots, u_N\}$  that form an orthonormal basis for the space  $\mathbb{R}^N$ . Further,

- 1. For every edge  $e\{u,v\} \in E$ , the sets of vectors B(u) and B(v) are almost the same up to some small perturbation. To be precise, there is a permutation  $\pi_e : [N] \mapsto [N]$ , such that  $\forall \ 1 \le i \le N$ ,  $\langle u_{\pi_e(i)}, v_i \rangle \ge 1 \eta$ . In other words, for every edge  $(u,v) \in E$ , the basis B(u) moves smoothly/continuously to the basis B(v).
- 2. For any labeling  $\lambda: V \mapsto [N]$ , i.e., assignment of an integer  $\lambda(u) \in [N]$  to every  $u \in V$ , for at least  $1 1/N^\eta$  fraction of the edges  $e\{u,v\} \in E$ , we have  $\lambda(u) \neq \pi_e(\lambda(v))$ . In other words, no matter how we choose to assign a vector  $u_{\lambda(u)} \in B(u)$  for every vertex  $u \in V$ , the movement from  $u_{\lambda(u)}$  to  $v_{\lambda(v)}$  is discontinuous for almost all edges  $e\{u,v\} \in E$ .
- 3. All vectors in  $\bigcup_{u \in V} B(u)$  have coordinates in the set  $\{1/\sqrt{N}, -1/\sqrt{N}\}$  and, hence, any three of them satisfy the triangle inequality constraint.

This UNIQUEGAMES integrality gap instance construction is rather non-intuitive (at least to the authors when this paper was first written): One can walk on the graph G by changing the basis B(u) continuously, but as soon as one picks a *representative vector* for each basis, the motion becomes discontinuous almost everywhere. Of course, one can pick these representatives in a continuous fashion for any small enough local sub-graph of G, but there is no way to pick representatives in a global fashion.

Before we present a high-level overview of our proofs and discuss the difficulties involved, we give a brief overview of related and subsequent works since the publication of our paper in 2005.

#### 1.3 Subsequent Works

For non-uniform BALANCEDEDGE-SEPARATOR and, hence, non-uniform SPARSESTCUT, our lower bound was improved to  $\Omega(\log\log n)$  by Krauthgamer and Rabani [33] and then to  $(\log n)^{\Omega(1)}$  in a sequence of papers by Lee and Naor [34] and Cheeger, Kleiner and Naor [15, 16]. For the uniform case, Devanur et al. [17] obtained the first super-constant lower bound of  $\Omega(\log\log n)$ , thus, disproving the ARV conjecture as well. This latter bound has been recently improved to  $2^{\Omega(\sqrt{\log\log n})}$  by Kane and Meka [26], building on the *short code* construction of Barak et al. [9]. At a high level, the constructions in [33, 17] are in the same spirit as ours<sup>5</sup> whereas the constructions in [34, 15, 16] are entirely different, based on the geometry of Heisenberg group.

An unsatisfactory aspect of our construction (and the subsequent ones in [33, 17]) is that the feasibility of the triangle inequality constraints is proved in a brute-force manner with little intuition. A more intuitive

<sup>&</sup>lt;sup>5</sup>Both [33] and [17] use a result of Kahn, Kalai and Linial [25] instead of Bourgain (Theorem 2.14) as in our paper.

proof along with more general results is obtained by Raghavendra and Steurer [42] and Khot and Saket [31]. As a non-embeddability result, these papers present an  $\ell_2^2$  metric that requires super-constant distortion to embed into  $\ell_1$ , but in addition, every sub-metric of it on a super-constant number of points is isometrically embeddable into  $\ell_1$ . The result of Kane and Meka also shares this stronger property. We remark that the Kane and Meka result can be viewed as a derandomization of results in our paper and those in [33, 17, 42, 31].

In hindsight, our paper may be best viewed as a scheme that translates a UGC-based hardness result into an integrality gap for a SDP relaxation with triangle inequality constraints. In the conference version of our paper [32], we applied this scheme to the MAXCUT and MINUNCUT problems as well. In particular, for MAXCUT, we showed that the integrality gap for the Goemans and Williamson's SDP relaxation [24] remains unchanged even after adding triangle inequality constraints. Subsequent works of Raghavendra and Steurer [42] and Khot and Saket [31] cited above extend this paradigm in two directions: Firstly, their SDP solution satisfies additional constraints given by a super-constant number of rounds of the so-called Sherali-Adams LP hierarchy and secondly, they demonstrate that the paradigm holds for every constraint satisfaction problem (CSP). Since these two works already present more general results and in a more intuitive manner, we omit our results for MAXCUT and MINUNCUT from this paper and keep the overall presentation cleaner by restricting only to SPARSESTCUT.

Further, a result of Raghavendra [41] shows that the integrality gap for a certain *canonical* SDP relaxation can be *translated* into a UGC-based hardness result with the same gap (this is a *translation* in the opposite direction as ours). Combined with the results in [42, 31], one concludes that the integrality gap for the basic SDP relaxation remains unchanged even after adding a super-constant number of rounds of the Sherali-Adams LP relaxation. Finally, our techniques have inspired integrality gap for problems that are strictly speaking not CSPs, e.g., integrality gap for the QUADRATICPROGRAMMING problem in [3, 30] and some new non-embeddability results, e.g., for the edit distance [29].

#### **Rest of the Introduction**

In Section 1.5, we give a high level overview of our  $\ell_2^2$  vs.  $\ell_1$  lower bound. The construction is arguably unusual and so is the construction of Lee and Naor [34] which is based on the geometry of Heisenberg group. The latter construction also needs rather involved mathematical machinery to prove its correctness, see [16]. In light of this, it seems worthwhile to point out the difficulties faced by the researchers towards proving the lower bound. Our discussion in Section 1.4 is informal, without precise statements or claims.

# 1.4 Difficulty in Proving $\ell_2^2$ vs. $\ell_1$ Lower Bound

**Difficulty in constructing**  $\ell_2^2$  **metrics:** To the best of our knowledge, no *natural* or *obvious* families of  $\ell_2^2$  metrics are known other than the Hamming metric on  $\{-1,1\}^k$ . The Hamming metric is an  $\ell_1$  metric and, hence, not useful for the purposes of obtaining  $\ell_1$  lower bounds. Certain  $\ell_2^2$  metrics can be constructed via Fourier analysis and one can also construct some by solving SDPs explicitly. The former approach has a drawback that metrics obtained via Fourier methods typically embed into  $\ell_1$  isometrically. The latter approach has limited scope, since one can only hope to solve SDPs of moderate size. Feige and Schechtman [22] show that selecting an appropriate number of points from the unit sphere gives an  $\ell_2^2$  metric. However, in this case, most pairs of points have distance  $\Omega(1)$  and, hence, the metric is likely to be  $\ell_1$ -embeddable with low distortion.

**Difficulty in proving**  $\ell_1$  **lower bounds:** The techniques to prove an  $\ell_1$ -embedding lower bound are limited. To the best of our knowledge, prior to this paper, the only interesting (super-constant) lower bound was due

to [7, 37], where it is shown that the shortest path metric on a constant degree expander requires  $\Omega(\log n)$  distortion to embed into  $\ell_1$ .

General theorems regarding group norms: A *group norm* is a distance function  $d(\cdot, \cdot)$  on a group  $(G, \circ)$ , such that d(x,y) depends only on the group difference  $x \circ y^{-1}$ . Using Fourier methods, it is possible to construct group norms that are  $\ell_2^2$  metrics. However, it is known that any group norm on  $\mathbb{R}^k$ , or on any group of characteristic 2, is isometrically  $\ell_1$ -embeddable (see [18]). Such a result might hold, perhaps allowing a small distortion, for every Abelian group (see [8]). Therefore, an approach via group norms would probably not succeed as long as the underlying group is Abelian. On the other hand, only in the Abelian case, Fourier methods work well.

The best known lower bounds for the  $\ell_2^2$  versus  $\ell_1$  question, prior to this paper, were due to Vempala (10/9 for a metric obtained by a computer search) and Goemans (1.024 for a metric based on the Leech Lattice), see [44]. Thus, it appeared that an entirely new approach was needed to resolve the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture. In this paper, we present an approach based on tools from complexity theory, namely, the UGC, PCPs, and Fourier analysis of Boolean functions. Interestingly, Fourier analysis is used both to construct the  $\ell_2^2$  metric, as well as, to prove the  $\ell_1$  lower bound.

## 1.5 Overview of Our $\ell_2^2$ vs. $\ell_1$ Lower Bound

In this section, we present a high level idea of our  $\ell_2^2$  versus  $\ell_1$  lower bound, i.e., Theorem 1.2. Given the construction of Theorem 1.4, it is fairly straight-forward to describe the candidate  $\ell_2^2$  metric: Let G(V, E) be the graph, and B(u) be the orthonormal basis for  $\mathbb{R}^N$  for every  $u \in V$  as in Theorem 1.4. For  $u \in V$  and  $x = (x_1, \dots, x_N) \in \{-1, 1\}^N$ , define the vector  $\mathbf{V}_{u,x}$  as follows:

$$\mathbf{V}_{u,x} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i u_i^{\otimes 8}.$$
 (1)

Note that since  $B(u) = \{u_1, \dots, u_N\}$  is an orthonormal basis for  $\mathbb{R}^N$ , every  $\mathbf{V}_{u,x}$  is a unit vector. Fix t to be a large odd integer, for instance  $2^{240} + 1$ , and consider the set of unit vectors

$$\mathscr{S} \stackrel{\text{def}}{=} \left\{ \mathbf{V}_{u,x}^{\otimes t} \mid u \in V, \ x \in \{-1,1\}^N \right\}.$$

Using, essentially, the fact that the vectors in  $\bigcup_{u \in V} B(u)$  are a *good* solution to the SDP relaxation of UNIQUEGAMES, we are able to show that every triple of vectors in  $\mathscr S$  satisfy the triangle inequality constraint and, hence,  $\mathscr S$  defines an  $\ell_2^2$  metric. One can also directly show that this  $\ell_2^2$  metric does not embed into  $\ell_1$  with distortion less than  $(\log N)^{1/6-\delta}$ .

However, we choose to present our construction in a different and an indirect way. The (lengthy) presentation goes through the UGC and the PCP reduction from UNIQUEGAMES integrality gap instance to BALANCEDEDGE-SEPARATOR. Hopefully, our presentation brings out the intuition as to why and how we came up with the above set of vectors, which happened to define an  $\ell_2^2$  metric. At the end, the reader should recognize that the idea of taking all +/- linear combinations of vectors in B(u) (as in Equation (1)) is directly inspired by the PCP reduction. Also, the proof of the  $\ell_1$  lower bound is hidden inside the *soundness analysis* of the PCP.

The overall construction can be divided into three steps:

<sup>&</sup>lt;sup>6</sup>We develop a Fourier analytic technique to prove an  $\ell_1$ -embedding lower bound that has been subsequently used in [33, 17, 29]. The approach of [15, 16] gives another technique, by developing an entire new theory of  $\ell_1$ -differentiability and its quantitative version.

<sup>&</sup>lt;sup>7</sup>For a vector  $x \in \mathbb{R}^N$  and an integer l, the l-th tensor of x,  $y \stackrel{\text{def}}{=} x^{\otimes l}$ , is a vector in  $(\mathbb{R}^N)^l$  defined such that for  $i_1, i_2, \dots, i_l \in [N]$ ,  $y_{i_1, i_2, \dots, i_l} \stackrel{\text{def}}{=} x_{i_1} x_{i_2} \cdots x_{i_l}$ . It follows that for  $x, z \in \mathbb{R}^N$ ,  $\langle x^{\otimes l}, z^{\otimes l} \rangle = \sum_{i_1, i_2, \dots, i_l \in [N]} (x_{i_1} x_{i_2} \cdots x_{i_l}) (z_{i_1} z_{i_2} \cdots z_{i_l}) = \left( \sum_{i \in [N]} x_i z_i \right)^l = \langle x, z \rangle^l$ .

- 1. A PCP reduction from UNIQUEGAMES to BALANCEDEDGE-SEPARATOR.
- 2. Constructing an integrality gap instance for a natural SDP relaxation of UNIQUEGAMES.
- 3. Combining the above two to construct an integrality gap instance of BALANCEDEDGE-SEPARATOR. This also gives an  $\ell_2^2$  metric that needs  $(\log \log n)^{1/6-\delta}$  distortion to embed into  $\ell_1$ .

We present an overview of each of these steps in three separate sections. Before we do that, let us summarize the precise notion of an integrality gap instance of BALANCEDEDGE-SEPARATOR. To keep things simple in this exposition, we pretend as if our construction works for the uniform version of BALANCEDEDGE-SEPARATOR as well. (Actually it does not; we have to work with the non-uniform version which complicates things a little.)

#### SDP Relaxation of Balanced Edge-Separator

Given a graph G'(V', E'), BALANCEDEDGE-SEPARATOR asks for a (1/2, 1/2)-partition of V' that cuts as few edges as possible (however, the algorithm is allowed to output a roughly balanced partition, say (1/4,3/4)partition). We denote an edge e between vertices i, j by  $e\{i, j\}$ . The SDP relaxation of BALANCEDEDGE-SEPARATOR appears in Figure 1.

Minimize 
$$\frac{1}{|E'|} \sum_{e' \{i,j\} \in E'} \frac{1}{4} ||v_i - v_j||^2$$
 (2)

Subject to

$$\forall i \in V' \qquad ||v_i||^2 = 1 \tag{3}$$

$$\forall i \in V' \qquad ||v_i||^2 = 1$$

$$\forall i, j, l \in V' \quad ||v_i - v_j||^2 + ||v_j - v_l||^2 \ge ||v_i - v_l||^2$$
(4)

$$\sum_{i < j} ||v_i - v_j||^2 \ge |V'|^2 \tag{5}$$

Figure 1: SDP relaxation of the uniform version of BALANCEDEDGE-SEPARATOR

Note that a  $\{+1,-1\}$ -valued solution represents a true partition and, hence, this is an SDP relaxation. Constraint (4) is the triangle inequality constraint and Constraint (5) stipulates that the partition be balanced.<sup>8</sup> The notion of integrality gap is summarized in the following definition:

**Definition 1.5** An integrality gap instance of BALANCEDEDGE-SEPARATOR is a graph G'(V', E') and an assignment of unit vectors  $i \mapsto v_i$  to its vertices such that:

- Every balanced partition (say (1/4,3/4)-partition, this choice is arbitrary) of V' cuts at least  $\alpha$  fraction of edges.
- The set of vectors  $\{v_i | i \in V'\}$  satisfy (3)-(5), and the SDP objective value in Equation (2) is at most  $\gamma$ .

The integrality gap is defined to be  $\alpha/\gamma$  (thus, we desire that  $\gamma \ll \alpha$ ).

The next three sections describe the three steps involved in constructing an integrality gap instance of BALANCEDEDGE-SEPARATOR. Once that is done, it follows from a folk-lore result that the resulting  $\ell_2^2$  metric (defined by vectors  $\{v_i | i \in V'\}$ ) requires distortion at least  $\Omega(\alpha/\gamma)$  to embed into  $\ell_1$ . This would prove Theorem 1.2 with an appropriate choice of parameters.

<sup>&</sup>lt;sup>8</sup>Notice that if a set of vectors  $\{v_i: i \in V'\}$  is such that for every vector in the set, its antipode is also in the set, then constraint (5) is automatically satisfied. Our construction obeys this property.

#### The PCP Reduction from Unique Games to Balanced Edge-Separator

An instance  $\mathscr{U} = (G(V, E), [N], \{\pi_e\}_{e \in E})$  of UNIQUEGAMES consists of a graph G(V, E) and permutations  $\pi_e : [N] \mapsto [N]$  for every edge  $e\{u, v\} \in E$ . The goal is to find a *labeling*  $\lambda : V \mapsto [N]$  that *satisfies* as many edges as possible. An edge  $e\{u, v\}$  is satisfied if  $\lambda(u) = \pi_e(\lambda(v))$ . Let opt( $\mathscr{U}$ ) denote the maximum fraction of edges satisfied by any labeling.

UGC (Informal Statement): It is NP-hard to decide whether an instance  $\mathscr{U}$  of UNIQUEGAMES has  $opt(\mathscr{U}) \ge 1 - \eta$  (YES instance) or  $opt(\mathscr{U}) \le \zeta$  (NO instance), where  $\eta, \zeta > 0$  can be made arbitrarily small by choosing N to be a sufficiently large constant.

It is possible to construct an instance of BALANCEDEDGE-SEPARATOR  $G'_{\varepsilon}(V',E')$  from an instance of UNIQUEGAMES. We describe only the high level idea here. The construction is parameterized by  $\varepsilon > 0$ . The graph  $G'_{\varepsilon}$  has a block of  $2^N$  vertices for every  $u \in V$ . This block contains one vertex for every point in the Boolean hypercube  $\{-1,1\}^N$ . Denote the set of these vertices by V'[u]. More precisely,

$$V'[u] \stackrel{\text{def}}{=} \{(u,x) \mid x \in \{-1,1\}^N\}.$$

We let  $V' \stackrel{\text{def}}{=} \cup_{u \in V} V'[u]$ . For every edge  $e\{u,v\} \in E$ , the graph  $G'_{\varepsilon}$  has edges between the blocks V'[u] and V'[v]. These edges are supposed to capture the constraint that the labels of u and v are consistent, i.e.,  $\lambda(u) = \pi_e(\lambda(v))$ . Roughly speaking, a vertex  $(u,x) \in V'[u]$  is connected to a vertex  $(v,y) \in V'[v]$  if and only if, after identifying the coordinates in [N] via the permutation  $\pi_e$ , the Hamming distance between the bit-strings x and y is about  $\varepsilon N$ . This reduction has the following two properties:

#### **Theorem 1.6** (PCP reduction: Informal statement)

- 1. (Completeness/YES case): If opt( $\mathscr{U}$ )  $\geq 1 \eta$ , then the graph  $G'_{\varepsilon}$  has a (1/2, 1/2)-partition that cuts at most  $\eta + \varepsilon$  fraction of its edges.
- 2. (Soundness/NO Case): If  $opt(\mathcal{U}) \leq 2^{-O(1/\epsilon^2)}$ , then every (1/4, 3/4)-partition of  $G'_{\varepsilon}$  cuts at least  $\sqrt{\varepsilon}$  fraction of its edges.

