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Robustness in Dynamical and Control Systems

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***Abstract*— We compile some results on robustness of dynamical and control systems. As control theory is preoccupied with stability problems, the robustness put forward in this paper is related to stability. We ask the question whether an asymptotically stable system remains asymptotically stable when perturbations are affecting it. We analyze robustness of control systems by examining vector fields in C^r topology, by studying associated Lyapunov functions, and by studying corresponding input-output maps. In the first case, we conclude that there is an open set of perturbations such that the system that is affected by them stays asymptotically stable. In the second case, we estimate the size of perturbations that do not destabilize the system. In the third and last case, we provide conditions on the gains of the interconnected systems such that the closed loop system has finite gain.**

I. Introduction

In this paper, we intend to gather central information about how issues related to robustness are addressed in control theory. Needless to say that this exposition will not be exhaustive. In brief, the aim of this paper is to extract the work that has been carried out in control theory in the light of current trends in computer science. Our particular attention has been [1], which defines a number of convenient metics for a robust concept of bi-simulation. To this end, we decided to present the concepts that involves metric spaces as this material might occur adaptable to computer science. Hence, in particular, robustness to parametric uncertainty is not considered in this article [2].

At the outset, we remark that control theory is preoccupied by the question of stability. Indeed, this is not without a reason, the majority of the questions of interest for control engineers can be re-phrased as questions about stability. As a consequence, the control theory predominantly deals with robustness related to stability. To this end, we suppose that the nominal system is stable, possibly by means of feedback control and ask the question if this system remains to be stable when affected by small perturbations.

To formalize the concept of a perturbation, we shall introduce several metrics in the course of this exposition. Subsequently, equipped with a convenient metric space, we wish to estimate the bounds of the perturbations that do not destabilize the system.

In this paper, we placed our emphasis on nonlinear systems. In Section III, we present the results on robustness of a particular class of vector fields - gradient-like vector fields. Any small perturbation of a gradient-like dynamical system is a system that qualitatively behaves just like the nominal system. In Section IV, we suppose that the system in hand is asymptotically stable at a singular (equilibrium) point and ask a question of which perturbations can be added to the nominal system such that the resulting dynamical system is asymptotically stable. In the remaining part of the paper, we refer to the theory of interconnected systems and input-output stability. In Section V, we put forward a version of the small gain theorem. It states that a closed loop system consisting of two input-output stable systems is input-output stable provided their gains are sufficiently small.

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II. Nomenclature

$\mathbb{Z}_+$ is the set of nonnegative integers, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R}_+$ is the set of nonnegative reals. Id stands for the identity map with the domain and the range matching the problem in hand.

In the first part of the exposition, we frequently use the notion of a smooth manifold [3]. Without loosing the thread of this exposition, the readers not acquainted with smooth manifolds may conceptually substitute a manifold, say M, of dimension m with a Euclidean space R^m, and Riemannian metric g withe the canonical scalar product on R^n, i.e., in coordinates $g(x, y) = x^T y$. Two simple manifolds, a sphere and a "deformed" cylinder are illustrated in Fig. 2. Equipped with a metric, one is able to compute lengths of a piecewise smooth curve $\gamma : [0, 1] \to M$ as

$$l(\gamma) \equiv \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt.$$

As a consequence, the distance between points p and q on M is the infimum of

$$d(p, q) \equiv \inf\{l(\gamma) : \ \gamma \in \Omega_{p,q}\}$$

over the set $\Omega_{p,q}$ of piecewise smooth curves defined on the unit interval $[0, 1]$ joining p and q. Again, on $\mathbb{R}^m$, $d(p, q) = ||p - q||$ with $|| \cdot ||$ denoting the Euclidean norm.

III. Robustness of gradient-like Dynamical Systems

Our first observation is that robustness in control theory is studied by considering the solution trajectories of a system indirectly. We distinguish at least three ways of doing so:

- to study the influence of a perturbation on a vector field;
- to study the influence of a perturbation on a function associated to a nominal vector field;
- to study influence of a perturbation on an input-to-ouput map, which describes a control system.

In this section, we address the first problem, i.e., the influence of small perturbations on a nominal vector field.

A. C^r topology

We follow [4], [5] and define C^r vector fields. For $r \in \mathbb{N} \cup \{\infty\}$, let $C^r(M,N)$ denote the space of C^r maps between two C^r manifolds M and N.

Let $f \in C^r(M,N)$, and let (ϕ, U) and (ψ, V) be charts on M and N respectively. Let $K \subset U$ be a compact set such that $f(K) \subset V$, and let $0 \leq \epsilon \leq \infty$. Define the subbasis element

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \epsilon)$$

to be the set of C^r maps $g : M \to N$ such that $g(K) \subset V$ and

$$\|d^k(\psi \circ f \circ \phi^{-1})(x) - d^k(\psi \circ g \circ \phi^{-1})(x)\| < \epsilon$$

for all $x \in \phi(K)$ and $k = 0, ..., r$. The C^r topology on $C^r(M,N)$ is defined to be the topology generated by the subbasis elements $\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \epsilon)$. The C^∞ topology on $C^\infty(M,N)$ is defined to be the union of topologies induced by the inclusions $C^\infty(M,N) \to C^r(M,N)$ for all $r \in \mathbb{N} \cup \{\infty\}$.