Remark 1.7 We were imprecise on two counts: (1) The soundness property holds only for those partitions that partition a constant fraction of the blocks V'[u] in a roughly balanced way. We call such partitions piecewise balanced. This is where the issue of uniform versus non-uniform version of BALANCEDEDGE-SEPARATOR arises. (2) For the soundness property, we can only claim that every piecewise balanced partition cuts at least  $\varepsilon^t$  fraction of edges, where any t > 1/2 can be chosen in advance. Instead, we write  $\sqrt{\varepsilon}$  for the simplicity of notation.

#### **Integrality Gap Instance for the Unique Games SDP Relaxation**

This has already been described in Theorem 1.4. The graph G(V,E) therein along with the orthonormal basis B(u), for every  $u \in V$ , can be used to construct an instance  $\mathscr{U} = (G(V,E),[N],\{\pi_e\}_{e\in E})$  of UNIQUEGAMES. For every edge  $e\{u,v\} \in E$ , we have an (unambiguously defined) permutation  $\pi_e:[N] \mapsto [N]$ , where  $\langle u_{\pi_e(i)}, v_i \rangle \geq 1 - \eta$ , for all  $1 \leq i \leq N$ .

Theorem 1.4 implies that  $\operatorname{opt}(\mathscr{U}) \leq 1/N^{\eta}$ . On the other hand, the fact that for every edge  $e\{u,v\}$ , the bases B(u) and B(v) are very close to each other means that the SDP objective value for  $\mathscr{U}$  is at least  $1-\eta$  (formally, the SDP objective value is defined to be  $\mathbb{E}_{e\{u,v\}\in E}\left[\frac{1}{N}\sum_{i=1}^N\langle u_{\pi_e(i)},v_i\rangle\right]$ ).

Thus, we have a concrete instance of UNIQUEGAMES with optimum at most  $1/N^{\eta} = o(1)$ , and which has an SDP solution with objective value at least  $1 - \eta$ . This is what an integrality gap example means: The SDP solution *cheats* in an *unfair* way.

#### Integrality Gap Instance for the Balanced Edge-Separator SDP Relaxation

Now we combine the two modules described above. We take the instance  $\mathscr{U} = (G(V,E),[N],\{\pi_e\}_{e\in E})$  as above and run the PCP reduction on it. This gives us an instance G'(V',E') of BALANCEDEDGE-SEPARATOR. We show that this is an integrality gap instance in the sense of Definition 1.5.

Since  $\mathscr{U}$  is a NO instance of UNIQUEGAMES, i.e.,  $\operatorname{opt}(\mathscr{U}) = o(1)$ , Theorem 1.6 implies that every (piecewise) balanced partition of G' must cut at least  $\sqrt{\varepsilon}$  fraction of the edges. We need to have  $1/N^{\eta} \le 2^{-O(1/\varepsilon^2)}$  for this to hold.

On the other hand, we can construct an SDP solution for the BALANCEDEDGE-SEPARATOR instance which has an objective value of at most  $O(\eta + \varepsilon)$ . Note that a typical vertex of G' is (u,x), where  $u \in V$  and  $x \in \{-1,1\}^N$ . To this vertex, we attach the unit vector  $\mathbf{V}_{u,x}^{\otimes t}$  (for  $t = 2^{240} + 1$ ), where

$$\mathbf{V}_{u,x} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i u_i^{\otimes 8}.$$

It can be shown that the set of vectors  $\left\{\mathbf{V}_{u,x}^{\otimes t} \mid u \in V, \ x \in \{-1,1\}^N\right\}$  satisfy the triangle inequality constraint and, hence, defines an  $\ell_2^2$  metric. Vectors  $\mathbf{V}_{u,x}^{\otimes t}$  and  $\mathbf{V}_{u,-x}^{\otimes t}$  are antipodes of each other and, hence, the SDP Constraint (5) is also satisfied. Finally, we show that the SDP objective value (Expression (2)) is  $O(\eta + \varepsilon)$ . It suffices to show that for every edge ((u,x),(v,y)) in G'(V',E'), we have

$$\langle \mathbf{V}_{u,x}^{\otimes t}, \mathbf{V}_{v,y}^{\otimes t} \rangle \geq 1 - O(t(\eta + \varepsilon)).$$

This holds because whenever ((u,x),(v,y)) is an edge of G', we have (after identifying the indices via the permutation  $\pi_e:[N]\mapsto[N]$ ):

- 1.  $\langle u_{\pi_{\sigma}(i)}, v_i \rangle \geq 1 \eta$  for all  $1 \leq i \leq N$  and
- 2. the Hamming distance between x and y is about  $\varepsilon N$ .

#### **Quantitative Parameters**

It follows from above discussion (see also Definition 1.5) that the integrality gap for BALANCEDEDGE-SEPARATOR is  $\Omega(1/\sqrt{\varepsilon})$  provided that  $\eta \approx \varepsilon$ , and  $N^{\eta} > 2^{O(1/\varepsilon^2)}$ . We can choose  $\eta \approx \varepsilon \approx (\log N)^{-1/3}$ . Since the size of the graph G' is at most  $n = 2^{2N}$ , we see that the integrality gap is  $\approx (\log \log n)^{1/6}$  as desired.

#### **Proving the Triangle Inequality**

As mentioned above, one can show that the set of vectors  $\{\mathbf{V}_{u,x}^{\otimes t} \mid u \in V, x \in \{-1,1\}^N\}$  satisfy the triangle inequality constraints. This is the most technical part of the paper, but we would like to stress that this is where the *magic* happens. In our construction, all vectors in  $\bigcup_{u \in V} B(u)$  happen to be points of the hypercube  $\{-1,1\}^N$  (up to a normalizing factor of  $1/\sqrt{N}$ ), and therefore, they define an  $\ell_1$  metric. The operation that takes their +/- combinations combined with tensoring leads to a metric that is  $\ell_2^2$  and non- $\ell_1$ -embeddable.

Our proof of the triangle inequality constraints is essentially brute-force. As we mentioned before, more recent works [42, 31] obtain a more intuitive proof.

#### 1.6 Organization of the Main Body of the Paper

In Section 2.1 we recall important definitions and results about metric spaces. Section 2.2 defines the cut optimization problems we are concerned about: SPARSESTCUT and BALANCEDEDGE-SEPARATOR. We

also give their SDP relaxations for which we construct integrality gap instances. Section 2.5 presents useful tools from Fourier analysis.

In Section 2.4, we present our overall strategy for disproving the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture. We give a disproval of the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture assuming an appropriate integrality gap for BALANCEDEDGE-SEPARATOR.

In Section 3 we present the UGC and our integrality gap instance for an SDP relaxation of UNIQUEGAMES.

In Section 4 we present our PCP reduction from UNIQUEGAMES to BALANCEDEDGE-SEPARATOR. The soundness proof this reduction is standard and appears in Appendix A.

We build on the UNIQUEGAMES integrality gap instance in Section 3 and the PCP reduction in Section 4 to obtain the integrality gap instance for BALANCEDEDGE-SEPARATOR. This is presented in Section 5. This section has two parts: In the first part (Section 5.1) we present the graph and in the second part (Section 5.2) we present the corresponding SDP solution and prove its properties.

Appendix B is where we establish the main technical lemma needed to show that the SDP solutions we construct satisfy the triangle inequality constraint.

#### 2 Preliminaries

## **2.1** The $(\ell_2^2, \ell_1, O(1))$ -Conjecture

We start with basics of metric embeddings. We are concerned with finite metric spaces which we denote by a pair (X,d), where X is the space and d is the metric on its points. We say that a space  $(X_1,d_1)$  embeds with distortion at most  $\Gamma$  into another space  $(X_2,d_2)$  if there exists a map  $\phi: X_1 \mapsto X_2$  such that for all  $x,y \in X_1$ 

$$d_1(x,y) \le d_2(\phi(x),\phi(y)) \le \Gamma \cdot d_1(x,y).$$

If  $\Gamma = 1$ , then  $(X_1, d_1)$  is said to *isometrically* embed in  $(X_2, d_2)$ .

An important class of metric spaces are those that arise by taking a finite subset X of  $\mathbb{R}^m$  for some  $m \ge 1$  and endowing it with the  $\ell_p$  norm as follows: For  $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in X$ ,

$$\ell_p(x,y) \stackrel{\text{def}}{=} \left( \sum_{i=1}^m |x_i - y_i|^p \right)^{1/p}.$$

When we call a metric  $\ell_1$  or  $\ell_2$ , an implicit underlying space is assumed.

A metric space (X,d) is said to be of negative type if  $(X,\sqrt{d})$  embeds isometrically into  $\ell_2$ . Formally, there is an integer m and a vector  $v_x \in \mathbb{R}^m$  for every  $x \in X$ , such that  $d(x,y) = \|v_x - v_y\|^2$  and the vectors satisfy the *triangle inequality*, i.e., for all  $x,y,z \in X$ ,

$$||v_x - v_y||^2 + ||v_y - v_z||^2 \ge ||v_x - v_z||^2.$$

The class of all negative type metrics is denoted by  $\ell_2^2$ . The following fact is easy to prove.

**Fact 2.1** [18] For every  $\ell_1$  metric space  $(X,\ell_1)$  there is a negative type metric space (Y,d) in which it embeds isometrically.

While the converse is not true, the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture asserts that the converse holds up to a universal constant.

**Conjecture 2.2**  $((\ell_2^2, \ell_1, O(1))$ -**Conjecture, [23, 36]**) For every negative type metric space (Y, d) there is a metric space  $(X, \ell_1)$  in which it embeds with at most a constant distortion. This constant is universal, i.e., independent of the metric space (Y, d).

#### 2.2 Balanced Edge-Separator, Sparsest Cut and their SDP Relaxations

In this section, we define the BALANCEDEDGE-SEPARATOR and the SPARSESTCUT problems and their SDP relaxations. All graphs are complete undirected graphs with non-negative *weights* or *demands* associated to its edges. For a graph G(V,E) and  $S \subseteq V$ , let  $E(S,\overline{S})$  denote the set of edges with one endpoint in S and other in  $\overline{S}$ . A cut  $(S,\overline{S})$  is called non-trivial if  $S \neq \emptyset$  and  $\overline{S} \neq \emptyset$ .

**Remark 2.3** The versions of SparsestCut and BalancedEdge-Separator that we define below are non-uniform versions with demands. The uniform version has all demands equal to 1, i.e., unit demand for every pair of vertices.

**Definition 2.4** (SPARSESTCUT) For a graph G(V, E) with a weight wt(e) and a demand dem(e) associated to each edge  $e \in E$ , the goal is to optimize

$$\min_{\emptyset \neq S \subsetneq V} \frac{\sum_{e \in E(S,\overline{S})} \operatorname{wt}(e)}{\sum_{e \in E(S,\overline{S})} \operatorname{dem}(e)}.$$

For a cut  $(S, \overline{S})$ , the ratio above is referred to as its sparsity.

The SDP relaxation for SPARSESTCUT appears in Figure 2. We note that this is indeed a relaxation: Any cut  $(S, \overline{S})$  corresponds to a feasible SDP solution by setting the vector  $v_x$  to be  $v_0$  or  $-v_0$  depending on whether  $x \in S$  or  $x \in \overline{S}$  and  $v_0$  is some fixed vector. The length of  $v_0$  is chosen so as to satisfy the last SDP constraint. The SDP objective is then the same as the sparsity of the cut.

Minimize 
$$\frac{1}{4} \sum_{e \in \{x,y\}} \operatorname{wt}(e) \|v_x - v_y\|^2$$

Subject to

$$\forall x, y, z \in V \quad \|v_x - v_y\|^2 + \|v_y - v_z\|^2 \ge \|v_x - v_z\|^2$$

$$\frac{1}{4} \sum_{e \in Y} |v_y| \operatorname{dem}(e) \|v_x - v_y\|^2 = 1$$

Figure 2: SDP relaxation of SPARSESTCUT

The integrality gap of this SDP relaxation is defined to be the largest ratio, as a function of the number of vertices n and over all possible instances, between the integral optimum and the SDP optimum. It is known (folklore) that the integrality gap f(n) of the SPARSESTCUT SDP relaxation is precisely the worst case distortion incurred to embed an n-point  $\ell_2^2$  metric into  $\ell_1$ . We need this observation (but only in one direction) in what follows. First, we formally introduce BALANCEDEDGE-SEPARATOR.

**Definition 2.5** (BALANCEDEDGE-SEPARATOR) For a graph G(V,E) with a weight  $\operatorname{wt}(e)$ , and a demand  $\operatorname{dem}(e)$  associated to each edge  $e \in E$ , let  $D \stackrel{\text{def}}{=} \sum_{e \in E} \operatorname{dem}(e)$  be the total demand. Let a balance parameter B be given where  $D/6 \leq B \leq D/2$ . The goal is to find a non-trivial cut  $(S,\overline{S})$  that minimizes  $\sum_{e \in E(S,\overline{S})} \operatorname{wt}(e)$ , subject to  $\sum_{e \in E(S,\overline{S})} \operatorname{dem}(e) \geq B$ . The cuts that satisfy  $\sum_{e \in E(S,\overline{S})} \operatorname{dem}(e) \geq B$  are called B-balanced cuts.

The SDP relaxation for BALANCEDEDGE-SEPARATOR appears in Figure 3. We note that this is indeed a relaxation: A *B*-balanced cut  $(S, \overline{S})$  corresponds to a feasible SDP solution by setting the vector  $v_x$  to be  $v_0$  or  $-v_0$  depending on whether  $x \in S$  or  $x \in \overline{S}$  and  $v_0$  is a fixed *unit* vector.

Minimize 
$$\frac{1}{4} \sum_{e\{x,y\}} \text{wt}(e) ||v_x - v_y||^2$$
 (6)

Subject to

$$\forall x \in V \qquad \|v_x\|^2 = 1 \tag{7}$$

$$\forall x \in V \qquad ||v_x||^2 = 1$$

$$\forall x, y, z \in V \quad ||v_x - v_y||^2 + ||v_y - v_z||^2 \ge ||v_x - v_z||^2$$
(8)

$$\frac{1}{4} \sum_{e\{x,y\}} \text{dem}(e) \|v_x - v_y\|^2 \ge B \tag{9}$$

Figure 3: SDP relaxation of BALANCEDEDGE-SEPARATOR with parameter B

An integrality gap instance for BALANCEDEDGE-SEPARATOR is a concrete instance along with a feasible B-balanced SDP solution such that the SDP objective is at most  $\gamma$  and the integral optimum over B/3-balanced cuts is at least  $\alpha$ . The integrality gap is  $\alpha/\gamma$ . Note that the SDP solution is B-balanced (in the sense of the last SDP constraint), but the integral optimum is allowed over B/3-balanced cuts, i.e., over a larger class of cuts than the B-balanced cuts.

# Relation Between $(\ell_2^2, \ell_1, O(1))$ -Conjecture, Sparsest Cut and Balanced Edge-Separator

Consider the following three statements:

- 1. Every *n*-point  $\ell_2^2$  metric embeds into  $\ell_1$  with distortion at most f(n).
- 2. The integrality gap of the SPARSESTCUT SDP relaxation is at most f(n).
- 3. The integrality gap of the BALANCEDEDGE-SEPARATOR SDP relaxation is at most O(f(n)).

It is known (folklore) that  $(1) \implies (2) \implies (3)$  (and in fact (1) is equivalent to (2)). We use the implication (1)  $\implies$  (3) to conclude our  $\ell_2^2$  vs.  $\ell_1$  lower bound from our integrality gap construction for BALANCEDEDGE-SEPARATOR. We summarize this implication below and present a sketch of its proof for the sake of completeness. The proof implicitly also proves the implication  $(1) \Longrightarrow (2)$ .

**Lemma 2.6** Suppose  $x \mapsto v_x$  is a solution for SDP of Figure 3 with objective value

$$\frac{1}{4}\sum_{e\{x,y\}}\operatorname{wt}(e)\|v_x-v_y\|^2\leq \varepsilon.$$

Assume that the negative type metric defined by the vectors  $\{v_x | x \in V\}$  embeds into  $\ell_1$  with distortion f(n)where n = |V|. Then, there exists a B'-balanced cut  $(S, \overline{S})$ ,  $B' \ge B/3$  such that

$$\sum_{e \in E(S,\overline{S})} \operatorname{wt}(e) \leq O(f(n) \cdot \varepsilon).$$

**Proof.** The idea is that the *good* SDP solution as given implies the existence of a cut with low sparsity. If this cut already cuts  $\Omega(B)$  of the demands, we are done. Otherwise the demands cut are *erased* (i.e., set to zero) and another sparse cut is found w.r.t. to the new (remaining) demands. This process is repeated until the sum of the demands cut in the sequence of cuts obtained so far is at least  $\Omega(B)$ . At this point, a random XOR of the cuts obtained so far yields a cut that cuts  $\Omega(B)$  of the demands, but does not cut too much of the edge weight. Formally, we begin by observing that there is a cut  $(S, \overline{S})$  with sparsity at most  $f(n) \cdot \varepsilon/B$ .

$$\begin{split} \min_{\emptyset \neq S \subsetneq V} \frac{\sum_{e \in E(S,\overline{S})} \operatorname{wt}(e)}{\sum_{e \in E(S,\overline{S})} \operatorname{dem}(e)} &= \min_{d \text{ is } \ell_1 \text{ embeddable}} \frac{\sum_{e \in x,y \}} \operatorname{wt}(e) d(x,y)}{\sum_{e \in x,y \}} \operatorname{dem}(e) d(x,y)} \\ &\leq f(n) \cdot \frac{\sum_{e \in x,y \}} \operatorname{wt}(e) \|v_x - v_y\|^2}{\sum_{e \in x,y \}} \operatorname{dem}(e) \|v_x - v_y\|^2} \leq f(n) \cdot \varepsilon / B. \end{split}$$

The first (in)equality uses the fact that optimizing over cuts is the same as optimizing over the cone of  $\ell_1$  embeddable metrics, see [18]. The second inequality uses the embedding of the metric  $||v_x - v_y||^2$  into  $\ell_1$  with distortion at most f(n). The third inequality uses the hypothesis that the SDP objective is at most  $\varepsilon$  and the SDP solution is B-balanced.

If the cut  $(S, \overline{S})$  happens to be B/3-balanced, then we are done since the edge weight cut by it is at most the sparsity (which is at most  $f(n) \cdot \varepsilon/B$ ) times the demands cut (which is at most  $D \le 6B$ ). Otherwise the demands cut by  $(S, \overline{S})$  is at most B/3. We rename the cut as  $(S_1, \overline{S}_1)$ , set all the demands cut to zero, and repeat the process. This leads to a sequence of cuts  $(S_1, \overline{S}_1), \ldots, (S_k, \overline{S}_k)$ . The process stops as soon as either

- (a) the cut just obtained cuts at least B/3 of the demands or else
- (b) the sum of the demands cut over these k cuts is at least 2B/3 (since a demand is set to zero as soon as it is cut, each original demand is counted at most once).

Note that prior to every step, at most  ${}^{2B}/3$  of the (original) demands has been set to zero, so the SDP solution w.r.t. to the remaining demands still qualifies as being  $B - {}^{2B}/3 = {}^{B}/3$  balanced. Thus, at every step, the cut obtained has sparsity at most  $f(n) \cdot {}^{\varepsilon}/({}^{B}/3)$ . We are done in the Case (a) as before and so we consider the Case (b).

To summarize, we have a sequence of cuts  $(S_1, \overline{S}_1), \ldots, (S_k, \overline{S}_k)$  such that the sum of the demands cut over these k cuts is at least  $2^B/3$ . Moreover, the sparsity of each of these cuts is at most  $O(f(n) \cdot \varepsilon/B)$  and, hence, the total edge weight cut by these cuts is at most  $O(f(n)\varepsilon)$  (an edge is considered cut if it is cut by at least one of the k cuts). Now we obtain our desired *balanced partition* by taking a random XOR of these cuts: The i-th cut is viewed as a  $\{0,1\}$ -valued function  $\phi_i$  on the vertices and the desired cut is given by the function  $\phi_A \stackrel{\text{def}}{=} \bigoplus_{i \in A} \phi_i$  where  $A \subseteq [k]$  is a uniformly random subset. We show that for some choice of the set A, we get a cut  $\phi_A$  that cuts at least B/3 of the demands and at most  $O(f(n)\varepsilon)$  of the edge weight. Clearly, the total edge weight cut is  $O(f(n)\varepsilon)$  irrespective of the set A. On the other hand, each demand in the sum total of at least B/3 gets cut with probability B/3 (this is the property of the random XOR). Thus, the expected demands cut by A/3 is at least B/3 and this expectation is achieved for some choice of A.

Remark 2.7 The proof above shows that if the integrality gap for SPARSESTCUT is upper bounded by f(n) then the gap for BALANCEDEDGE-SEPARATOR is bounded by O(f(n)). The same proof implicitly also shows that if there is an f(n) approximation algorithm for SPARSESTCUT, then the algorithm can be used iteratively a polynomial number of times to achieve O(f(n)) (pseudo-)approximation for BALANCEDEDGE-SEPARATOR, see also [45, Chapter 7]. Given an instance of BALANCEDEDGE-SEPARATOR that has a B-balanced cut that cuts an edge weight  $\alpha$  and  $\beta \geq D/6$  where  $\beta$  is the total demand, the algorithm finds a  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$ . In the contrapositive, a  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive, a  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive, a  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive, a  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  in the contrapositive  $\beta$ -balanced cut that cuts an edge weight  $O(f(n)\alpha)$  is the cut that  $O(f(n)\alpha)$  is th

#### 2.4 Our Integrality Gap Instance for Balanced Edge-Separator

With the preliminaries for negative type metrics and SDPs in place, we now state the main result regarding the construction of the integrality gap for BALANCEDEDGE-SEPARATOR which suffices to disprove the  $(\ell_2^2, \ell_1, O(1))$ -Conjecture using Lemma 2.6. The instance has two parts: (1) The graph and (2) The SDP solution. The graph construction is described in Section 5.1, while the SDP solution appears in Section 5.2. We construct a complete weighted graph  $G(V, \operatorname{wt})$ , with vertex set V and weight  $\operatorname{wt}(e)$  on edge e, and with  $\sum_e \operatorname{wt}(e) = 1$ . The vertex set is partitioned into sets  $V_1, V_2, \ldots, V_r$ , each of size |V|/r (think of  $r \approx \sqrt{|V|}$ ). A cut A in the graph is viewed as a function  $A: V \mapsto \{-1,1\}$ . We are interested in cuts that cut many sets  $V_i$  in a somewhat balanced way. The notation  $s \in_R S$  would mean that s is a uniformly random element of S.

**Definition 2.8** For  $0 \le \theta \le 1$ , a cut  $A : V \mapsto \{-1, 1\}$  is called  $\theta$ -piecewise balanced if

$$\mathbb{E}_{i\in_{R}[r]}\mid \mathbb{E}_{x\in_{R}V_{i}}[A(x)]\mid \leq \theta.$$

We also assign a unit vector to every vertex in the graph. Let  $v_x$  denote the vector assigned to vertex x. Our construction of the graph G(V, wt) and the vector assignment  $x \mapsto v_x$  can be summarized as follows:

**Theorem 2.9 (Main Theorem)** Fix any 1/2 < t < 1. For every sufficiently small  $\varepsilon > 0$ , there exists a graph  $G(V, \operatorname{wt})$ , with a partition  $V = \bigcup_{i=1}^{r} V_i$ , and a vector assignment  $x \mapsto v_x$  for every  $x \in V$ , such that

- 1.  $|V| \leq 2^{2^{O(1/\varepsilon^3)}}$ .
- 2. Every  $\frac{5}{6}$ -piecewise balanced cut A must cut  $\varepsilon^t$  fraction of edges, i.e., for any such cut

$$\sum_{e \in E(A,\overline{A})} \operatorname{wt}(e) \geq \varepsilon^t.$$

3. The unit vectors  $\{v_x \mid x \in V\}$  define a negative type metric, i.e., the following triangle inequality is satisfied:

$$\forall x, y, z \in V, \|v_x - v_y\|^2 + \|v_y - v_z\|^2 \ge \|v_x - v_z\|^2$$
.

4. For each part  $V_i$ , the vectors  $\{v_x \mid x \in V_i\}$  are well-separated, i.e.,

$$\frac{1}{2}\mathbb{E}_{x,y\in_{R}V_{i}}\left[\|v_{x}-v_{y}\|^{2}\right]=1.$$

5. The vector assignment gives a low SDP objective value, i.e.,

$$\frac{1}{4}\sum_{e\{x,y\}}\operatorname{wt}(e)\|v_x-v_y\|^2\leq \varepsilon.$$

**Proof of Theorem 1.2.** We show how the construction in Theorem 2.9 implies Theorem 1.2. Suppose that the negative type metric defined by vectors  $\{v_x | x \in V\}$  embeds into  $\ell_1$  with distortion  $\Gamma$ . We show that  $\Gamma = \Omega(1/\epsilon^{1-t})$  using Lemma 2.6.

Construct an instance of BALANCEDEDGE-SEPARATOR as follows. The graph  $G(V, \operatorname{wt})$  is as in Theorem 2.9. The demands  $\operatorname{dem}(e)$  depend on the partition  $V = \bigcup_{i=1}^r V_i$ . We let  $\operatorname{dem}(e) = 1$  if e has both endpoints in the same part  $V_i$  for some  $1 \le i \le r$  and  $\operatorname{dem}(e) = 0$  otherwise. Clearly, the total demand is  $D \stackrel{\text{def}}{=} \sum_e \operatorname{dem}(e) = r \cdot \binom{|V|/r}{2}$ .

Now,  $x \mapsto v_x$  is an assignment of unit vectors that satisfy the triangle inequality constraints. This is a solution to the SDP of Figure 3. Property (4) of Theorem 2.9 guarantees that

$$\frac{1}{4} \sum_{e = \{x, y\}} \operatorname{dem}(e) ||v_x - v_y||^2 = \frac{1}{4} \cdot r \cdot \binom{|V|/r}{2} \cdot 2 = \frac{D}{2}.$$

Letting  $B \stackrel{\text{def}}{=} D/2$ , the SDP solution is *B*-balanced and its objective value is at most  $\varepsilon$ . Using Lemma 2.6, we get a *B'*-balanced cut  $(A, \overline{A})$ ,  $B' \ge B/3$  such that  $\sum_{e \in E(A, \overline{A})} \text{wt}(e) \le O(\Gamma \cdot \varepsilon)$ .

**Claim**: The cut  $(A, \overline{A})$  must be a  $\frac{5}{6}$ -piecewise balanced cut.

**Proof of Claim.** Let  $p_i \stackrel{\text{def}}{=} \Pr_{x \in V_i}[A(x) = 1]$ . The total demand cut by  $(A, \overline{A})$  is equal to  $\sum_{i=1}^r p_i (1 - p_i) |V_i|^2$ . This is at least  $B' \geq B/3$  since  $(A, \overline{A})$  is B'-balanced. Hence,

$$\sum_{i=1}^{r} p_i (1 - p_i) \cdot \frac{|V|^2}{r^2} \ge \frac{1}{6} r \cdot \binom{|V|/r}{2}.$$

Thus,  $\sum_{i=1}^{r} p_i (1-p_i) \ge r/12$ . By Cauchy-Schwarz inequality,

$$\mathbb{E}_{i \in_{R}[r]} \left| \mathbb{E}_{x \in_{R} V_{i}}[A(x)] \right| = \frac{1}{r} \sum_{i=1}^{r} |1 - 2p_{i}| \le \sqrt{\frac{1}{r} \sum_{i=1}^{r} (1 - 2p_{i})^{2}} = \sqrt{1 - \frac{4}{r} \sum_{i=1}^{r} p_{i} (1 - p_{i})} \le \sqrt{\frac{2}{3}} < \frac{5}{6}.$$

Hence,  $(A, \overline{A})$  must be a 5/6-piecewise balanced cut. However, Property (2) of Theorem 2.9 says that such a cut must cut at least  $\varepsilon^t$  fraction of edges. This implies that  $\Gamma = \Omega(1/\varepsilon^{1-t})$ . Theorem 1.2 now follows by noting that t > 1/2 is arbitrary and  $n = |V| \le 2^{2^{O(1/\varepsilon^3)}}$ .

#### 2.5 Fourier Analysis

Consider the real vector space of all functions  $f: \{-1,1\}^n \to \mathbb{R}$ , where the addition of two functions is defined to be pointwise addition. For  $f,g: \{-1,1\}^n \to \mathbb{R}$ , define the following inner product:

$$\langle f, g \rangle_2 \stackrel{\text{def}}{=} 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)g(x).$$

For a set  $S \subseteq [n]$ , define the *Fourier character*  $\chi_S(x) \stackrel{\text{def}}{=} \prod_{i \in S} x_i$ . It is well-known (and easy to prove) that the set of all Fourier characters forms an orthonormal basis with respect to the above inner product. Hence, every function  $f: \{-1,1\}^n \mapsto \mathbb{R}$  has a (unique) representation as  $f = \sum_{S \subseteq [n]} \widehat{f_S} \chi_S$ , where  $\widehat{f_S} \stackrel{\text{def}}{=} \langle f, \chi_S \rangle_2$  is the Fourier coefficient of f w.r.t. S. The following is a simple but useful fact.

**Fact 2.10 (Parseval's Identity)** For any  $f : \{-1,1\}^n \mapsto \{-1,1\}, \sum_{S \subseteq [n]} \widehat{f}_S^2 = 1.$ 

The proof of this follows from the following sequence of equalities:

$$1 = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} f^2(x) = \langle f, f \rangle_2 = \left\langle \sum_{S \subseteq [n]} \widehat{f}_S \chi_S, \sum_{T \subseteq [n]} \widehat{f}_T \chi_T \right\rangle_2 = \sum_{S \subseteq [n]} \widehat{f}_S^2,$$

where the last equality follows from the orthonormality of the characters  $\{\chi_S\}_{S\subseteq[n]}$  with respect to the inner product  $\langle\cdot,\cdot\rangle_2$ .

For the analysis of our UNIQUEGAMES integrality gap instance presented in Section 3, we need the following notion of an  $\ell_p$  norm of a Boolean function. For  $f: \{-1,1\}^n \mapsto \mathbb{R}$  and  $p \ge 1$ , let

$$||f||_p \stackrel{\text{def}}{=} \left(\frac{1}{2^n} \sum_{x \in \{-1,1\}^n} |f(x)|^p\right)^{1/p}.$$

We also need to define the so-called Bonami-Beckner operator whose input is a Boolean function f and whose output is again a Boolean function (which is supposed to be a *smoothened* version of f).

**Definition 2.11 (Hyper-contractive Operator)** For each  $\rho \in [-1,1]$ , the Bonami-Beckner operator  $T_{\rho}$  is a linear operator that maps the space of functions  $\{-1,1\}^n \mapsto \mathbb{R}$  into itself via

$$T_{\rho}[f] \stackrel{\text{def}}{=} \sum_{S \subseteq [n]} \rho^{|S|} \widehat{f}_S \chi_S.$$

The following theorem shows that the Bonami-Beckner operator indeed smoothens f: It allows us to upper bound a higher norm of  $T_0[f]$  of f with a lower norm of f under certain conditions.

**Theorem 2.12 (Bonami-Beckner Inequality [40])** Let  $f: \{-1,1\}^n \to \mathbb{R}$  and 1 . Then

$$||T_{\rho}[f]||_q \leq ||f||_p$$

for all 
$$0 \le \rho \le \left(\frac{p-1}{q-1}\right)^{1/2}$$
.

The last set of preliminaries are important for the PCP reduction in Section 4.

**Definition 2.13 (Long Code [10])** The Long Code over a domain [N] is indexed by all  $x \in \{-1,1\}^N$ . The Long Code f of an element  $j \in [N]$  is defined to be  $f(x) \stackrel{\text{def}}{=} \chi_{\{j\}}(x) = x_j$ , for all  $x = (x_1, \dots, x_N) \in \{-1,1\}^N$ .

Thus, a Long Code is simply a Boolean function that is a dictatorship, i.e., it depends only on one coordinate. In particular, if f is the Long Code of  $j \in [N]$ , then  $\widehat{f}_{\{j\}} = 1$  and all other Fourier coefficients are zero.

The following theorem (quantitatively) shows that if a Boolean function is such that its Fourier mass is concentrated on sets of small size, then it must be close to a *junta*. In other words, its Fourier mass on sets with *small* Fourier coefficients is *small*.

**Theorem 2.14 (Bourgain's Junta Theorem [12])** Fix any 1/2 < t < 1. Then, there exists a constant  $c_t > 0$ , such that, for all positive integers k, for all  $\gamma > 0$  and for all Boolean functions  $f : \{-1,1\}^n \mapsto \{-1,1\}$ ,

$$if \sum_{S : |S| > k} \widehat{f}_S^2 < c_t k^{-t} \quad then \quad \sum_{S : |\widehat{f}_S| < \gamma 4^{-k^2}} \widehat{f}_S^2 < \gamma^2.$$

## 3 The Integrality Gap Instance for Unique Games

In this section, we present the integrality gap construction for a natural SDP relaxation of the UNIQUEGAMES problem. We start with defining the UNIQUEGAMES problem, the UGC of Khot [28] along with the related preliminaries towards our construction.

#### 3.1 The Unique Games Problem, its SDP Relaxation and the UGC

**Definition 3.1** (UNIQUEGAMES) An instance  $\mathscr{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$  of UNIQUEGAMES is defined as follows: G(V, E) is a graph with a set of vertices V and a set of edges E. An edge e with endpoints v and w is written as  $e\{v, w\}$ . For every  $e\{v, w\} \in E$ , there is a bijection  $\pi_e : [N] \mapsto [N]$  and a weight  $\text{wt}(e) \in \mathbb{R}^+$ . The goal is to assign a label from the set [N] to every vertex of the graph so as to satisfy the constraints given by bijective maps  $\pi_e$ . A labeling  $\lambda : V \mapsto [N]$  satisfies an edge  $e\{v, w\}$ , if  $\lambda(v) = \pi_e(\lambda(w))$ . Let  $\text{val}(\lambda)$  denote the total weight of the edges satisfied by a labeling  $\lambda$ :

$$\operatorname{val}(\lambda) \stackrel{\text{def}}{=} \sum_{e\{v,w\} \in E: \lambda \text{ satisfies } e} \operatorname{wt}(e).$$

The optimum  $opt(\mathcal{U})$  of the UNIQUEGAMES instance is defined to be the maximum weight of edges satisfied by any labeling:

$$\operatorname{opt}(\mathscr{U}) \stackrel{\text{def}}{=} \max_{\lambda: V \mapsto [N]} \operatorname{val}(\lambda).$$

We assume w.l.o.g that  $\sum_{e \in E} \operatorname{wt}(e) = 1$  so that the weights define a probability distribution over edges. A choice of a random edge refers to an edge chosen from this distribution. We also assume that the graph is regular in the sense that the sum of weights of edges incident on a vertex is the same for all vertices. A choice of a random edge incident on a vertex v refers to a choice of a random edge conditional on having one endpoint as v.

**Conjecture 3.2 (UGC [28])** For every pair of constants  $\eta, \zeta > 0$ , there exists a sufficiently large constant  $N = N(\eta, \zeta)$  such that given a UNIQUEGAMES instance  $\mathscr{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$ , it is NP-hard to distinguish whether:

- opt( $\mathscr{U}$ )  $\geq 1 \eta$ , or
- opt( $\mathscr{U}$ )  $\leq \zeta$ .