By Theorem 2.4.4 in [5], $C^r(M,N)$, $r \in \mathbb{N} \cup \{\infty\}$, with the C^r topology arises from a complete metric. In the following, we shall construct a metric for the space $C^r(M^n, \mathbb{R}^s)$ with M a closed C^r manifold such that this metric generates the topology, which coincides with the C^r topology.

The space $C^r(M, \mathbb{R}^s)$ has a canonical vector space structure: For $f,\ g \in C^r(M, \mathbb{R}^s)$ and a real λ we define

$$(f+g)(p) = f(p) + g(p),\ (\lambda f)(p) = \lambda f(p) \ \text{ for all } p \in M.$$

We shall take a finite open cover $\{V_i\}_{i=1,...,k}$ of M such that each V_i is contained in the domain of a local chart (ψ_i, U_i) with $\psi_i(U_i) = D_2^n$ and $\psi_i(V_i) = D_1^n$, where D_r^n denotes the open ball of radius r with center 0 in $\mathbb{R}^n$. We shall use the notation

$$f^i \equiv f \circ \psi_i^{-1} :\ D_2^n \to \mathbb{R}^s,$$

and define a norm

$$||f||_r \equiv \max_i \sup\{||f^i(u)||, ||df^i(u)||, ..., ||d^r f^i(u)|| | u \in D_1^n\}.$$

Indeed, the norm $\|\cdot\|_r$ generates the C^r topology on $C^r(M, \mathbb{R}^s)$, Section 1.2 in [6]. We let $\mathcal{X}^r(M)$ be the real vector space of C^r vector fields on the (closed) manifold M with C^r topology.

B. Singular Points

Suppose $\xi \in \mathfrak{X}^r(M)$ and a is a singular point of ξ, that is $\xi(a) = 0$. Consider a local chart (ψ, U) with $a \in U$ and $\psi(a) = 0$. In these local coordinates, ξ is represented by

$$\hat{\xi} \equiv d\psi \xi \circ \psi^{-1}.$$

The singular point a is called hyperbolic if and only if the differential of $\hat{\xi}$ at 0, $d\hat{\xi}(0) : \mathbb{R}^n \to \mathbb{R}^n$ is hyperbolic, i.e., $d\hat{\xi}(0)$ does not have any complex eigenvalues whose real part is zero, Section2.3 in [6].

We denote a flow line of ξ by $\phi_x^\xi(t)$, that is

$$\frac{d}{dt}\phi_x^\xi(t) = \xi\left(\phi_x^\xi(t)\right) \ \text{ with } \phi_x^\xi(0) = x.$$

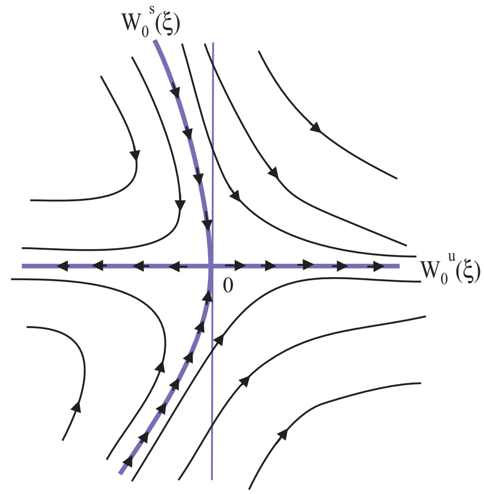

Fig. 1. Stable and unstable manifolds for $\dot{x} = x + y^2,\ \dot{y} = -y$

The manifold M is compact thus the vector field ξ generates a 1-parameter group $\phi_t^\xi : M \to M,\ t \in \mathbb{R}$, of diffeomorphisms and the smooth flow map $\Phi^\xi : \mathbb{R} \times M \to M$ related in the following way to each other

$$\Phi^\xi(t,x) \equiv \phi_t^\xi(x) \equiv \phi_x^\xi(t).$$

The stable manifold of ξ at a singular point a, [6], is the set of all initial values $x \in M$ such that the flow $\phi_x^\xi(t)$ converges to a with t going to infinity,

$$W_a^s(\xi) \equiv \{x \in M | \lim_{t \to +\infty} \phi_x^\xi(t) = a\}.$$

The *unstable manifold* of ξ at a is

$$W_a^u(\xi) \equiv \{x \in M | \lim_{t \to -\infty} \phi_x^\xi(t) = a\}.$$

The stable and unstable manifolds of the vector field

$$\xi(x,y) = \left(x + y^2, -y\right)$$

at the origin are depicted in Fig. 1.

C. Structural Stability - Robustness

We are ready to define the elements of a subset $\mathfrak{G}^r(M) \subset \mathfrak{X}^r(M)$ which play an important role in dynamical systems. We will see that they are robust to small perturbations.

Definition 1: A vector field ξ on M will be called gradient-like provided it satisfies the following four conditions:

1) The vector field ξ has a finite number of singular points, say $\beta_1, ..., \beta_k$, each hyperbolic.
2) Let

$$\alpha(x) \equiv \bigcap_{\tau \leq 0} \overline{\bigcup_{t \leq \tau} \phi_t^\xi(x)} \ \text{ and}$$
$$\omega(x) \equiv \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi_t^\xi(x)},$$

where $\overline{A}$ stands for the closure of A.

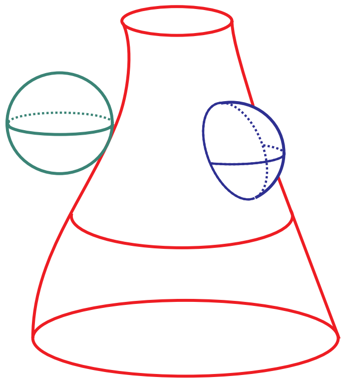

Fig. 2. The right sphere intersects transversally the "deformed cylinder" in $\mathbb{R}^3$; whereas, the left sphere does not.

Then for each $x \in M$, $\alpha(x) = \{\beta_i\}$ and $\omega(x) = \{\beta_j\}$ for some i and j.

3) Let $\Omega(\xi)$ be the set of nonwandering points[1] for ξ, then $\Omega(\xi) = \{\beta_1, ..., \beta_k\}$.
4) The stable and unstable manifolds associated with the β_i have transversal intersection, Section 3.2 in [5]. For the illustration of a transversal intersection see Fig. 2.

The set of gradient-like vector fields on M is denoted by $\mathfrak{G}^r(M)$.

We remark that a gradient-like vector field is a Morse-Smale vector field, Section 4.1 in [6], which does not have any closed orbits.

In the remaining of this section, we will discuss robustness of gradient-like vector fields. To this end, we define topological equivalence.

Definition 2: Two vector fields $\xi, \eta \in \mathfrak{X}^r(M)$ are *topologically equivalent* if there exists a homeomorphism $h : M \to M$ such that

1) $h \circ \phi^\xi(\mathbb{R}, x) = \phi^\eta(\mathbb{R}, h(x))$ for each $x \in M$,
2) h preserves the orientation, that is if $x \in M$ and $\delta > 0$ there exists $\epsilon > 0$ such that, for $0 < t < \delta$, $h \circ \phi^\xi(t, x) = \phi^\eta(\tau, h(x))$ for some $0 < \tau < \epsilon$, see Fig 3.

The first condition of the definition states that the homeomorphism h takes orbits into orbits. The second states that a stable manifold of ξ goes to a stable manifold of η. Specifically, for a pair of topologically equivalent vector fields ξ and η via a homeomorphism $h : M \to M$ and a singular point p we have $W^s_\xi(p) = h^{-1}(W^s_\eta(h(p)))$.

Example 1: Consider two vector fields ξ, η on $\mathbb{R}^2$ given by

$$\xi(x, y) = (x, y) \text{ and } \eta(x, y) = (x + y, -x + y).$$

[1]We say that $p \in M$ is a wandering point for ξ if there exists a neighborhood V of p and a number t_0 such that $\phi^\xi_t(V) \cap V = \emptyset$ for $|t| > t_0$. Otherwise we say that p is nonwandering.

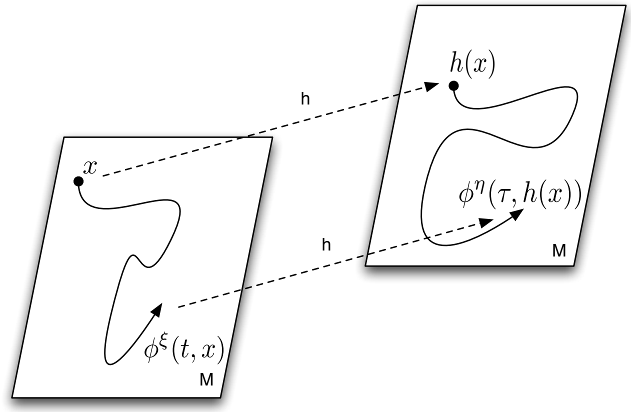

Fig. 3. An orbit of the vector field ξ goes to an orbit of the vector field η, $h \circ \Phi^\xi(t, x) = \Phi^\eta(\tau, h(x))$.

The origin corresponds to a node source for the vector field ξ, and a spiral source for η. By the Grobman-Hartman Theorem, Proposition 2.14 in [6], any two linear hyperbolic vector fields with the same indices are topologically equivalent; in particular, ξ and η have index 0. Thus, they are topologically equivalent.

As mentioned before, a vector field that is robust is one whose orbits do not change qualitatively under small perturbations.

Definition 3: A vector field $\xi \in \mathfrak{X}^r(M)$ is structurally stable if there exists an open neighborhood U of ξ in $\mathfrak{X}^r(M)$ such that every $\eta \in U$ is topologically equivalent to ξ.

If $\xi \in \mathfrak{X}^r(M)$ is a gradient-like (or more generally Morse-Smale) vector field then ξ is structurally stable, Theorem 4.1 in [6]. This means that small perturbations of a gradient-like vector field behave qualitatively the same. In particular, asymptotically stable dynamical systems are robust to small perturbations. In the next section, we will estimate the size of perturbations that do not destabilize the system.