Consider a UNIQUEGAMES instance  $\mathscr{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$ . Khot [28] proposed the SDP relaxation in Figure 4 (inspired by a paper of Feige and Lovász [21]). Here, for every  $v \in V$ , we associate a set of N orthogonal vectors  $\{v_1, \ldots, v_N\}$ . The intention is that if  $i_0 \in [N]$  is a label for vertex  $v \in V$ , then  $v_{i_0} = \sqrt{N}\mathbf{1}$ , and  $v_i = \mathbf{0}$  for all  $i \neq i_0$ . Here,  $\mathbf{1}$  is some fixed unit vector and  $\mathbf{0}$  is the zero-vector. However, once we take the SDP relaxation, this may no longer be true and  $\{v_1, v_2, \ldots, v_N\}$  could be any set of orthogonal vectors.

#### The Noisy Hypercube and an Overview of the Integrality Gap Instance

With a UNIQUEGAMES instance with N labels, one can associate a related graph called the *label extended graph*. It turns out that the optimum of the UNIQUEGAMES instance is closely related to the expansion of *small* sets, namely those of relative size 1/N, in the label extended graph. In particular, if all sets of size 1/N in the label extended graph have a near-full expansion, then the optimum of the UNIQUEGAMES instance is low. Our integrality gap construction starts with a so-called *noisy hypercube graph* on vertex set  $\{-1,1\}^N$  and obtain a UNIQUEGAMES instance from it so that the former is precisely the label extended graph of the latter. The fact that the UNIQUEGAMES instance has low optimum then follows directly from the observation that the noisy hypercube graph is a small set expander (its proof via the Bonami-Beckner inequality was pointed out to us by Ryan O'Donnell). The SDP solution for the UNIQUEGAMES instance is constructed using the vertices of the hypercube thought of as vectors in  $\mathbb{R}^N$ .

<sup>&</sup>lt;sup>9</sup>We consider the edges to be undirected, but there is an implicit direction when we write the edge as  $e\{v,w\}$  and it is reflected in the bijective constraint that  $\lambda(v) = \pi_e(\lambda(w))$ . The edge could be written in reverse by reversing the bijection.

Maximize 
$$\sum_{e \{v, w\} \in E} \operatorname{wt}(e) \cdot \frac{1}{N} \left( \sum_{i=1}^{N} \left\langle v_{\pi_e(i)}, w_i \right\rangle \right)$$
 (10)

Subject to

$$\forall v \in V \qquad \sum_{i=1}^{N} \langle v_i, v_i \rangle = N \tag{11}$$

$$\forall v \in V \ \forall i \neq j \qquad \langle v_i, v_i \rangle = 0 \tag{12}$$

$$\forall v \in V \qquad \sum_{i=1}^{N} \langle v_i, v_i \rangle = N$$

$$\forall v \in V \ \forall i \neq j \qquad \langle v_i, v_j \rangle = 0$$

$$\forall v, w \in V \ \forall i, j \qquad \langle v_i, w_j \rangle \geq 0$$

$$(11)$$

$$\forall v, w \in V \quad \sum_{1 < i, j < N} \langle v_i, w_j \rangle = N \tag{14}$$

Figure 4: SDP for UNIQUEGAMES

**Remark 3.3** The idea of the label extended graph and the implication that the small set expansion in the label extended graph implies low optimum for the UNIQUEGAMES instance were implicit in the conference version of this paper [32]. We choose to make this more explicit here for the ease of presentation as well as in light of recent works that we briefly mention. Raghavendra and Steurer recently proposed the Small Set Expansion Conjecture [43] and showed that it implies the UGC. The former states that for every constant  $\varepsilon > 0$ , there exists a constant  $\delta > 0$  such that given an n-vertex graph that has a small non-expanding set, i.e., of size  $\delta n$  and with edge expansion at most  $\varepsilon$ , it is NP-hard to find a set of size (roughly)  $\delta n$  that is even somewhat non-expanding, i.e., with expansion at most  $1-\varepsilon$ . The SSE Conjecture has led to many interesting works including a new algorithm for UNIQUEGAMES by Arora, Barak and Steurer [2] and the construction of the short code [9].

**Definition 3.4** Given a UNIQUEGAMES instance  $\mathcal{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$ , the corresponding label extended graph G'(V', E', wt') is defined as follows:

- $\bullet V' = V \times [N].$
- $\forall e\{v,w\} \in E, i \in [N], we let e'\{(v,\pi_e(i)),(w,i)\} \in E' \text{ and } wt'(e') = wt(e).$

Note that  $\sum_{e' \in E'} \operatorname{wt}'(e') = N$ .

It is helpful to view the label extended graph as being obtained from the UNIQUEGAMES graph by replacing every vertex v by a group of N vertices representing labels to v and replacing every edge  $e\{v,w\}$  by an edgebundle of N edges that form a perfect matching between the two groups and capture the bijective constraint  $\pi_e$ .

The expansion  $\Phi(S')$  of a set  $S' \subseteq V'$  in the label extended graph is defined to be the probability of leaving S' when a random vertex in S' and then a random edge leaving that vertex (w.r.t. the weights wt') is chosen. Note that  $\Phi(S') \in [0,1]$ . Any labeling  $\lambda : V \mapsto [N]$  to a UNIQUEGAMES instance corresponds to the set  $S'_{\lambda} \subseteq V'$  as follows:

$$S_{\lambda}' \stackrel{\text{def}}{=} \{ (v, \lambda(v)) \mid v \in V \}.$$

An easy observation is that the (weighted) fraction of edges satisfied by a labeling  $\lambda$  is related to the expansion of the set  $S'_{\lambda}$ :

$$val(\lambda) = 1 - \Phi(S'_{\lambda}). \tag{15}$$

Here is a quick proof of the above equality. Pick a random vertex  $(v, \lambda(v))$  in  $S'_{\lambda}$  by choosing a random vertex  $v \in V$ . Choosing a random edge incident on  $(v, \lambda(v))$  (w.r.t. wt') amounts to choosing a random edge  $e\{v,w\}$  incident on v (w.r.t. wt) and outputting  $\{(v,\lambda(v)),(w,\pi_e^{-1}(\lambda(v)))\}$ . The expansion of  $S'_\lambda$  is now related to the event that  $(w,\pi_e^{-1}(\lambda(v)))\in S'_\lambda$  which is same as the event that  $\pi_e^{-1}(\lambda(v))=\lambda(w)$  which is same as the event that  $\lambda$  satisfies the edge  $e\{v,w\}$ .

As remarked before, our construction starts with the noisy hypercube graph and uses the fact that the graph is a small set expander. A natural way to describe this graph is by describing one step of the random walk on it (which then naturally leads to edge-weights with unit total weight).

**Definition 3.5** The noisy hypercube graph H with parameters N and  $0 < \eta < 1/2$  has

- the vertex set  $\{-1,1\}^N$  with uniform distribution and
- for any vertex  $x \in \{-1,1\}^N$ , choosing a random edge (x,y) incident on x amounts to flipping every bit of x with probability  $\eta$  independently and letting y to be the string so obtained.

**Lemma 3.6** Let H be the noisy hypercube with parameters N and  $\eta$  and  $S \subseteq \{-1,1\}^N$  be a set of relative size 1/N. Then  $1 - \Phi(S) \le 1/N^{\eta + \eta^2}$ .

**Proof.** Let  $f: \{-1,1\}^N \mapsto \{0,1\}$  be the indicator function of the set S so that  $||f||_p^p = 1/N$  for any  $1 \le p < \infty$ . An application of Bonami-Beckner inequality gives (the probability is taken over choice of a random vertex x and a random edge (x,y) incident on it)

$$\begin{split} 1 - \Phi(S) & = & \Pr[y \in S \mid x \in S] = \frac{\Pr[x \in S, \ y \in S]}{\Pr[x \in S]} = N \cdot \Pr[x \in S, \ y \in S] \\ & = & N \cdot \mathbb{E}_{x,y}[f(x)f(y)] = N \cdot \sum_{\alpha \subseteq [N]} \widehat{f}_{\alpha}^{2} (1 - 2\eta)^{|\alpha|} \stackrel{\text{Def. 2. } 11}{=} N \cdot ||T_{\sqrt{1 - 2\eta}}f||_{2}^{2} \\ & \stackrel{\text{Thm. 2. } 12}{\leq} & N \cdot ||f||_{2 - 2\eta}^{2} = N \cdot \left(\frac{1}{N}\right)^{2/(2 - 2\eta)} \leq N \cdot \frac{1}{N^{1 + \eta + \eta^{2}}} = \frac{1}{N^{\eta + \eta^{2}}}. \end{split}$$

Call an edge (x,y) of the noisy hypercube *typical* if the Hamming distance between x and y is close to  $\eta N$ , say between  $\frac{\eta}{2}N$  and  $2\eta N$ . By the Chernoff bound, the (weighted) fraction of edges which are not *typical* is at most  $2^{-\Omega(\eta N)}$  which is negligible in our context. We delete all these edges (mainly for the ease of presentation) and observe that the conclusion of Lemma 3.6 still holds with the bound  $1 - \Phi(S) \le 1/N^{\eta}$ . The weights of the edges change slightly, due to a re-normalization to preserve the unit total weight, but we ignore this issue.

We are now ready to construct an integrality gap instance for the SDP in Figure 4. To be precise, for parameters N and  $\eta$ , we construct an instance  $\mathcal{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$  of UNIQUEGAMES such that

- (Soundness) opt( $\mathscr{U}$ )  $\leq 1/N^{\eta}$  and
- (Completeness) There is an SDP solution with objective value at least  $1-9\eta$ .

This construction is used later to construct integrality gap instances for cut problems. As mentioned earlier, the UNIQUEGAMES instance is constructed precisely so that the noisy hypercube graph happens to be its label extended graph and then the soundness guarantee follows from Lemma 3.6. The vertex set of the noisy hypercube graph is  $\{-1,1\}^N$  where  $N=2^k$ . It is convenient for us to identify a point in  $\{-1,1\}^N$  as a Boolean function  $f: \{-1,1\}^k \mapsto \{-1,1\}$ . We describe the construction formally now.

#### 3.2 The Integrality Gap Instance

Let  $\mathscr{F}$  denote the family of all Boolean functions on  $\{-1,1\}^k$ . For  $f,g\in\mathscr{F}$ , define the product fg as

$$(fg)(x) \stackrel{\text{def}}{=} f(x)g(x).$$

Consider the equivalence relation  $\equiv$  on  $\mathscr{F}$  defined to be  $f \equiv g$  if and only if there is an  $S \subseteq [k]$ , such that  $f = g\chi_S$  (recall that  $\chi_S$  is the Fourier character function). This relation partitions  $\mathscr{F}$  into equivalence classes  $\mathscr{P}_1, \ldots, \mathscr{P}_m$ , each class containing exactly  $N = 2^k$  functions. We denote by  $[\mathscr{P}_i]$  one arbitrarily chosen function in  $\mathscr{P}_i$  as its representative. Thus, by definition,

$$\mathscr{P}_i = \{ [\mathscr{P}_i] \chi_S \mid S \subseteq [k] \}.$$

It follows from the orthogonality of the characters  $\{\chi_S\}_{S\subseteq [k]}$ , that all the functions in any class are also mutually orthogonal. Further, for a function  $f\in \mathscr{F}$ , let  $\mathscr{P}(f)$  denote the class  $\mathscr{P}_i$  in which f belongs.

Let  $\mu \in_{\eta} \mathscr{F}$  denote a random *perturbation* function on  $\{-1,1\}^k$  where for every  $x \in \{-1,1\}^k$ , independently,  $\mu(x) = 1$  with probability  $1 - \eta$ , and -1 with probability  $\eta$ . Let H be the noisy hypercube graph: It is a graph with vertex set  $\mathscr{F}$  and for Boolean functions  $f,g \in \mathscr{F}$ , the weight of the edge  $\{f,g\}$  is defined as follows:

$$\operatorname{wt}'(\{f,g\}) \stackrel{\operatorname{def}}{=} \Pr_{h \in \mathscr{F}, \ \mu \in n\mathscr{F}} \left[ \left( (f=h) \land (g=h\mu) \right) \lor \left( (f=h\mu) \land (g=h) \right) \right],$$

where h is a uniformly random function and  $\mu$  is a random perturbation function. Note that the sum of weights over all (undirected) edges is 1. Moreover, for any  $S \subseteq [k]$ , we have  $\operatorname{wt}'(\{f,g\}) = \operatorname{wt}'(\{f\chi_S, g\chi_S\})$ . We delete all edges  $\{f,g\}$  such that the Hamming distance between f and g is outside the range  $[\frac{\eta}{2}N, 2\eta N]$  without really affecting anything as observed before.

The UNIQUEGAMES instance  $\mathscr{U}=(G(V,E),[N],\{\pi_e\}_{e\in E},\text{wt})$  is now obtained by taking the noisy hypercube graph H as above with a *grouping* of its vertices into classes  $\mathscr{P}_1,\ldots,\mathscr{P}_m$ . The edges of H are grouped neatly into edge-bundles: A typical bundle is a set of N edges between  $\mathscr{P}_i$  and  $\mathscr{P}_j$ , all with the same weight, and forming a perfect matching between the N vertices in each group. With this grouping in mind, the graph can now be naturally thought of as a label extended graph. The UNIQUEGAMES instance is obtained by thinking of each class  $\mathscr{P}_i$  as a (super-)vertex, each function  $f \in \mathscr{P}_i$  as a potential label to it, and the edge bundle between  $\mathscr{P}_i, \mathscr{P}_j$  as defining the bijective constraint between them. Here is a formal (somewhat tedious) description.

The UNIQUEGAMES graph G(V,E) is defined as follows. The set of vertices is  $V \stackrel{\text{def}}{=} \{\mathscr{P}_1,\ldots,\mathscr{P}_m\}$  as above. For every  $f,g \in \mathscr{F}$  with Hamming distance in the range  $[\frac{\eta}{2}N,2\eta N]$ , there is an edge in E between the vertices  $\mathscr{P}(f)$  and  $\mathscr{P}(g)$  with weight

$$\operatorname{wt}(\{\mathscr{P}(f),\mathscr{P}(g)\}) \stackrel{\operatorname{def}}{=} N \cdot \operatorname{wt}'(\{f,g\})$$

(the factor of N reflects the fact that there are N pairs of functions that define the same edge). The set of labels for the UNIQUEGAMES instance is  $2^{[k]} \stackrel{\text{def}}{=} \{S : S \subseteq [k]\}$ , i.e., the set of labels [N] is identified with the set  $2^{[k]}$  (and by design  $N = 2^k$ ). Note that  $f = [\mathcal{P}_i]\chi_S$  and  $g = [\mathcal{P}_j]\chi_T$  for some sets  $S, T \subseteq [k]$ . The bijection  $\pi_e$ , for the edge  $e\{\mathcal{P}_i, \mathcal{P}_j\}$ , can now be defined:

$$\pi_e(T \star U) \stackrel{\text{def}}{=} S \star U, \quad \forall \ U \subseteq [k].$$

Here,  $\star$  is the symmetric difference operator on sets. Note that  $\pi_e: 2^{[k]} \mapsto 2^{[k]}$  is a permutation on the set of allowed labels. An alternate view is that the potential labels to class  $\mathscr{P}_i$  are really the functions in that class and for the edge defined by a pair  $f \in \mathscr{P}_i$  and  $g \in \mathscr{P}_j$  as above,  $\pi_e$  designates  $(f\chi_U, g\chi_U)$  as a matching pairs of labels for all  $U \subseteq [k]$ . We emphasize that every matching pair of labels corresponds to a pair of functions with Hamming distance in  $[\frac{\eta}{2}N, 2\eta N]$ .

#### **Soundness: No Good Labeling**

Using Lemma 3.6 and Equation (15), i.e., the connection between the optimum of UNIQUEGAMES and the small set expansion of the label extended graph, it follows immediately that any labeling to the UNIQUEGAMES instance described above achieves an objective of at most  $1/N^{\eta}$ .

#### **Completeness: A Good SDP Solution**

For  $f \in \mathscr{F}$ , let  $u_f$  denote the unit vector (w.r.t. the  $\ell_2$  norm) corresponding to the truth-table of f. Formally, indexing the vector  $u_f$  with coordinates  $x \in \{-1,1\}^k$ ,

$$(u_f)_x \stackrel{\mathrm{def}}{=} \frac{f(x)}{\sqrt{N}}.$$

Recall that in the SDP relaxation of UNIQUEGAMES (Figure 4), for every vertex in V, we need to assign a set of orthogonal vectors. For every vertex  $\mathscr{P}_i \in V$ , we choose a function  $f \in \mathscr{P}_i$  arbitrarily, and with  $\mathscr{P}_i$ , we associate the set of vectors  $\left\{u_{f\chi_s}^{\otimes 2}\right\}_{S \subset [k]}$ . The following facts are easily verified:

1. 
$$\sum_{S\subseteq[k]}\left\langle u_{f\chi_S}^{\otimes 2}, u_{f\chi_S}^{\otimes 2}\right\rangle = \sum_{S\subseteq[k]}\left\langle u_{f\chi_S}, u_{f\chi_S}\right\rangle^2 = N.$$

2. For 
$$S \neq T \subseteq [k]$$
,  $\left\langle u_{f\chi_S}^{\otimes 2}, u_{f\chi_T}^{\otimes 2} \right\rangle = \left\langle u_{f\chi_S}, u_{f\chi_T} \right\rangle^2 = \left\langle u_{\chi_S}, u_{\chi_T} \right\rangle^2 = 0$ .

3. For 
$$f,g \in \mathscr{F}$$
 and  $S,T \subseteq [k], \left\langle u_{f\chi_S}^{\otimes 2}, u_{g\chi_T}^{\otimes 2} \right\rangle = \left\langle u_{f\chi_S}, u_{g\chi_T} \right\rangle^2 \ge 0$ .