IV. Robustness of Asymptotically Stable Systems

We shall briefly introduce the Lyapunov stability theory. The aim is to provide necessary conditions for a dynamic system to be asymptotically stable and relate asymptotic stability to robustness. To this end, we associate a function decreasing along the flow lines.

One of the most important concepts in control is stability of a singular point. We say that a system ξ is stable at a if for any open neighborhood U of a there is an open neighborhood V of a (possibly "smaller" than U) such that for any initial value x_0 in V, the flow line $\phi^\xi_{x_0}(t)$ remains in U for all $t > 0$. More desirable property of dynamical systems is

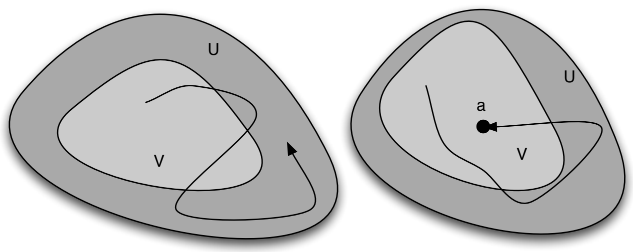

Fig. 4. Illustration of stability (to the left) and asymptotic stability (to the right); in the figure to the left, the flow line with the initial value in V stays forever in U. In the figure to the right, the flow line converges to the singular point a.

asymptotic stability. This concept combines stability with the convergence of the flow lines to a, Fig 4.

A. Asymptotic and Exponential Stability

Definition 4 (Definition 2.1.24 in [7], Sec. 2.7 in [12]): Let M be a Riemannian manifold with a metric g. Suppose a is a singular point of $\xi \in \mathfrak{X}^r(M)$.

1) The point a is stable if for any neighborhood U of a, there is a neighborhood V of a such that if $x \in V$ then $\bigcup_{t \geq 0} \phi_t^\xi(x) \subset U$.
2) The point a is asymptotically stable if it is stable and there is a neighborhood V' of a such that if $x \in V'$, then
$$\lim_{t \to +\infty} \phi_t^\xi(x) = a.$$
3) The point is exponentially stable if it is stable and there are a neighborhood V'' of a and numbers $\omega < 0$ and $A > 0$ such that if $x \in V''$, then
$$d(\phi_t^\xi(x), a) \leq A e^{\omega t} d(x, a),$$
where d is the Riemannian distance.

Exponential stability is an important classification of stability as it has an intrinsic robustness property. For illustration, consider the following example.

Example 2: A dynamical system $\xi : \mathbb{R} \to \mathbb{R}$, $\xi(x) = -x^3$ is asymptotically stable at 0 on $\mathbb{R}$. Indeed, the solution of the Cauchy problem
$$\dot{z} = -z^3, \text{ for the initial value } x \text{ at } t = 0$$
is given by
$$(\Phi_x^\xi(t))^2 = \frac{x^2}{2tx^2 + 1}.$$
Thus, $|\Phi_x^\xi(t)| < |\Phi_x^\xi(t')|$ for $t > t'$, and $\lim_{t\to+\infty} \phi_t^\xi(x) = 0$. On the other hand, it is not exponentially stable.

Notice that asymptotic stability can be deduced from the specific form of the vector field ξ. In truth, the vector field $\xi : x \mapsto -x^3$ is negative for $x > 0$ and positive for $x < 0$. Thus, at each x, it points toward 0.

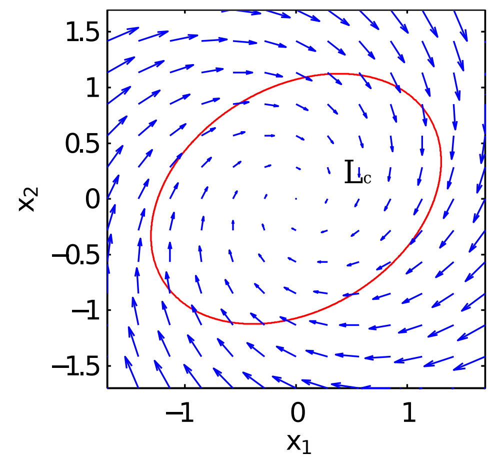

Fig. 5. The vector field ξ is transversal to the set $L_c = \{x \in \mathbb{R}^n | \ f(x) \leq c\}$ and points into it.

Consider now, a perturbation $\eta(x) = \alpha^2 x$ of ξ with arbitrarily small but fixed nonzero real α. Observe that
$$\xi(x) + \eta(x) = -x(x - \alpha)(x + \alpha),$$
which is positive for $x \in]0, \alpha[$; in other words, $\xi(x) + \eta(x)$ points out of 0. Thus, the perturbed system $\xi + \eta$ is unstable for arbitrary small parameter α.

In the example, we have explicitly solved the differential equation describing the system dynamics. However, for more complex system, this approach is not tangible. An alternative approach is to associate a function f that is decreasing along the flow lines of ξ, and which has its minimum precisely at a singular point a. As a consequence, by studying the directional derivative of the function along the vector field alone,
$$\xi(f)(x) \equiv df(x)\xi(x) = \sum \xi_i(x) df_i(x),$$
it is possible to determine if the system is stable. Locally in $\mathbb{R}^n$, we formulate the following sufficient conditions for stability.