4. For  $f \in \mathcal{P}_i$ ,  $g \in \mathcal{P}_j$  for  $i \neq j$ ,

$$\sum_{S,T\subseteq[k]}\left\langle u_{f\chi_S}^{\otimes 2},u_{g\chi_T}^{\otimes 2}\right\rangle = \sum_{S,T\subseteq[k]}\left\langle u_{f\chi_S},u_{g\chi_T}\right\rangle^2 = \sum_{T\subseteq[k]}\left\|u_{g\chi_T}\right\|^2 = N.$$

Here, the second last equality follows from the fact that, for any  $f \in \mathscr{F}$ ,  $\{u_{f\chi_S}\}_{S\subseteq [k]}$  forms an orthonormal basis for  $\mathbb{R}^N$ .

Hence, all the conditions (11)-(14) of the SDP are satisfied. Next, we show that this vector assignment has an objective at least  $1-9\eta$ . Consider any UNIQUEGAMES edge defined by a pair f,g with Hamming distance in the range  $[\frac{\eta}{2}N, 2\eta N]$ . For any  $S \subseteq [k]$ , note that the same edge is defined by the pair  $f\chi_S, g\chi_S$  with the same Hamming distance and

$$\left\langle u_{f\chi_S}^{\otimes 2}, u_{g\chi_S}^{\otimes 2} \right\rangle = \left\langle u_{f\chi_S}, u_{g\chi_S} \right\rangle^2 \geq (1 - 4\eta)^2 \geq 1 - 8\eta.$$

Since the pairs  $(f\chi_S, g\chi_S)$  are precisely the matching pairs of labels for the UNIQUEGAMES constraint, it follows that the objective of this SDP solution is at least  $1-9\eta$  (accounting possibly for the *non-typical* pairs f,g with Hamming distance outside of range  $\left[\frac{\eta}{2}N, 2\eta N\right]$  that were deleted and ignored throughout). Finally, note that since all the vectors have coordinates either 1 or -1 (up to a normalization factor), any three vectors u, v, w among those described above satisfy the triangle inequality:

$$1 + \langle u, v \rangle \ge \langle v, w \rangle + \langle u, w \rangle.$$

#### **Summarizing and Abstracting the Unique Games Instance**

For future reference, we summarize and abstract out the key properties of the integrality gap construction in the theorem below. Therein, for every vertex  $v \in V$  of the UNIQUEGAMES instance, there is an associated set of vectors  $\{v_i^{\otimes 2}\}_{i\in[N]}$ . Moreover, [N] has a group structure with addition operator  $\oplus$  (the group being  $\mathbb{F}_2^k$  and  $i\in[N]$  identified with the corresponding group element). Additionally, we keep track of the parameter  $\eta$  and denote the instance by  $\mathscr{U}_n$ .

**Theorem 3.7** For any  $0 < \eta < 1/2$  and any integer N that is a power of 2, there is a UNIQUEGAMES instance  $\mathcal{U}_{\eta} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$  along with a set of vectors  $\{v_i^{\otimes 2}\}_{i \in [N]}$  for every vertex such that:

1. 
$$|V| = \tilde{n} = 2^N/N$$
 and  $\operatorname{opt}(\mathcal{U}_{\eta}) \leq \log^{-\eta} \tilde{n}$ .

#### 2. Orthonormal Basis

The set of vectors  $\{v_i\}_{i\in[N]}$  forms an orthonormal basis for the space  $\mathbb{R}^N$ . Hence, for any vector  $w\in\mathbb{R}^N$ ,  $||w||^2=\sum_{i\in[N]}\langle w,v_i\rangle^2$ .

#### 3. Triangle Inequality

For any  $u, v, w \in V$ , and any  $i, j, \ell \in [N]$ ,  $1 + \langle u_i, v_j \rangle \ge \langle u_i, w_\ell \rangle + \langle v_j, w_\ell \rangle$ .

#### 4. Matching Property

For any  $v, w \in V$ , and  $i, j, \ell \in [N]$ ,  $\langle v_i, w_j \rangle = \langle v_{i \oplus \ell}, w_{j \oplus \ell} \rangle$ .

#### 5. Closeness Property

For any  $e\{v,w\} \in E$ , there are  $i_0, j_0 \in [N]$  such that  $\langle v_{i_0}, w_{j_0} \rangle \geq 1 - 4\eta$ . Moreover, if  $\pi_e$  is the bijection corresponding to this edge, then  $i_0 \oplus \ell = \pi_e(j_0 \oplus \ell)$  for all  $\ell \in [N]$ .

## 4 A PCP Reduction from Unique Games to Balanced Edge-Separator

This section presents the reduction from UNIQUEGAMES to non-uniform BALANCEDEDGE-SEPARATOR which underlies the proof of Theorem 1.3. Remark 2.7 implies that if non-uniform BALANCEDEDGE-SEPARATOR is hard to approximate within a factor of C, then so is non-uniform SPARSESTCUT up to a factor  $\Omega(C)$ . Hence, Theorem 1.3 can be strengthened as follows.

**Theorem 4.1** Assuming the UGC, it is NP-hard to approximate (non-uniform versions of) BALANCEDEDGE-SEPARATOR and SPARSESTCUT to within any constant factor.

We present the reduction and the proof of this theorem, modulo the soundness proof of the PCP reduction. The soundness proof is (by now) standard and relegated to Appendix A. The reduction underlying the proof of this theorem is used in the construction of the integrality gap for BALANCEDEDGE-SEPARATOR presented in Section 5.

#### **Overview of the Reduction**

The reduction starts with a UNIQUEGAMES instance  $\mathscr{U}=(G(V,E),[N],\{\pi_e\}_{e\in E},\mathrm{wt})$ . Each vertex  $v\in V$  is replaced with a block of vertices  $\{(v,x):x\in\{-1,1\}^N\}$ . The reduction has a parameter  $\varepsilon$  which is to be thought of as a small constant. For each edge  $e\{v,w\}$  in  $\mathscr{U}$ , a bundle of weighted edges are put between the two corresponding blocks of vertices taking into account the permutation  $\pi_e$  corresponding to that edge. The weight of the edge between (v,x) and (w,y) is equal to the product of the weight of the edge  $e\{v,w\}$  and the probability that, if we flip each bit of x independently with probability  $\varepsilon$ , we obtain  $y\circ\pi_e$ . Here  $y\circ\pi_e$  is the reordering of the coordinates of y as dictated by  $\pi_e$ ; formally,  $(y\circ\pi_e)_i=y_{\pi_e(i)}$  for all  $i\in[N]$ .

Note that if we contract the vertices of the two hypercubes after identifying the coordinates according to  $\pi_e$ , we obtain exactly the noisy hypercube introduced in Definition 3.5. To complete the reduction, we need to specify the demand pairs. For reasons that will become clear in a bit, any pair of vertices in the same block is set to have demand one and the remaining pairs have demand zero.

Our reduction has the property that if the UNIQUEGAMES instance has a good labeling then there is a cut that cuts a constant fraction of the demand pairs and the weight of the edges crossing the cut is small. This is by construction: If the UNIQUEGAMES instance  $\mathscr{U}$  has a good labeling, i.e., a  $\lambda: V \mapsto [N]$  which satisfies at least a  $1-\varepsilon$  fraction of the constraints of  $\mathscr{U}$ , then we consider the cut in the reduced graph whose one side consists of the vertices (v,x) such that  $x_{\lambda(v)} = 1$  and the other side with vertices (v,x) such that  $x_{\lambda(v)} = -1$ . It is easy to see that the weight of the edges that cross this cut is  $1 - (1-\varepsilon)(1-\varepsilon) = O(\varepsilon)$ . Moreover, the number of demand pairs cut is half that of the total demand pairs as the cut described above cuts each hypercube along a coordinate into two equal parts. This is the completeness of the reduction.

For soundness, we show that if every labeling of the UNIQUEGAMES instance satisfies a negligible (as a function of  $\varepsilon$ ) fraction of the constraints, any cut in the reduced graph that cuts a constant fraction of demand pairs must have about  $\sqrt{\varepsilon} \gg \varepsilon$  weight of edges crossing it. Since the reduction is local in the sense that it replaces each vertex in  $\mathcal{U}$  by a set of vertices, and each edge in  $\mathcal{U}$  by a bundle of edges between the corresponding sets, the weighted graph obtained by applying this reduction on  $\mathcal{U}$  inherits connectivity properties of  $\mathcal{U}$ . For instance, if  $\mathcal{U}$  is disconnected, then there is a cut in the reduced graph which has no edges crossing it. Such a cut, however, puts each hypercube entirely on one side of the cut or the other, thus, cutting no demand pair. Hence, the way we have enforced demands essentially ensures that each cut in the reduced graph that cuts a constant fraction of demand pairs cuts most of the hypercubes into two roughly equal parts. Hence, for each vertex v in  $\mathcal{U}$  we can look at the restriction of this cut to the corresponding hypercube and assign to v the label corresponding to the dimension of the hypercube which is the most correlated with the cut restricted to that hypercube. Since  $\mathcal{U}$  does not have a good labeling, this strategy of converting a cut in the reduced graph to a labeling for  $\mathcal{U}$  should not be good. Hence, one can deduce that, for any cut that cuts a constant fraction of the demand in the reduced graph, its restrictions to most hypercubes must not be well-correlated to any coordinate cut. This is where Bourgain's Junta theorem (Theorem 2.14) comes in. It essentially implies that such a cut must be close to a majority cut in most hypercubes. This allows us to deduce that such a cut has at least  $\sqrt{\varepsilon}$  weight edges crossing it, giving us the hardness of approximation ratio  $\approx \sqrt{\varepsilon}/\varepsilon$  which can be made larger than any constant by choosing  $\varepsilon$  small enough.

We now describe the reduction formally. Here, it is instructive to break the reduction into two parts: The first consists of presenting a PCP verifier for UNIQUEGAMES and the second step involves translating the PCP verifier into a BALANCEDEDGE-SEPARATOR instance. The completeness and the soundness of this verifier give us the proof of Theorem 4.1.

#### 4.1 The PCP Verifier

For  $\varepsilon \in (0,1)$ , we present a PCP verifier which given a UNIQUEGAMES instance  $\mathscr{U} = (G(V,E),[N],\{\pi_e\}_{e\in E})$  decides whether  $\operatorname{opt}(\mathscr{U}) \sim 1$  or  $\operatorname{opt}(\mathscr{U}) \sim 0$ . The verifier  $V_{\varepsilon}$  expects, as a proof, the Long Code (see Definition 2.13) of the label of every vertex  $v \in V$ . Formally, a proof  $\Pi$  is  $\{A^v\}_{v\in V}$ , where each  $A^v:\{-1,1\}^N \mapsto \{-1,1\}$  is the supposed Long Code of the label of v. The actions of  $V_{\varepsilon}$  on  $\Pi$  are as follows.

- 1. Pick  $e\{v, w\} \in E$  with probability wt(e).
- 2. Pick a random  $x \in_{1/2} \{-1,1\}^N$  and  $\mu \in_{\varepsilon} \{-1,1\}^N$ .
- 3. Let  $\pi_e : [N] \mapsto [N]$  be the bijection corresponding to  $e\{v, w\}$ . Accept if and only if

$$A^{\nu}(x) = A^{\nu}((x\mu) \circ \pi_e).$$

The completeness of verifier is easy and we provide a proof here.

**Lemma 4.2** (Completeness) For every  $\varepsilon \in (0,1)$ , if  $\operatorname{opt}(\mathscr{U}) \geq 1 - \eta$ , there is a proof  $\Pi$  such that

$$\Pr[V_{\varepsilon} \text{ accepts } \Pi] \ge (1 - \eta)(1 - \varepsilon).$$

Moreover, every table  $A^{\nu}$  in  $\Pi$  is balanced, i.e., exactly half of its entries are +1 and the rest are -1.

**Proof.** Since  $\operatorname{opt}(\mathscr{U}) \geq 1 - \eta$ , there is a labeling  $\lambda$  for which the total weight of the edges satisfied is at least  $1 - \eta$ . Hence, if we pick an edge  $e\{v,w\}$  with probability  $\operatorname{wt}(e)$ , with probability at least  $1 - \eta$ , we have  $\lambda(v) = \pi_e(\lambda(w))$ . Let the proof consist of Long Codes of the labels assigned by  $\lambda$  to the vertices. With probability  $1 - \varepsilon$ , we have  $\mu_{\lambda(v)} = 1$ . Hence, with probability at least  $(1 - \eta)(1 - \varepsilon)$ ,

$$A^{\nu}(x) = x_{\lambda(\nu)} = (x\mu)_{\pi_e(\lambda(w))} = A^{w}((x\mu) \circ \pi_e).$$

Noting that a Long Code is balanced, this completes the proof.

The soundness of the reduction involves more work and, since [28, 32], has become standard. We state the result here and the proof appears in Appendix A. We say that a proof  $\Pi = \{A^{\nu}\}_{\nu \in V}$  is  $\theta$ -piecewise balanced if

$$\mathbb{E}_{v}\left[|\widehat{A}_{\emptyset}^{v}|
ight] \leq heta.$$

Here,  $\widehat{A}_{\emptyset}^{\nu}$  is the Fourier coefficient corresponding to the empty set of the Boolean function  $A^{\nu}$  and the expectation is over a uniformly random vertex  $\nu \in V$ .

**Lemma 4.3 (Soundness)** For every  $t \in (1/2,1)$ , there exists a constant  $b_t > 0$  such that the following holds: Let  $\varepsilon > 0$  be sufficiently small and let  $\mathscr U$  be an instance of UNIQUEGAMES with  $\operatorname{opt}(\mathscr U) < 2^{-O(1/\varepsilon^2)}$ . Then, for every 5/6-piecewise balanced proof  $\Pi$ ,

$$\Pr[V_{\varepsilon} \text{ accepts } \Pi] < 1 - b_t \varepsilon^t.$$

#### 4.2 From the PCP Verifier to a Balanced Edge-Separator Instance

The reduction from the PCP verifier to an instance  $\mathscr{I}_{\varepsilon}$  of non-uniform BALANCEDEDGE-SEPARATOR is as follows. Replace the bits in the proof by vertices and replace every (2-query) PCP test by an edge of the graph. The weight of the edge is equal to the probability that the test is performed by the PCP verifier. Formally, we start with a UNIQUEGAMES instance  $\mathscr{U} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$ , and replace each vertex  $v \in V$  by a *block* of vertices (v, x) for each  $x \in \{-1, 1\}^N$ . For an edge  $e\{v, w\} \in E$ , there is an edge in  $\mathscr{I}_{\varepsilon}$  between (v, x) and (w, y), with weight

$$\operatorname{wt}(e) \cdot \Pr_{\substack{x' \in 1/2^{\{-1,1\}^N} \\ \mu \in_{\mathcal{E}} \{-1,1\}^N}} \left[ \left( x = x' \right) \wedge \left( y = x' \mu \circ \pi_e \right) \right].$$

This is exactly the probability that  $V_{\varepsilon}$  picks the edge  $e\{v,w\}$ , and decides to look at the x-th (resp. y-th) coordinate in the Long Code of the label of v (resp. w).

The demand function  $dem(\cdot)$  is 1 for any edge between vertices in the same block, and 0 otherwise. Let  $B \stackrel{\text{def}}{=} \frac{1}{2} \cdot |V| \cdot {2^N \choose 2}$  be half of the total demand.

Assuming the UGC, for any  $\eta, \zeta > 0$ , for a sufficiently large N, it is NP-hard to determine whether an instance  $\mathscr U$  of UNIQUEGAMES has  $\operatorname{opt}(\mathscr U) \geq 1 - \eta$  or  $\operatorname{opt}(\mathscr U) \leq \zeta$ . We choose  $\eta = \varepsilon$  and  $\zeta \leq 2^{-O(1/\varepsilon^2)}$  so that

- (a) when  $\operatorname{opt}(\mathscr{U}) \geq 1 \eta$ , there is a (piecewise balanced) proof that the verifier accepts with probability at least  $1 2\varepsilon$  and
- (b) when opt( $\mathscr{U}$ )  $\leq \zeta$ , the verifier does not accept any 5/6-piecewise balanced proof with probability more than  $1 b_t \varepsilon^t$ .

Note that  $b_t$  is defined as in the statement of Lemma 4.3.

Suppose that  $\operatorname{opt}(\mathscr{U}) \geq 1 - \eta$ . Let  $\lambda$  be a labeling that achieves the optimum. Consider the partition  $(S, \overline{S})$  in  $\mathscr{I}_{\varepsilon}$  such that S consists of all vertices (v, x) with the property that the Long Code of  $\lambda(v)$  evaluated at x is +1. Clearly, the demands cut by this partition is exactly equal to B. Moreover, it follows from Lemma 4.2 that this partition cuts edges with weight at most  $\eta + \varepsilon = 2\varepsilon$ .

Now, suppose that  $\operatorname{opt}(\mathscr{U}) \leq \zeta$ . Then, it follows from Lemma 4.3, that any B'-balanced partition, with  $B' \geq B/3$ , cuts at least  $b_t \varepsilon^t$  fraction of the edges. This is due to the following: Any partition  $(S, \overline{S})$  in  $\mathscr{I}_{\varepsilon}$  corresponds to a proof  $\Pi$  in which we let the (supposed) Long Code of the label of v to be +1 at the point x if  $(v,x) \in S$ , and -1 otherwise. Since  $B' \geq B/3$ , as in the proof of Theorem 2.9,  $\Pi$  is 5/6-piecewise balanced and we apply Lemma 4.3.

Thus, we get a hardness factor of  $\Omega(1/\epsilon^{1-t})$  for BALANCEDEDGE-SEPARATOR and, hence, by Remark 2.7, for SPARSESTCUT as well. This completes the proof of Theorem 4.1.

## 5 The Integrality Gap Instance for Balanced Edge-Separator

In this section, we describe the integrality gap instance for BALANCEDEDGE-SEPARATOR along with its SDP solution and prove Theorem 2.9. As pointed out in Section 2.3, this also implies an integrality gap for non-uniform SPARSESTCUT. The following is, thus, a strengthening of Theorem 1.3.