Theorem 1 (Theorem 4.1 in [8]): Let 0 be a singular point of a vector field $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$ ($r \geq 1$). If there exist an open neighborhood U of 0 and a C^1 function $f : U \to \mathbb{R}$ such that $f(0) = 0$, $f(x) > 0$ for $x \in U - \{0\}$, and $-\xi(f)(x) \geq 0$ for $x \in U$. Then 0 is stable. Moreover, if $-\xi(f)(x) > 0$ for $x \in U - \{0\}$ then 0 is asymptotically stable.

Figure 5 illustrates that the geometrical interpretation of the condition $-\xi(f)(x) > 0$ that the vector field ξ is transversal to a (sub-level) set $L_c = \{x \in \mathbb{R}^n | \ f(x) \leq c\}$ and points into it.

Corollary 1 (Theorem 4.2 in [8]): Let 0 be a singular point of a $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$. If there is a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$

such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, $-\xi(f)(x) > 0$ for $x \in \mathbb{R}^n$ and $f(x) \to +\infty$ as $\|x\| \to +\infty$. Then 0 is asymptotically stable on $\mathbb{R}^n$, i.e. for any $x \in \mathbb{R}^n$, $\lim_{t\to+\infty} \phi_t^\xi(x) = 0$.

A very interesting class of dynamical systems consists of linear systems, i.e., their vector fields are of the form

$$L : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto Lx,$$

where L is a linear operator, $L \in \mathcal{L}(\mathbb{R}^n)$. The reason for this interest is that the closed form solution of any linear differential equations is known. Furthermore, the "behaviors" of the flow line as time goes to infinity is completely characterized by the placement of eigenvalues. Lastly, by studying linearization of a generic nonlinear systems, it is possible to determine if the nonlinear systems is asymptotically stable in sufficiently small neighborhood of a singular point. As a consequence, a linearized nonlinear system is stable if and only if it is exponentially stable.

The singular point 0 of a linear vector field $L \in \mathcal{L}(\mathbb{R}^n)$ is asymptotically stable if and only if all the eigenvalues of L have negative real part. In the next theorem, we shall relate asymptotic stability to the solution of a certain equation.

Theorem 2 (Lyapunov Stability Theorem 3.2 in [9], [10]): The singular point 0 of a linear vector field $L \in \mathcal{L}(\mathbb{R}^n) \equiv \{A : \mathbb{R}^n \to \mathbb{R}^n | \, A \text{ is linear}\}$ is asymptotically stable if and only if, for any selfadjoint positive definite matrix Q there exists a unique selfadjoint positive definite matrix P satisfying the Lyapunov equation

$$L^{\mathrm{T}}P + PL = -Q. \tag{1}$$

The Lyapunov function f from Theorem 1 and the solution P of the Lyapunov equation (1) are related by

$$f(x) = x^{\mathrm{T}}Px, \tag{2}$$

since

$$-L(f)(x) = -x^{\mathrm{T}}(L^{\mathrm{T}}P + PL)x = x^{\mathrm{T}}Qx > 0$$

for $x \neq 0$.

We suppose that $L = d\xi(0)$. By Taylor expansion, Sec. XIII.6 in [11], ξ may be considered as a perturbation of a linear ordinary differential equation of the form

$$\begin{aligned} \frac{d}{dt}\phi_x^\xi(t) &= \xi \circ \phi_x^\xi(t) = L\phi_x^\xi(t) + \eta \circ \phi_x^\xi(t), \qquad (3)\\ \phi_x^\xi(0) &= x \end{aligned}$$

in some open neighborhood U of 0 in $\mathbb{R}^n$, where $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a C^{r-1} map that satisfies

$$\begin{aligned} \eta(0) &= 0 \\ \|\eta(x) - \eta(y)\| &\leq \delta(\epsilon)\|x - y\| \quad \text{for} \quad \|x\|, \|y\| < \epsilon \qquad (4) \end{aligned}$$

with the function $\delta : [0, \infty) \to [0, \infty)$ continuous and monotonically increasing.

In the next corollary, we relate asymptotic stability of a vector field to asymptotic stability of its linearization.

Corollary 2: Let 0 be a singular point of a vector field $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$, $r \geq 1$. Suppose $L = d\xi(0)$. If L is asymptotically stable, then the point 0 is asymptotically stable for ξ.

Proof: The linear system L is asymptotically stable; thus, for any selfadjoint positive definite Q, there is a unique solution P to the Lyapunov equation. Define a map $f : \mathbb{R}^n \to \mathbb{R}$ by $x \mapsto x^{\mathrm{T}}Px$. By the Taylor expansion of ξ, we have

$$-\xi(f)(x) = x^{\mathrm{T}}Qx - 2x^{\mathrm{T}}P\eta(x).$$

The matrix Q is selfadjoint positive definite, therefore by the Spectral Theorem, $x^{\mathrm{T}}Qx \geq c\|x\|$ where c is the smallest eigenvalue of Q. Furthermore, we use the estimate

$$|x^{\mathrm{T}}P\eta(x)| \leq \|x\| \; \|P\| \; \|\eta(x)\| \leq \delta(\epsilon)\|P\|\|x\|^2,$$

where δ is continuous and monotonically nondecreasing as in (4). Therefore, we can choose ϵ such that $\delta(\epsilon) < d$, where d is an arbitrary real number. For $\|x\| < \epsilon$, we have

$$\begin{aligned} -\xi(f)(x) = x^{\mathrm{T}}Qx - 2x^{\mathrm{T}}P\eta(x) &\geq c\|x\|^2 - 2|x^{\mathrm{T}}P\eta(x)| \\ &\geq (c - 2\delta(\epsilon)\|P\|)\,\|x\|^2. \end{aligned}$$