**Theorem 5.1** Non-uniform versions of SparsestCut and BalancedEdge-Separator have an integrality gap of at least  $(\log \log n)^{1/6-\delta}$ , where  $\delta > 0$  is arbitrary. The integrality gaps hold for standard SDPs with triangle inequality constraints.

We present a proof of this theorem (by proving Theorem 2.9). The fact that our SDP solution satisfies the triangle inequality constraints relies on a technical lemma whose proof is via an extensive case analysis and is not very illuminating, hence, relegated to Appendix B.

#### Overview of the Integrality Gap Instance

The integrality gap instance for non-uniform BALANCEDEDGE-SEPARATOR has two parts: A (weighted) graph  $(V^*, E^*)$  on n vertices along with demand pairs and a unit vector  $\mathbf{V}_u$  for each vertex  $u \in V^*$ . The integrality gap instance is parameterized by  $\varepsilon > 0$  and  $\mathscr{I}_{\varepsilon}$  denotes the instance. We show that

- 1. every cut in  $V^*$  that cuts a constant fraction of the demand pairs must have at least  $\sqrt{\varepsilon}$  fraction of edges crossing it and that
- 2. the set of vectors  $\{\mathbf{V}_u\}_{u\in V^*}$  satisfy the constraints in the SDP in Figure 3 and have an objective value  $O(\varepsilon)$ , thus, giving us an integrality gap of  $\Omega(\sqrt{\varepsilon})$ .

The smallest value  $\varepsilon$  can take turns out to be  $(\log \log n)^{-1/3}$ , giving us the lower bound  $\Omega((\log \log n)^{-1/6})$ .

The graph in  $\mathscr{I}_{\varepsilon}$  is obtained by applying the reduction from UNIQUEGAMES to BALANCEDEDGE-SEPARATOR presented in Section 4 to the UNIQUEGAMES integrality gap instance  $\mathscr{U}_{\eta}$  from Section 3, see

Theorem 3.7 for a summary. Recall that  $\mathscr{U}_{\eta}$  consists of the constraint graph  $(G(V,E),[N],\{\pi_e\}_{e\in E},\text{wt})$  and a set of vectors  $\{v_i^{\otimes 2}\}_{i\in[N]}$  for each vertex  $v\in V$ . Further,  $\tilde{n}=|V|=2^N/N$  and  $\text{opt}(\mathscr{U}_{\eta})\leq \log^{-\eta}\tilde{n}$ .

The reduction implies that  $n=|V^*|=2^N\cdot |V|\leq O(\tilde{n}^2\log\tilde{n})$  and, hence,  $\log^{-\eta}\tilde{n}\approx \log^{-\eta}n$  up to a constant. Thus, if  $\log^{-\eta}n\leq 2^{-O(1/\varepsilon^2)}$ , then it follows from Lemma 4.3 and the discussion in Section 4.2 that every cut in  $\mathscr{I}_{\varepsilon}$  that cuts at least a constant fraction of demand pairs cuts at least  $\sqrt{\varepsilon}$  fraction of edges. This proves the first claim. A constraint on  $\eta$ , as we see shortly, is that  $\eta\leq\varepsilon$ . Thus, choosing  $\eta=\varepsilon$  implies that in order to ensure  $\log^{-\varepsilon}n\leq 2^{-O(1/\varepsilon^2)}$ , it is sufficient to set  $\varepsilon$  to be  $(\log\log n)^{-1/3}$ .

Thus, to complete the proof of Theorem 5.1, it remains to construct vectors  $\mathbf{V}_u$  for each vertex  $u \in V^*$  that satisfy the required constraints and have a small objective value. This is the focus of this section. Here again the starting point is the SDP solution to the UNIQUEGAMES integrality gap  $\mathcal{U}_{\eta}$ . Recall that the vectors  $\{v_i\}_{i\in[N]}$  form an orthonormal basis of  $\mathbb{R}^N$  for each  $v\in V$  and, in addition satisfy Triangle Inequality, the Matching Property and the Closeness Property in Theorem 3.7. In addition, the SDP objective value of these vectors for  $\mathcal{U}_{\eta}$  is  $1-9\eta$ .

For each vertex  $v \in V$  there is a block of vertices  $\{(v,x) : x \in \{-1,1\}^N\}$  in  $V^*$ . Thus, we need a unit vector for each (v,x). A choice for such a vector is

$$\mathbf{V}_{(v,x)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i \in [N]} x_i v_i^{\otimes 2}. \tag{16}$$

The fact that this is a unit vector is easy to see. Recall that for a typical edge in  $\mathcal{U}_{\eta}$ , the basis vectors are  $\eta$ -close when matched according to the permutation corresponding to that edge. Further, recall that for an edge between (v,x) and (w,y), there must be an edge between v and w in  $\mathcal{U}_{\eta}$ . Moreover, for a typical edge in  $\mathcal{I}_{\varepsilon}$ , except with probability  $\varepsilon$ , the relative Hamming distance between x and y is at most  $2\varepsilon$  (after taking into account the permutation between v and w in  $\mathcal{U}_{\eta}$ ). This easily implies that for a typical edge in  $\mathcal{I}_{\varepsilon}$ ,

$$\langle \mathbf{V}_{(v,x)}, \mathbf{V}_{(w,y)} \rangle \geq 1 - O(\eta + \varepsilon).$$

Since the vectors are of unit length, this implies that

$$\|\mathbf{V}_{(v,x)}-\mathbf{V}_{(w,y)}\|^2 \leq O(\eta+\varepsilon).$$

This is what dictates the choice of  $\eta = \varepsilon$  and we obtain that our SDP solution to  $\mathscr{I}_{\varepsilon}$  has an objective value at most  $O(\varepsilon)$ . To see the well-separatedness of this SDP solution, observe that for each  $v \in V$ ,  $\mathbf{V}_{(v,x)}$  and  $\mathbf{V}_{(v,-x)}$  are unit vectors in opposite direction.

It remain to prove that the vectors  $\{V_{(\nu,x)}\}$  satisfy the triangle inequality. This is the technically hardest part of the paper and is shown via an extensive case analysis that repeatedly uses the fact that the vectors for  $\mathcal{U}_{\eta}$  satisfy the properties they do. In fact, we do not know whether the vectors described above work for this proof. We need to modify the vectors in (16) as follows

$$\left(\frac{1}{\sqrt{N}}\sum_{i\in[N]}x_iv_i^{\otimes 8}\right)^{\otimes(2^{240}+1)}.$$

While the inner tensor, which goes to 8 from 2, is a minor modification, it ensures that when we take inner products of the form

$$\left\langle \frac{1}{\sqrt{N}} \sum_{i \in [N]} v_i^{\otimes 8}, \frac{1}{\sqrt{N}} \sum_{i' \in [N]} w_{i'}^{\otimes 8} \right\rangle,$$

and if  $\langle v_i, w_i \rangle \approx 1 - \eta$  for all  $i \in [N]$ , then the contribution of the cross terms is negligible and the inner product remains around  $1 - \eta$ . This 8-th tensor also implies the converse: If

$$\left\langle \frac{1}{\sqrt{N}} \sum_{i \in [N]} v_i^{\otimes 8}, \frac{1}{\sqrt{N}} \sum_{i' \in [N]} w_{i'}^{\otimes 8} \right\rangle \ge 1 - \eta,$$

then there is a permutation  $\pi: [N] \mapsto [N]$  such that for all  $i \in [N]$ ,

$$|\langle v_{\pi(i)}, w_i \rangle| \geq 1 - 2\eta$$
.

This latter property and the outer tensor are crucial in the proof of the triangle inequality. This new SDP solution is also easily seen to satisfy the properties satisfied by the previous SDP solution up to a loss of an additional constant factor.

We conclude this overview by giving the reader some idea of why we have the outer tensor. Start by noting that proving the triangle inequality is the same as showing

$$1 + \langle \mathbf{V}_{u,x}, \mathbf{V}_{v,y} \rangle^t \ge \langle \mathbf{V}_{u,x}, \mathbf{V}_{w,z} \rangle^t + \langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle^t$$

since all the vectors have unit length. If none of the dot-products has magnitude at least 1/3 the inequality holds trivially. Thus, we may assume that one of the inner products, say,  $|\langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle|^t \geq 1/3$ . This implies that  $|\langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle| = 1 - O(1/t)$ . By the converse property mentioned earlier, it can be deduced that, for some  $i_0, j_0 \in [N], |\langle v_{i_0}, w_{j_0} \rangle| = 1 - O(1/t)$  which can be made very close to 1 by picking t large enough. This turns out to be convenient towards proving the triangle inequality via a case analysis, see Lemma 5.8.

Unfortunately, we cannot provide much more intuition than this and, as mentioned in the introduction, for a more intuitive proof of the triangle inequality one can refer to the papers [31, 42]. We now present the graph construction and the SDP solution formally and prove the claims above for the SDP solution.

#### 5.1 The Graph

We recall the following notations which are needed. For a permutation  $\pi:[N] \mapsto [N]$  and a vector  $x \in \{-1,1\}^N$ , the vector  $x \circ \pi$  is defined to be the vector with its j-th entry as  $(x \circ \pi)_j \stackrel{\text{def}}{=} x_{\pi(j)}$ . For  $\varepsilon > 0$ , the notation  $x \in_{\varepsilon} \{-1,1\}^N$  means that the vector x is a random  $\{-1,1\}^N$  vector, with each of its bits independently set to -1 with probability  $\varepsilon$ , and set to 1 with probability  $1-\varepsilon$ .

The Balancededeceseparator instance has a parameter  $\varepsilon > 0$  and we refer to it as  $\mathscr{I}_{\varepsilon}(V^*, E^*)$ . We start with the UniqueGames instance  $\mathscr{U}_{\eta} = (G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$  of Theorem 3.7. In  $\mathscr{I}_{\varepsilon}$ , each vertex  $v \in V$  is replaced by a block of vertices denoted by  $V^*[v]$ . This block consists of vertices (v, x) for each  $x \in \{-1, 1\}^N$ . Thus, the set of vertices for the Balancededeceseparator instance is

$$V^* \stackrel{\text{def}}{=} \{(v, x) \mid v \in V, \ x \in \{-1, 1\}^N\}$$
 and  $V^* = \bigcup_{v \in V} V^*[v].$ 

The edges in the BALANCEDEDGE-SEPARATOR instance are defined as follows: For  $e\{v,w\} \in E$ , there is an edge  $e^*$  in  $\mathscr{I}_{\mathcal{E}}$  between (v,x) and (w,y), with weight

$$\operatorname{wt}_{\mathrm{BS}}(e^*) \stackrel{\mathrm{def}}{=} \operatorname{wt}(e) \cdot \Pr_{\substack{x' \in_{1/2} \{-1,1\}^N \\ \mu \in_{\mathcal{E}} \{-1,1\}^N}} \left[ \left( x = x' \right) \wedge \left( y = x' \mu \circ \pi_e \right) \right].$$

Notice that the size of  $\mathscr{I}_{\varepsilon}$  is  $|V^*| = |V| \cdot 2^N = O(\tilde{n}^2 \log \tilde{n})$ . The following theorem establishes that every cut in  $\mathscr{I}_{\varepsilon}$  that cuts a constant fraction of the demand cuts a large fraction of the edges. It is a restatement of Lemma 4.3. See Section 4 for details.

<sup>&</sup>lt;sup>10</sup>This property has also been key in the results of Arora et al. [4].

**Theorem 5.2 (No Small Balanced Cut)** For every  $t \in (1/2, 1)$ , there exists a constant  $c_t > 0$  such that the following holds: Let  $\varepsilon > 0$  be sufficiently small and let  $\mathcal{U}_{\eta}(G(V, E), [N], \{\pi_e\}_{e \in E}, \text{wt})$  be an instance of UNIQUEGAMES with  $\text{opt}(\mathcal{U}_{\eta}) < 2^{-O(1/\varepsilon^2)}$ . Let  $\mathcal{I}_{\varepsilon}$  be the corresponding instance of BALANCEDEDGE-SEPARATOR as defined above. Let  $V^* = \bigcup_{v \in V} V^*[v]$  be the partition of its vertices as above. Then, any 5/6-piecewise balanced cut  $(A, \overline{A})$  in  $\mathcal{I}_{\varepsilon}$  (in the sense of Definition 2.8) satisfies

$$\sum_{e^* \in E^*(A,\overline{A})} \operatorname{wt}_{\mathrm{BS}}(e^*) \ge c_t \varepsilon^t.$$

#### 5.2 The SDP Solution

Now we present an SDP solution for  $\mathscr{I}_{\varepsilon}(V^*, E^*, \operatorname{wt}_{BS})$  that satisfies Properties (3), (4) and (5) of Theorem 2.9. This proves Theorem 2.9 and, hence, Theorem 5.1.

We begin with the SDP solution of Theorem 3.7. Recall that [N] is identified with the group  $\mathbb{F}_2^k$  where  $N=2^k$ , and  $\oplus$  is the corresponding group operation. We construct the following unit vectors, one for each pair (v,x), where  $v \in V$  and  $x \in \{-1,1\}^N$  (note that V is the set of vertices of the UNIQUEGAMES instance of Theorem 3.7):

$$\mathbf{V}_{\nu,x} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i \in [N]} x_i \nu_i^{\otimes 8}. \tag{17}$$

For  $(v,x) \in V^*$ , we associate the vector  $\mathbf{V}_{v,x}^{\otimes t}$ , where  $t = 2^{240} + 1$ . We start by noting that this vector is indeed a unit vector. Since  $\{v_i\}_{i \in [N]}$  is an orthonormal basis for  $\mathbb{R}^N$  and  $x_i \in \{-1,1\}$ ,

$$\langle \mathbf{V}_{\nu,x}, \mathbf{V}_{\nu,x} \rangle = \frac{1}{N} \sum_{i \in [N]} x_i^2 \langle \nu_i, \nu_i \rangle^8 = \frac{1}{N} \sum_{i \in [N]} 1 = 1.$$

Hence, for every  $v \in V$  and  $x \in \{-1, 1\}^N$ ,

$$\|\mathbf{V}_{v,x}^{\otimes t}\| = 1. \tag{18}$$

Next, we show Property (5) in Theorem 2.9 which establishes that the SDP solution has value  $O(\varepsilon)$  when  $\eta = \varepsilon$ .

$$\textbf{Theorem 5.3 (Low Objective Value)} \ \ \textstyle \sum_{e^*\{(\nu,x),(w,y)\} \in E^*} \operatorname{wt}_{BS}(e^*) \|\mathbf{V}_{\nu,x}^{\otimes t} - \mathbf{V}_{w,y}^{\otimes t}\|^2 \leq O(\eta + \varepsilon).$$

The proof of this theorem uses the following lemma which shows that, if  $e\{v,w\}$  is an edge in the UNIQUEGAMES instance  $\mathcal{U}_{\eta}$  so that the corresponding orthonormal bases are  $\eta$ -close (via the permutation  $\pi$ ), then  $\mathbf{V}_{v,x}$  and  $\mathbf{V}_{w,y}$  are also close if  $x \circ \pi$  and y are close.

**Lemma 5.4** Let  $0 < \eta < 1/2$  and assume that for  $v, w \in V$  and  $i_0, j_0 \in [N], \langle v_{i_0}, w_{j_0} \rangle = 1 - \eta$ . Let  $\pi : [N] \mapsto [N]$  be defined to be  $\pi(j_0 \oplus j) \stackrel{\text{def}}{=} i_0 \oplus j \ \forall \ j \in [N]$ . Then,

- Lower Bound:  $(1-\eta)^8(1-2\Delta(x\circ\pi,y))-(2\eta)^4\leq \langle \mathbf{V}_{v,x},\mathbf{V}_{w,y}\rangle$ .
- Upper Bound:  $\langle \mathbf{V}_{\nu,x}, \mathbf{V}_{w,y} \rangle \leq (1-\eta)^8 (1-2\Delta(x\circ\pi,y)) + (2\eta)^4$ .

*Here,*  $\Delta(x,y)$  *denotes the fraction of points where x and y differ.* 

We first show how Lemma 5.4 implies Theorem 5.3.

**Proof.** [of Theorem 5.3] It is sufficient to prove that for an edge  $e\{v,w\} \in E$  picked with probability  $\operatorname{wt}(e)$  (from the UNIQUEGAMES instance  $\mathcal{U}_{\eta}$ ),  $x \in_{1/2} \{-1,1\}^N$ , and  $\mu \in_{\varepsilon} \{-1,1\}^N$ ,

$$\mathbb{E}_{e\{v,w\}}\left[\mathbb{E}_{x\in_{1/2}\{-1,1\}^N}\left\langle \mathbf{V}_{v,x}^{\otimes t}, \mathbf{V}_{w,x\mu\circ\pi_e}^{\otimes t}
ight
angle
ight]\geq 1-O(t(\eta+arepsilon)).$$

Since  $e\{v,w\}$  is an edge of  $\mathcal{U}_{\eta}$ , we know from the Closeness Property of Theorem 3.7, that there are  $i_0, j_0 \in [N]$  such that  $\langle v_{i_0}, w_{j_0} \rangle \geq 1 - O(\eta)$ . Moreover,  $\pi_e(j_0 \oplus j) = i_0 \oplus j$ ,  $\forall j \in [N]$ . Further, it follows from a simple Chernoff Bound argument that, except with probability  $\varepsilon$ ,  $\Delta(x,x\mu) \leq 2\varepsilon$ . Thus, using the lower bound estimate from Lemma 5.4, we get that

$$\langle \mathbf{V}_{v,x}^{\otimes t}, \mathbf{V}_{w,x\mu \circ \pi_e}^{\otimes t} \rangle \geq 1 - O(t(\eta + \varepsilon)).$$

This completes the proof.

We now present the proof of Lemma 5.4.