We shrink ϵ such that $\kappa \equiv c - 2\delta(\epsilon)\|P\| > 0$ and get

$$-\xi(f)(x) \geq \kappa\|x\|^2, \; \forall \; \|x\| < \epsilon.$$

Thus, by Theorem 1, the singular point 0 of ξ is asymptotically stable. ■

To prove converse to Corollary 2, we make stronger assumption. Indeed, in the next theorem, we assume that the singular point 0 is exponentially stable.

As a consequence, 0 is an asymptotically stable singular point of ξ. Theorem 3 below shows that it is in fact an exponentially stable singular point.

Theorem 3 (Theorem 2.3 in [12]): 0 is an exponentially stable singular point of $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$ if and only if it is exponentially stable singular point of $L = d\xi(0)$. Furthermore, the stability exponents $\omega < 0$ and $A > 0$ are the same for ξ and L,

$$||\phi_t^\xi(x)|| \leq Ae^{\omega t}||x||, \tag{5}$$
$$||\phi_t^L(x)|| \leq Ae^{\omega t}||x|| \tag{6}$$

on some $V \subset \mathbb{R}^n$.

To address robustness, we assume that 0 is an asymptotically stable singular point of ξ, and we let $\eta \in \mathfrak{X}^r(\mathbb{R}^n)$ be such that $\eta(0) = 0$. The vector field η plays the role of a perturbation. Subsequently, we ask for what perturbations η the singular point 0 is asymptotically stable of $\xi+\eta$. An answer is provided by the following proposition.

Proposition 1 (Lemma 2.7 in [12]): If 0 is an exponentially stable singular point of $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$ with the stability exponents $\omega < 0$ and $A > 0$, and

$$||d\eta(0)|| < \frac{|\omega|}{A},$$

where $||L|| \equiv \sup\{||Lx|| \mid ||x|| = 1\}$ is the greatest singular value of L, then 0 is exponentially stable singular point of $\xi + \eta$ (in sufficiently small open neighborhood of 0).

Example 3: Let ξ by an exponentially stable planar dynamical system with a singular point 0. Thus, the two eigenvalues of its linearization $L = d\xi(0)$ have negative real parts. There exists a linear isomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$, [13], such that L in the new coordinates can be represented as one of the following three case

$$L_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\ L_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},\ \text{and } L_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$

with $-\lambda, -\lambda_1, -\lambda_2, -\alpha \in \mathbb{R}_+$ The solution of $\dot{z} = L_1 z$ with the initial value x is

$$\phi_x^{L_1}(t) = (x_1 e^{\lambda_1 t}, x_2 e^{\lambda_2 t})$$

hence $||\phi_x^{L_1}(t)|| \leq e^{\max\{\lambda_1,\lambda_2\}t}||x||$. The solution of $\dot{z} = L_2 z$ is

$$\phi_x^{L_2}(t) = (x_1 e^{\lambda t}, x_2 t e^{\lambda t}).$$

Observe that there is $T > 0$ such that $e^{-\frac{\lambda}{2}t} > t$. Therefore,

$$|x_2| t e^{\lambda t} < |x_2| T e^{\frac{\lambda}{2}t},$$

and $||\phi_x^{L_2}(t)|| \leq T e^{\frac{\lambda}{2}t}||x||$. Finally, the solution of $\dot{z} = L_3 z$ is

$$\begin{aligned}\phi_x^{L_3}(t) = (&x_1 e^{\alpha t}\cos\beta t + x_2 e^{\alpha t}\sin\beta t,\\ &x_1 e^{\alpha t}\sin\beta t + x_2 e^{\alpha t}\cos\beta t).\end{aligned}$$

Thus, $||\phi_x^{L_3}(t)|| \leq e^{\alpha t}||x||$. In conclusion, if the perturbation η has a linear term with $||d\eta(0)|| < -\max\{\lambda_1, \lambda_2\}$ in the first case, $||d\eta(0)|| < -\frac{\lambda}{2T}$ in the second, and $||d\eta(0)|| < -\alpha$ in the third, then the system $\xi + \eta$ is exponentially stable in a sufficiently small open neighborhood of the singular point 0.

V. Robustness of Interconnected Systems

In this section, we will address robustness of input-output maps. We look upon a control system as an input-output map. The inputs are interpreted as disturbances and outputs as commodities to be kept constant. Subsequently, the task of controller is to make the outputs deviations small when the system is affected by disturbances. We fomulate the framework for studying robustness as an interconnection of two input-output maps, which represent the nominal systems and unmodeled dynamics.

A. Extended Signal Spaces

For each $p \in \mathbb{N} \cup \{\infty\}$, let L_m^p be the L^p space $L^p(\mathbb{R}^m)$ with the norm $||\cdot||_p$ [14],

$$\begin{aligned} ||f||_p &= \left(\int_0^\infty ||f||^p dt\right)^{\frac{1}{p}} \text{ for } p \in \mathbb{N}, \\ ||f||_\infty &= \inf\{C \geq 0 |\ ||f(t)|| \leq C \text{ almost every } t \geq 0\}. \end{aligned}$$

Let $\Gamma \equiv \Gamma^m \equiv \{f : \mathbb{R}_+ \to \mathbb{R}^m\}$ be a set of measurable maps. For each $T \in \mathbb{R}_+$, we define a map $\ _T : \Gamma \to \Gamma$ given by

$$f \mapsto \ _T(f) \equiv f_T = \begin{cases} f(t) & 0 \leq t < T \\ 0 & t \geq t. \end{cases}$$

The extended space L_{me}^p is

$$L_{me}^p \equiv L_e^p(\mathbb{R}^m) = \{f \in \Gamma |\ f_T \in L_m^p \text{ for all } T \in \mathbb{R}_+\}.$$

To define a topology on L_{me}^p, we observe that for each $T \in \mathbb{R}_+$, $||\cdot||_{pT}$, given by $||f||_{pT} = ||f_T||_p$, is a seminorm on L_{me}^p. Suppose $g \in L_{me}^p$, and let

$$B_T^r(g) \equiv \{f \in L_{me}^p |\ ||f||_{pT} < r\}.$$

Subsequently, the collection

$$\mathcal{B} \equiv \{B_T^r(g) |\ g \in L_{me}^p,\ T \in \mathbb{R}_+,\ r > 0\}$$

forms a base of a topology $\mathcal{T}$ on L_{me}^p. In fact, L_{me}^p is a metric space with a metric

$$d(f,g) = \sum_{r \in \mathbb{N}} 2^{-n} \frac{||f-g||_{pT_n}}{1 + ||f-g||_{pT_n}},$$

where the sequence $\{T_n\} \equiv \{T_n |\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}_+$ [15]. The metric d is translation invariant, i.e.,

$$d(f+h, g+h) = d(f,g), \text{ for all } f, g, h \in L_{me}^p.$$

Furthermore, the metric d is compatible with $\mathcal{T}$. In particular, a sequence $\{f_n\}$ converges to u if and only if for all $T \in \mathbb{R}_+$ the sequence $\{f_{nT}\}$ converges to f_T.

B. Input-Output Stability

So far in this paper, we have met two forms of stability: structural stability of vector fields and asymptotic stability of dynamical systems. Below, we want to formalize input-output stability. Informally, we say that a system system is input-output stable if small input signals generate small output signals.

At the outset, we define the concept of causality. We say that a map $G : L_{me}^p \to L_{ne}^p$ is causal [16] if and only if

$$(G(u))_T = (G(u_T))_T \text{ for all } T \in \mathbb{R}_+ \text{ and } u \in L_{me}^p,$$

or equivalently for all $u, v \in L_{me}^p$ and $T \in \mathbb{R}_+$

$$(u_T = v_T) \implies ((G(u))_T = (G(v))_T).$$

A map $G : L_{me}^p \to L_{ne}^p$ is said to have finite L^p gain if there exist γ and β both in $\mathbb{R}_+$ such that for all $T \in \mathbb{R}_+$,

$$||G(u)||_{pT} \leq \gamma ||u||_{pT} + \beta \text{ for all } u \in L_{me}^p.$$

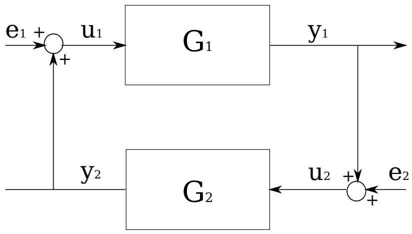

Fig. 6. Feedback configuration

We shall refer to the constant γ as a gain and β as a bias.

Proposition 2: If G has finite L^p gain with $\beta = 0$ then the map G is causal.

Proof: Since for all $T \in \mathbb{R}_+$,

$$||G(\mathrm{Id} - {}_T)u||_{pT} = ||{}_T(G(\mathrm{Id} - {}_T)u)||_T \leq \gamma||(\mathrm{Id} - {}_T)u)||,$$

but $(\mathrm{Id} - {}_T)u) = 0$; thus, ${}_T(G(\mathrm{Id} - {}_T)u) = 0$, and $(G(u))_T = (G(u_T))_T$ ■

Finally, we say that a map $G : L^p_{me} \to L^p_{ne}$ is L^p-stable if for any $u \in L^p_m$, $G(u) \in L^p_n$.

If G is causal then G is L^p stable is equivalent to G has finite L^p-gain, Proposition 1.2.2 in [17].