**Proof.** [of Lemma 5.4] Note that

$$\langle \mathbf{V}_{v,x}, \mathbf{V}_{w,y} \rangle = \frac{1}{N} \sum_{i,i' \in [N]} x_i y_{i'} \langle v_i, w_{i'} \rangle^8$$
$$= \frac{1}{N} \sum_{i,i' \in [N]} x_{i_0 \oplus i} y_{j_0 \oplus i'} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^8.$$

We first show that in the above summation, terms with i=i' dominate the summation. Since  $\langle v_{i_0}, w_{j_0} \rangle = 1-\eta$ , the Matching Property implies that for all  $i \in [N]$ ,  $\langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle = 1-\eta$ . Further, since the vectors  $\{w_{i'}\}_{i' \in [N]}$  form an orthonormal basis for  $\mathbb{R}^N$ ,  $\sum_{i' \in [N]} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^2 = 1$ . Hence,

$$\sum_{i' \in [N], i' \neq i} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^8 \le \left(1 - (1 - \eta)^2\right)^4 = (2\eta - \eta^2)^4 \le (2\eta)^4.$$

Now,  $\langle \mathbf{V}_{v,x}, \mathbf{V}_{w,y} \rangle$  is at least

$$\frac{1}{N} \sum_{i \in [N]} x_{i_0 \oplus i} y_{j_0 \oplus i} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle^8 - \frac{1}{N} \sum_{\substack{i, i' \in [N] \\ i \neq i'}} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^8,$$

and at most

$$\frac{1}{N} \sum_{i \in [N]} x_{i_0 \oplus i} y_{j_0 \oplus i} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle^8 + \frac{1}{N} \sum_{\substack{i,i' \in [N] \\ i \neq i'}} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^8.$$

The first term in both these expressions is

$$\frac{1}{N} \sum_{i \in [N]} x_{i_0 \oplus i} y_{j_0 \oplus i} (1 - \eta)^8 = (1 - 2\Delta(x \circ \pi, y)) (1 - \eta)^8.$$

The second term is bounded by  $(2\eta)^4$  as seen above. This completes the proof of the lemma.

The well-separatedness of the SDP solution, or Property (4) in Theorem 2.9, follows from the following lemma.

**Lemma 5.5 (Well Separatedness)** For any odd integer t > 0,

$$\frac{1}{2} \mathbb{E}_{x \in \mathbb{I}_{/2} \{-1,1\}^N, \ y \in \mathbb{I}_{/2} \{-1,1\}^N} \left[ \| \mathbf{V}_{\nu,x}^{\otimes t} - \mathbf{V}_{\nu,y}^{\otimes t} \|^2 \right] = 1.$$

**Proof.** Observe that

$$\frac{1}{2}\mathbb{E}_{x,y}\left[\|\mathbf{V}_{v,x}^{\otimes t} - \mathbf{V}_{v,y}^{\otimes t}\|^{2}\right] = \mathbb{E}_{x,y}\left[1 - \langle \mathbf{V}_{v,x}^{\otimes t}, \mathbf{V}_{v,y}^{\otimes t}\rangle\right]$$

$$= 1 - \mathbb{E}_{x,y}\left[\left(\frac{1}{N}\sum_{i,j\in[N]}x_{i}y_{j}\langle v_{i}, v_{j}\rangle^{8}\right)^{t}\right]$$

$$= 1.$$

The last equality follows from the fact that the contribution of (x, y) to the expectation is canceled by that of (x, -y).

Finally, the following theorem establishes that our SDP solution satisfies the triangle inequality, Property (3) of Theorem 2.9.

**Theorem 5.6 (Triangle Inequality)** For  $t = 2^{240} + 1$ , the set of vectors  $\{\mathbf{V}_{v,x}^{\otimes t}\}_{v \in V, x \in \{-1,1\}^N}$  give rise to a negative-type metric.

**Proof of Theorems 2.9 and 5.1.** Before we go into the proof of Theorem 5.6, we note that Theorem 5.2 and Theorem 5.3, along with (18), Lemma 5.5 and Theorem 5.6, for the choices  $\varepsilon = (\log \log n)^{-1/3}$  and  $\eta = O(\varepsilon)$  complete the proof of Theorem 2.9 and Theorem 5.1 (note that  $\operatorname{opt}(\mathcal{U}_{\eta}) \leq \log^{-\eta} \tilde{n} \leq 3\log^{-\eta} n$ ).

**Proof of Theorem 5.6.** Theorem 5.6 requires proving that any three vectors  $\mathbf{V}_{u,x}^{\otimes t}$ ,  $\mathbf{V}_{v,y}^{\otimes t}$  and  $\mathbf{V}_{w,z}^{\otimes t}$  satisfy

$$1 + \langle \mathbf{V}_{u,x}^{\otimes t}, \mathbf{V}_{v,y}^{\otimes t} \rangle \ge \langle \mathbf{V}_{u,x}^{\otimes t}, \mathbf{V}_{w,z}^{\otimes t} \rangle + \langle \mathbf{V}_{v,y}^{\otimes t}, \mathbf{V}_{w,z}^{\otimes t} \rangle. \tag{19}$$

We can assume that at least one of the dot-products has magnitude at least 1/3; otherwise, the inequality holds trivially. Assume, w.l.o.g., that

$$|\langle \mathbf{V}_{v,v}^{\otimes t}, \mathbf{V}_{w,z}^{\otimes t} \rangle| \ge 1/3.$$

This implies that  $|\langle \mathbf{V}_{v,v}, \mathbf{V}_{w,z} \rangle|^t \ge 1/3$ , and therefore,

$$|\langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle| = 1 - \eta',$$

for some  $\eta' = O(1/t)$ . It follows that, for some  $i, j \in [N]$ ,  $|\langle v_i, w_j \rangle| = 1 - \beta$  for some  $\beta \le 2^{-160}$ . We give a quick proof of this. Let  $i_0, j_0$  be  $\arg\max_{i,j} |\langle v_i, w_j \rangle|$  and  $1 - \beta = |\langle v_{i_0}, w_{j_0} \rangle|$ . Then,

$$|\langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle| = \left| \frac{1}{N} \sum_{i,j \in [N]} y_i z_j \langle v_i, w_j \rangle^8 \right| \leq \frac{1}{N} \sum_{i \in [N]} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle^8 + \frac{1}{N} \sum_{i \neq j \in [N]} \langle v_{i_0 \oplus i}, w_{j_0 \oplus j} \rangle^8.$$

By the Matching Property,  $\langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle = \langle v_{i_0}, w_{j_0} \rangle$  for all  $i \in [N]$ . Hence,

$$\frac{1}{N} \sum_{i \in [N]} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i} \rangle^8 = (1 - \beta)^8.$$

Moreover, by orthonormality, for all  $i \in [N]$ ,

$$\sum_{i' \in [N], i' \neq i} \langle v_{i_0 \oplus i}, w_{j_0 \oplus i'} \rangle^8 \le \left(1 - (1 - \beta)^2\right)^4 = (2\beta - \beta^2)^4 \le (2\beta)^4.$$

Thus,

$$1 - \eta' = |\langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle| \le (1 - \beta)^8 + (2\beta)^4,$$

giving us the claimed upper bound on  $\beta$ . By relabeling, if necessary, we may assume that  $|\langle v_1, w_1 \rangle| = 1 - \beta$ . Note that (19) is equivalent to showing that

$$1 + \langle \mathbf{V}_{u,x}, \mathbf{V}_{v,y} \rangle^t \ge \langle \mathbf{V}_{u,x}, \mathbf{V}_{w,z} \rangle^t + \langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle^t.$$

The following elementary lemma, whose proof appears at the end of this section, implies that it is sufficient to prove that

$$1 + \langle \mathbf{V}_{u,x}, \mathbf{V}_{v,y} \rangle \ge \langle \mathbf{V}_{u,x}, \mathbf{V}_{w,z} \rangle + \langle \mathbf{V}_{v,y}, \mathbf{V}_{w,z} \rangle. \tag{20}$$

**Lemma 5.7** Let  $a,b,c \in [-1,1]$  such that  $1+a \ge b+c$ . Then,  $1+a^t \ge b^t+c^t$  for every odd integer  $t \ge 1$ .

Equation (20) is the same as showing

$$N + \sum_{i,j=1}^{N} x_i y_j \langle u_i, v_j \rangle^8 \ge \sum_{i,j=1}^{N} x_i z_j \langle u_i, w_j \rangle^8 + \sum_{i,j=1}^{N} y_i z_j \langle v_i, w_j \rangle^8.$$

As noted before, we may assume that  $|\langle v_1, w_1 \rangle| = 1 - \beta$  and, hence, by the Matching Property,

$$\langle v_1, w_1 \rangle = \langle v_2, w_2 \rangle = \cdots = \langle v_N, w_N \rangle = \pm (1 - \beta).$$

Let  $\lambda \stackrel{\text{def}}{=} \max_{1 \leq i,j \leq N} |\langle u_i, w_j \rangle|$ . We may assume, w.l.o.g., that the maximum is achieved for  $u_1, w_1$ , and again by the Matching Property,

$$\langle u_1, w_1 \rangle = \langle u_2, w_2 \rangle = \cdots = \langle u_N, w_N \rangle = \pm \lambda.$$

Now, Theorem 5.6 follows from the following lemma.

**Lemma 5.8** Let  $\{u_i\}_{i=1}^N, \{v_i\}_{i=1}^N, \{w_i\}_{i=1}^N$  be three sets of unit vectors in  $\mathbb{R}^N$ , such that the vectors in each set are mutually orthogonal. Assume that any three of these vectors satisfy the triangle inequality. Assume, moreover, that

$$\langle u_1, v_1 \rangle = \langle u_2, v_2 \rangle = \dots = \langle u_N, v_N \rangle,$$

$$\lambda \stackrel{\text{def}}{=} \langle u_1, w_1 \rangle = \langle u_2, w_2 \rangle = \dots = \langle u_N, w_N \rangle \ge 0,$$

$$\forall 1 \le i, j \le N, \ |\langle u_i, w_j \rangle| \le \lambda,$$

$$1 - \beta \stackrel{\text{def}}{=} \langle v_1, w_1 \rangle = \langle v_2, w_2 \rangle = \dots = \langle v_N, w_N \rangle,$$

where  $0 \le \beta \le 2^{-160}$ . Let  $x_i, y_i, z_i \in \{-1, 1\}$  for  $1 \le i \le N$ . Define unit vectors

$$u \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i u_i^{\otimes 8}, \quad v \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} y_i v_i^{\otimes 8} \quad w \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} z_i w_i^{\otimes 8}.$$

Then, the vectors u, v, w satisfy the triangle inequality  $1 + \langle u, v \rangle \ge \langle u, w \rangle + \langle v, w \rangle$ , i.e.,

$$N + \sum_{i,j=1}^{N} x_i y_j \langle u_i, v_j \rangle^8 \ge \sum_{i,j=1}^{N} x_i z_j \langle u_i, w_j \rangle^8 + \sum_{i,j=1}^{N} y_i z_j \langle v_i, w_j \rangle^8.$$

Note that we only have  $|\langle v_1, w_1 \rangle| = 1 - \beta$  but we can remove the absolute value and use this lemma as it holds for *all* sign patterns  $x_i, y_i, z_i$ . The proof of this lemma is very technical and appears in Appendix B. We conclude with a proof of Lemma 5.7.

**Proof.** [of Lemma 5.7] First, we notice that it is sufficient to prove this inequality when  $0 \le a, b, c \le 1$ . Suppose that b < 0 and c < 0, then  $b^t + c^t < 0 \le 1 + a^t$ . Hence, without loss of generality assume that  $b \ge 0$ . If c < 0 and  $a \ge 0$ , then  $b^t + c^t < b^t \le 1 + a^t$ . If c < 0 and a < 0, by hypothesis,  $1 - c \ge b - a$ , which is the same as  $1 + |c| \ge b + |a|$ , and proving  $1 + a^t \ge b^t + c^t$  is equivalent to proving  $1 + |c|^t \ge b^t + |a|^t$ . Hence, we may assume that  $c \ge 0$ . If a < 0, then  $1 + a^t = 1 - |a|^t \ge 1 - |a| = 1 + a \ge b + c \ge b^t + c^t$ . Hence, we may assume that 0 < a, b, c < 1.

Further, we may assume that  $a < b \le c$ . Since, if  $a \ge b$ , then  $1 + a^t \ge c^t + b^t$ .  $1 + a \ge b + c$  implies that  $1 - c \ge b - a$ . Notice that both sides of this inequality are positive. It follows from the fact that  $0 \le a < b \le c \le 1$ , that  $\sum_{i=0}^{t-1} c^i \ge \sum_{i=0}^{t-1} a^i b^{t-1-i}$ . Multiplying these two inequalities, we obtain  $1 - c^t \ge b^t - a^t$ , which implies that  $1 + a^t \ge b^t + c^t$ . This completes the proof.

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#### A Proof of Soundness of the PCP Reduction

**Lemma A.1 (Same as Lemma 4.3)** For every  $t \in (1/2, 1)$ , there exists a constant  $b_t > 0$  such that the following holds: Let  $\varepsilon > 0$  be sufficiently small and let  $\mathscr{U}$  be an instance of UNIQUEGAMES with  $\operatorname{opt}(\mathscr{U}) < 2^{-O(1/\varepsilon^2)}$ . Then, for every 5/6-piecewise balanced proof  $\Pi$ ,

$$\Pr[V_{\varepsilon} \text{ accepts } \Pi] < 1 - b_t \varepsilon^t$$
.

**Proof.** The proof is by contradiction: We assume that there is a 5/6-piecewise balanced proof  $\Pi$ , which the verifier accepts with probability at least  $1 - b_t \varepsilon^t$ , and deduce that  $\operatorname{opt}(\mathscr{U}) \geq 2^{-O(1/\varepsilon^2)}$ . We let  $b_t \stackrel{\text{def}}{=} \frac{1 - e^{-2}}{96} c_t$ , where  $c_t$  is the constant in Bourgain's Junta theorem.

The probability of acceptance of the verifier is

$$\frac{1}{2} + \frac{1}{2} \mathbb{E}_{\nu, e\{\nu, w\}, x, \mu} \left[ A^{\nu}(x) A^{w}(x \mu \circ \pi_{e}) \right].$$

Using the Fourier expansion  $A^{\nu} = \sum_{\alpha} \widehat{A}_{\alpha}^{\nu} \chi_{\alpha}$  and  $A^{w} = \sum_{\beta} \widehat{A}_{\beta}^{w} \chi_{\beta}$ , and the orthonormality of characters, we get that this probability is

$$\frac{1}{2} + \frac{1}{2} \mathbb{E}_{v,e\{v,w\}} \left[ \sum_{\alpha} \widehat{A}^{v}_{\alpha} \widehat{A}^{w}_{n_{e}^{-1}(\alpha)} (1 - 2\varepsilon)^{|\alpha|} \right].$$

Here  $\alpha \subseteq [N]$ . Hence, the acceptance probability is

$$\frac{1}{2} + \frac{1}{2} \mathbb{E}_{\nu} \left[ \sum_{\alpha} \widehat{A}^{\nu}_{\alpha} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)} \right] (1 - 2\varepsilon)^{|\alpha|} \right].$$

If this acceptance probability is at least  $1 - b_t \varepsilon^t$ , then,

$$\mathbb{E}_{v}\left[\sum_{\alpha} \widehat{A}_{\alpha}^{v} \mathbb{E}_{e\{v,w\}}\left[\widehat{A}_{\pi_{e}^{-1}(\alpha)}^{w}\right] (1-2\varepsilon)^{|\alpha|}\right] \geq 1-2b_{t}\varepsilon^{t}.$$

Hence, over the choice of v, with probability at least  $\frac{23}{24}$ ,

$$\sum_{\alpha} \widehat{A}_{\alpha}^{\nu} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}_{\pi_e^{-1}(\alpha)}^{w} \right] (1 - 2\varepsilon)^{|\alpha|} \ge 1 - 48b_t \varepsilon^t.$$

Call such vertices  $v \in V$  good. Fix a good vertex v. Using the Cauchy-Schwarz inequality we get,

$$\sum_{\alpha} \widehat{A}_{\alpha}^{\nu} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}_{\pi_e^{-1}(\alpha)}^{w} \right] (1 - 2\varepsilon)^{|\alpha|} \leq \sqrt{\sum_{\alpha} \left( \widehat{A}_{\alpha}^{\nu} \right)^2 (1 - 2\varepsilon)^{2|\alpha|} \sum_{\alpha} \mathbb{E}_{e\{\nu,w\}}^2 \left[ \widehat{A}_{\pi_e^{-1}(\alpha)}^{w} \right]}.$$

Combining Jensen's inequality and Parseval's identity, we get that

$$\sum_{\alpha} \mathbb{E}^{2}_{e\{v,w\}} \left[ \widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)} \right] \leq 1.$$

Hence,

$$1 - 96b_t \varepsilon^t \le \sum_{\alpha} \left( \widehat{A}_{\alpha}^{\nu} \right)^2 (1 - 2\varepsilon)^{2|\alpha|}.$$

Now we combine Parseval's identity with the fact that  $1 - x \le e^{-x}$  to obtain

$$\sum_{\alpha : |\alpha| > 1/\varepsilon} \left( \widehat{A}_{\alpha}^{\nu} \right)^2 \leq \frac{96}{1 - e^{-2}} b_t \varepsilon^t = c_t \varepsilon^t.$$

Hence, by Bourgain's Junta theorem

$$\sum_{\alpha \ : \ |\widehat{A}^{\nu}_{\alpha}| \leq \frac{1}{50} 4^{-1/\epsilon^2}} \left( \widehat{A}^{\nu}_{\alpha} \right)^2 \leq \frac{1}{2500}.$$

Call  $\alpha$  good if  $\alpha\subseteq [N]$  is nonempty,  $|\alpha|\leq \varepsilon^{-1}$  and  $|\widehat{A}_{\alpha}^{\nu}|\geq \frac{1}{50}4^{-1/\varepsilon^2}$ .

#### Bounding the contribution due to large sets.

Using the Cauchy-Schwarz inequality, Parseval's identity and Jensen's inequality, we get

$$\left|\sum_{\alpha \ : \ |\alpha| > 1/\varepsilon} \widehat{A}^{\nu}_{\alpha} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)} \right] (1 - 2\varepsilon)^{|\alpha|} \right| \leq \sqrt{\sum_{\alpha \ : \ |\alpha| > 1/\varepsilon} \left( \widehat{A}^{\nu}_{\alpha} \right)^{2}} < \sqrt{c_{t} \varepsilon^{t}}.$$

We can choose  $\varepsilon$  to be small enough so that the last term above is less than 1/50.

#### Bounding the contribution due to small Fourier coefficients.

Similarly, we use  $\sum_{\alpha : |\widehat{A}_{\alpha}^{\nu}| \leq \frac{1}{50}4^{-1/\epsilon^2}} \left(\widehat{A}_{\alpha}^{\nu}\right)^2 \leq 1/2500$ , and get

$$\left|\sum_{\alpha \ : \ |\widehat{A}^{\nu}_{\alpha}| \leq \frac{1}{50} 4^{-1/\varepsilon^2}} \widehat{A}^{\nu}_{\alpha} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)} \right] (1 - 2\varepsilon)^{|\alpha|} \right| \leq \frac{1}{50}.$$

#### Bounding the contribution due to the empty set.

Since  $\mathbb{E}_{v}\left[|\widehat{A}_{\emptyset}^{v}|\right] \leq 5/6$ ,  $\mathbb{E}_{v}\left[\mathbb{E}_{e\{v,w\}}\left[|\widehat{A}_{\emptyset}^{v}\widehat{A}_{\emptyset}^{w}|\right]\right] \leq 5/6$ . This is because each  $|\widehat{A}_{\emptyset}^{v}| \leq 1$ . Hence, with probability at least 1/12 over the choice of v,  $\mathbb{E}_{e\{v,w\}}\left[|\widehat{A}_{\emptyset}^{v}\widehat{A}_{\emptyset}^{w}|\right] \leq 10/11$ . Hence, with probability at least 1/24 over the choice of v, v is good and  $\mathbb{E}_{e\{v,w\}}\left[|\widehat{A}_{\emptyset}^{v}\widehat{A}_{\emptyset}^{w}|\right] \leq 10/11$ . Call such a vertex  $very\ good$ .

#### Lower bound for a very good vertex with good sets.

Hence, for a very good v,

$$\sum_{\alpha \text{ is good}} \widehat{A}_{\alpha}^{\nu} \mathbb{E}_{e\{\nu,w\}} \left[ \widehat{A}_{\pi_e^{-1}(\alpha)}^{w} \right] (1 - 2\varepsilon)^{|\alpha|} \ge 1 - \frac{1}{50} - \frac{1}{50} - \frac{10}{11} \ge \frac{1}{22}.$$
 (21)

#### The labeling.

Now we define a labeling for the UNIQUEGAMES instance  $\mathscr{U}$  as follows: For a vertex  $v \in V$ , pick  $\alpha$  with probability  $\left(\widehat{A}_{\alpha}^{\nu}\right)^{2}$ , pick a random element of  $\alpha$  and define it to be the label of  $\nu$ .

Let *v* be a very good vertex. It follows that the weight of the edges adjacent to *v* satisfied by this labeling is at least

$$\mathbb{E}_{e\{v,w\}}\left[\sum_{\alpha \text{ is good}} \left(\widehat{A}^{v}_{\alpha}\right)^{2} \left(\widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)}\right)^{2} \frac{1}{|\alpha|}\right] \geq \varepsilon \; \mathbb{E}_{e\{v,w\}}\left[\sum_{\alpha \text{ is good}} \left(\widehat{A}^{v}_{\alpha}\right)^{2} \left(\widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)}\right)^{2}\right].$$

This is at least

$$arepsilon \, rac{1}{2500} 4^{-2/arepsilon^2} \, \mathbb{E}_{e\{v,w\}} \left[ \sum_{lpha ext{ is good}} \left( \widehat{A}_{\pi_e^{-1}(lpha)}^w 
ight)^2 
ight],$$

which is at least

$$arepsilon \, rac{1}{2500} 4^{-2/arepsilon^2} \, \mathbb{E}_{e\{
u,w\}} \left[ \sum_{lpha ext{ is good}} \left( \widehat{A}^w_{\pi_e^{-1}(lpha)} 
ight)^2 (1-2arepsilon)^{|lpha|} 
ight].$$

It follows from the Cauchy-Schwarz inequality and Parseval's identity that this is at least

$$arepsilon \, rac{1}{2500} 4^{-2/arepsilon^2} \, \mathbb{E}_{e\{v,w\}} \left[ \left| \sum_{lpha ext{ is good}} \widehat{A}^v_lpha \widehat{A}^w_{\pi^{-1}_e(lpha)} (1-2arepsilon)^{|lpha|} 
ight|^2 
ight].$$

Using Jensen's inequality, we get that this is at least

$$\varepsilon \; \frac{1}{2500} 4^{-2/\varepsilon^2} \; \left( \mathbb{E}_{e\{\nu,w\}} \left[ \sum_{\alpha \text{ is good}} \widehat{A}^{\nu}_{\alpha} \widehat{A}^{w}_{\pi_{e}^{-1}(\alpha)} (1-2\varepsilon)^{|\alpha|} \right] \right)^2 \geq \varepsilon \; \frac{1}{2500} 4^{-2/\varepsilon^2} \frac{1}{484}.$$

Here, the last inequality follows from our estimate in Equation (21). Since, with probability at least 1/24 over the choice of v, v is very good, our labeling satisfies edges with total weight at least  $\Omega\left(\varepsilon \, 4^{-2/\varepsilon^2}\right)$ . This completes the proof of the lemma.

### B Proof of Lemma 5.8 (Triangle Inequality Constraint)

**Lemma B.1** [Same as Lemma 5.8 up to a renaming of variables] Let  $\{u_i\}_{i=1}^N$ ,  $\{v_i\}_{i=1}^N$ ,  $\{w_i\}_{i=1}^N$  be three sets of unit vectors in  $\mathbb{R}^N$ , such that the vectors in each set are mutually orthogonal. Assume that any three of these vectors satisfy the triangle inequality. Assume, moreover, that

$$\langle u_1, v_1 \rangle = \langle u_2, v_2 \rangle = \dots = \langle u_N, v_N \rangle,$$
 (22)

$$\lambda \stackrel{\text{def}}{=} \langle u_1, w_1 \rangle = \langle u_2, w_2 \rangle = \dots = \langle u_N, w_N \rangle \ge 0, \tag{23}$$

$$\forall 1 \le i, j \le N, \ |\langle u_i, w_j \rangle| \le \lambda, \tag{24}$$

$$1 - \eta \stackrel{\text{def}}{=} \langle v_1, w_1 \rangle = \langle v_2, w_2 \rangle = \dots = \langle v_N, w_N \rangle, \tag{25}$$

where  $0 \le \eta \le 2^{-160}$ . Let  $s_i, t_i, r_i \in \{-1, 1\}$  for  $1 \le i \le N$ . Define unit vectors

$$u \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} s_i u_i^{\otimes 8}, \quad v \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} t_i v_i^{\otimes 8} \quad w \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} r_i w_i^{\otimes 8}.$$

Then, the vectors u, v, w satisfy the triangle inequality  $1 + \langle u, v \rangle \ge \langle u, w \rangle + \langle v, w \rangle$ , i.e.,

$$N + \sum_{i,j=1}^{N} s_i t_j \langle u_i, v_j \rangle^8 \ge \sum_{i,j=1}^{N} s_i r_j \langle u_i, w_j \rangle^8 + \sum_{i,j=1}^{N} t_i r_j \langle v_i, w_j \rangle^8.$$
 (26)

**Proof.** It suffices to show that for every  $1 \le j \le N$ ,

$$1 + \sum_{i=1}^{N} s_i t_j \langle u_i, v_j \rangle^8 \ge \sum_{i=1}^{N} s_i r_j \langle u_i, w_j \rangle^8 + t_j r_j \langle v_j, w_j \rangle^8 + \sum_{1 \le i \le N, i \ne j} \langle v_i, w_j \rangle^8.$$

$$(27)$$

We consider four cases depending on value of  $\lambda$ .

(Case 1)  $\lambda \leq \eta$ : Since  $\langle v_j, w_j \rangle = 1 - \eta$ , and  $\sum_{1 \leq i \leq N} \langle v_i, w_j \rangle^2 = 1$ , we have

$$\sum_{1 \le i \le N; i \ne j} \langle v_i, w_j \rangle^8 \le (2\eta - \eta^2)^4.$$

Also,  $\sum_{i=1}^{N} \langle u_i, w_i \rangle^8 \le \lambda^6 \le \eta^6$ . Moreover, for any  $1 \le i \le N$ , by the triangle inequality,

$$1 \pm \langle u_i, v_i \rangle \geq \langle v_i, w_i \rangle \pm \langle u_i, w_i \rangle \geq 1 - \eta - \lambda \geq 1 - 2\eta$$

and therefore,

$$|\langle u_i, v_j \rangle| \leq 2\eta$$
.

Therefore,  $\sum_{i=1}^{N} \langle u_i, v_i \rangle^8 \le (2\eta)^6$ . Thus, it suffices to prove that

$$1 \ge (2\eta)^6 + \eta^6 + (1-\eta)^8 + (2\eta - \eta^2)^4$$

This is true when  $\eta \leq 2^{-160}$ .

(Case 2)  $\eta \le \lambda \le 1 - \sqrt{\eta}$ : We show that

$$1 + \sum_{i=1}^{N} s_i t_j \langle u_i, v_j \rangle^8 \ge \sum_{i=1}^{N} s_i r_j \langle u_i, w_j \rangle^8 + t_j r_j (1 - \eta)^8 + (2\eta - \eta^2)^4.$$
 (28)

(**Subcase i**)  $t_j \neq r_j$ : In this case it suffices to show that

$$1 + (1 - \eta)^8 \ge \sum_{i=1}^N \langle u_i, v_j \rangle^8 + \sum_{i=1}^N \langle u_i, w_j \rangle^8 + (2\eta - \eta^2)^4.$$

Again, as before, we have that for every  $1 \le i \le N$ ,

$$|\langle u_i, w_j \rangle| \le \lambda \le 1 - \sqrt{\eta},$$

and

$$|\langle u_i, v_j \rangle| \leq \lambda + \eta \leq 1 - \sqrt{\eta} + \eta.$$

Thus, it suffices to prove that

$$1 + (1 - \eta)^8 \ge (1 - \sqrt{\eta} + \eta)^6 + (1 - \sqrt{\eta})^6 + (2\eta - \eta^2)^4.$$

This also holds when  $\eta \leq 2^{-160}$ .

(**Subcase ii**)  $t_i = r_i$ : We need to prove (28). It suffices to show that

$$1 - (1 - \eta)^8 - (2\eta - \eta^2)^4 \ge \sum_{i=1}^N |\langle u_i, w_j \rangle|^8 - \langle u_i, v_j \rangle^8| = \sum_{i=1}^N |\theta_i^8 - \mu_i^8|$$

where  $\theta_i \stackrel{\text{def}}{=} |\langle u_i, w_j \rangle|$ ,  $\mu_i \stackrel{\text{def}}{=} |\langle u_i, v_j \rangle|$ . Clearly,

$$|\theta_i - \mu_i| \le |\langle u_i, v_j \rangle - \langle u_i, w_j \rangle| \le 1 - \langle v_i, w_j \rangle = \eta.$$

Here, we used the assumption that  $(u_i, v_j, w_j)$  satisfy the triangle inequality. Note also that  $\max_{1 \le i \le N} \theta_i = \lambda$  and  $\sum_{i=1}^N \theta_i^2 = 1$ . Let  $J \stackrel{\text{def}}{=} \{i \mid \theta_i \le \eta\}$  and  $I \stackrel{\text{def}}{=} \{i \mid \theta_i \ge \eta\}$ . We have,

$$\begin{split} \sum_{i=1}^N |\theta_i^8 - \mu_i^8| & \leq & \sum_{i \in J} (\theta_i^8 + \mu_i^8) + \sum_{i \in I} ((\theta_i + \eta)^8 - \theta_i^8) \\ & \leq & (\eta)^6 + (2\eta)^6 + \sum_{i \in J} ((\theta_i + \eta)^8 - \theta_i^8). \end{split}$$

Lemma B.2, which appears after this proof, implies that the summation on the last line above is bounded by

$$\sum_{l=1}^{6} {8 \choose l} \lambda^{6-l} \eta^l + 9\eta^6.$$

Thus, it suffices to show that

$$1-(1-\eta)^8-(2\eta-\eta^2)^4 \, \geq \, \sum_{l=1}^6 inom{8}{l} \lambda^{6-l} \eta^l + (4\eta)^6.$$

This is true if

$$8\eta - \sum_{l=2}^{8} {8 \choose l} \eta^l - (2\eta - \eta^2)^4 \ge 8\lambda^5 \eta + \sum_{l=2}^{8} {8 \choose l} \eta^l + (4\eta)^6.$$

This is true if  $8\eta(1-\lambda^5) \ge \eta^2(2^8+2^8+1+4^8)$ . This is true if  $8\eta\sqrt{\eta} \ge \eta^2\cdot 4^9$ , which holds when  $\eta \le 2^{-160}$ . Note that we used the fact that  $\lambda \le 1-\sqrt{\eta}$ .

(Case 3)  $1 - \sqrt{\eta} \le \lambda \le 1 - \eta^2$ : We have  $\langle v_j, w_j \rangle = 1 - \eta$ ,  $\langle u_j, w_j \rangle = \lambda =: 1 - \zeta$ . This implies that  $\langle u_j, v_j \rangle = 1 - \delta$ , where by the triangle inequality

$$\eta \leq \zeta + \delta, \ \delta \leq \eta + \zeta, \ \zeta \leq \eta + \delta.$$

Thus, to prove (27), it suffices to show that

$$1 + s_i t_i \langle u_i, v_i \rangle^8 \geq s_i r_i \langle u_i, w_i \rangle^8 + t_i r_i \langle v_i, w_i \rangle^8 + (2\eta - \eta^2)^s + (2\zeta - \zeta^2)^4 + (2\delta - \delta^2)^4.$$

Depending on signs  $s_j, t_j, r_j$ , this reduces to proving one of the three cases:

$$\begin{split} &1+(1-\delta)^8 \geq (1-\zeta)^8 + (1-\eta)^8 + (2\eta-\eta^2)^4 + (2\zeta-\zeta^2)^4 + (2\delta-\delta^2)^4.\\ &1+(1-\eta)^8 \geq (1-\zeta)^8 + (1-\delta)^8 + (2\eta-\eta^2)^4 + (2\zeta-\zeta^2)^4 + (2\delta-\delta^2)^4.\\ &1+(1-\zeta)^8 \geq (1-\eta)^8 + (1-\delta)^8 + (2\eta-\eta^2)^4 + (2\zeta-\zeta^2)^4 + (2\delta-\delta^2)^4. \end{split}$$

We prove the first case, and the remaining two are proved in a similar fashion. We have that

$$\begin{array}{ll} 1 + (1-\delta)^8 - (1-\zeta)^8 - (1-\eta)^8 & \geq & 1 + (1-(\zeta+\eta))^8 - (1-\zeta)^8 - (1-\eta)^8 \\ & \geq & 8 \cdot 7 \cdot \zeta \eta - \sum_{\substack{3 \leq i+j \leq 8 \\ i \geq 1, j \geq 1}} \binom{8}{i+j} \binom{i+j}{i} \zeta^i \eta^j \\ & \geq & 8 \cdot 7 \cdot \zeta \eta - 2^{32} \zeta \eta \cdot \max\{\zeta, \eta, \delta\} \\ & \geq & \min\{\zeta \eta, \eta \delta, \zeta \delta\}, \end{array}$$

provided that  $2^{32}\max\{\zeta,\eta,\delta\}\leq 1$ . Thus, it suffices to have

$$\min\{\zeta\eta,\eta\delta,\zeta\delta\} \geq (2\eta-\eta^2)^4 + (2\zeta-\zeta^2)^4 + (2\delta-\delta^2)^4.$$

This is clearly true if  $\zeta, \eta, \delta$  are within a quadratic factor of each other, and  $\eta \leq 2^{-160}$ . On the contrary if  $\delta < \eta^2$ , since we already have  $\delta \leq \eta + \zeta$  from the triangle inequality, it reduces to Case (2) by setting  $\eta$  to  $\delta$  and setting  $\lambda$  to  $1 - \eta$ .

(Case 4)  $1 - \eta^2 \le \lambda$ : This is essentially same as Case (2). Just interchange  $1 - \eta$  with  $\lambda$  and interchange  $u_i, v_i$  for every i. This completes the proof of the lemma.

**Lemma B.2** Let  $\eta$ ,  $\lambda$  and  $\{\theta_i\}_{i=1}^N$  be non-negative reals, such that  $\sum_{i=1}^N \theta_i^2 \leq 1$ , and for all  $i, \eta \leq \theta_i \leq \lambda$ .

$$\sum_{i=1}^{N} ((\theta_i + \eta)^8 - \theta_i^8) \le \sum_{l=1}^{6} {8 \choose l} \lambda^{6-l} \eta^l + 9\eta^6.$$

**Proof.** Clearly,  $N \leq 1/\eta^2$ .

$$\begin{split} \sum_{i=1}^{N} (\theta_{i} + \eta)^{8} - \theta_{i}^{8} &= \sum_{i=1}^{N} \sum_{l=1}^{8} {8 \choose l} \theta_{i}^{8-l} \eta^{l} \\ &= \sum_{l=1}^{8-2} {8 \choose l} \sum_{i=1}^{N} \theta_{i}^{8-l} \eta^{l} + 8 \cdot \left( \sum_{i=1}^{N} \theta_{i} \right) \eta^{7} + N \eta^{8} \\ &\leq \sum_{l=1}^{6} {8 \choose l} \lambda^{6-l} \eta^{l} + 8 \cdot \sqrt{N} \eta^{7} + N \eta^{8} \\ &\leq \sum_{l=1}^{6} {8 \choose l} \lambda^{6-l} \eta^{l} + 9 \eta^{6}. \end{split}$$