The most popular norm that control theory works with is $|| \cdot ||_2$; partly, because it has a physical interpretation as the energy of a signal, partly, because L^2 has a structure of a Hilbert space. This has been explored in many branches of control theory, for example passivity theory [18]. In linear control theory [2], if a control system is modeled as a real, rational transfer function $\tilde{G}(s)$ then

$$||G(u)||_2 \leq \gamma||u||_2, \text{ where } \gamma = \sup_{\omega} |\tilde{G}(j\omega)|.$$

C. Robustness of Input-Output Maps

The focus is on the stability analysis of interconnected systems as in Fig. 6., i.e., the closed loop system Σ_{G_1,G_2}, for $G_1 : L^p_{m,\mathrm{e}} \to L^p_{n,\mathrm{e}}$ and $G_2 : L^p_{n,\mathrm{e}} \to L^p_{m,\mathrm{e}}$, is defined by

$$\begin{aligned} y_1 &= G_1(u_1), \quad y_2 = G_2(u_2), \\ u_1 &= e_1 + y_2, \quad u_2 = e_2 + y_1. \end{aligned} \tag{7}$$

We identify $L^p_{(m+n),\mathrm{e}} \approx L^p_{m,\mathrm{e}} \times L^p_{n,\mathrm{e}} \approx L^p_{n,\mathrm{e}} \times L^p_{m,\mathrm{e}}$, and define two maps $G,\ F :\ L^p_{(m+n),\mathrm{e}} \to L^p_{m+n,\mathrm{e}}$ by

$$G = G_1 \times G_2,\ (u_1, u_2) \mapsto (G_1(u_1), G_2(u_2)),$$

and

$$F = \pi_2 \times \pi_1,\ (u_1, u_2) \mapsto (u_2, u_1),$$

where $\pi_1(u_1, u_2) = u_1,\ \pi_2(u_1, u_2) = u_2$ are canonical projections. As a consequence, the following equality holds

$$y = G(u + Fy). \tag{8}$$

In (8), we have used the standard notation $v = (v_1, v_2)$. The equality (8) gives rise to the following relation on $L^p_{(m+n),\mathrm{e}} \times L^p_{(m+n),\mathrm{e}}$

$$R_{ey} \equiv \{(e, y) \in L^p_{(m+n),\mathrm{e}} \times L^p_{(m+n),\mathrm{e}} |\ y = G(u + Fy)\}.$$

We say that a relation $R \subset L^p_{q,\mathrm{e}} \times L^p_{q,\mathrm{e}}$ has finite L^p gain, Definition 1.2.3 in [17], if there exist $\gamma, \beta \in \mathbb{R}_+$ such that for all $(u, y) \in R$

$$||y||_{pT} \leq \gamma||u||_{pT} + \beta \text{ for all } T \in \mathbb{R}_+.$$

Perhaps, the most commonly used robustness criterion for the closed loop system Σ_{G_1,G_2} in Fig. 6 is the small gain theorem: If both components in the feedback loop have finite gain and their product is less than one then the closed loop system has finite gain.

Theorem 4 (Theorem 2.1 in [17]): Let $p \in \mathbb{N} \cup \{\infty\}$. Suppose that G_1 has finite L^p gain γ_1, and G_2 has finite L^p gain γ_2. The closed-loop system Σ_{G_1,G_2} in (7) has finite L^p gain if $\gamma_1\gamma_2 < 1$.

The small gain theorem is a consequence of the graph separation theorem [19]. Let $\mathcal{G}_G \subset L^p_{\mathrm{me}} \times L^p_{\mathrm{ne}}$ denote the graph of a map $G : L^p_{\mathrm{me}} \to L^p_{\mathrm{ne}}$. The inverse graph $\mathcal{G}^I_G \subset L^p_{\mathrm{ne}} \times L^p_{\mathrm{me}}$ is defined by

$$\mathcal{G}^I_G \equiv \{(y, u) \in L^p_{\mathrm{ne}} \times L^p_{\mathrm{me}} |\ (u, y) \in \mathcal{G}_G\}.$$

For each $T \in \mathbb{R}_+$, we have introduced a seminorm $|| \cdot ||_{pT}$, which we use to define a Huasdorff distance on the subsets of $L^p_{\mathrm{me}} \times L^p_{\mathrm{ne}}$. In particular, the distance between a point $x \in L^p_{\mathrm{me}} \times L^p_{\mathrm{ne}}$ and the graph $\mathcal{G}_G$ is

$$d_T(x, \mathcal{G}_G) = \inf\{||x - z||_{pT} |\ z \in \mathcal{G}_G\}$$

Theorem 5 ([19]): The feedback system (7) has finite L^p gain with 0 bias if and only if there exists $\gamma \in \mathbb{R}_+$ such that for any $x \in \mathcal{G}^I_{G_2}$,

$$||x||_{pT} \leq \gamma d_T(x, \mathcal{G}^I_{G_2}).$$

The small gain theorem has a version for linear systems $G_1 : L^2_n \to L^2_m$ and $G_2 : L^2_m \to L^2_n$. If G_1 and G_2 are modeled as elements $\tilde{G}_1,\ \tilde{G}_2$ in the space RH_∞ of proper, real rational stable transfer matrices, Section 4.3 in [2], the induced norm $|| \cdot ||_\infty$ ($||G||_\infty \equiv \sup\{||Gu||^2_n|\ ||u||^2_m = 1\}$) can be computed (Parseval's theorem) by

$$||G||_\infty = \sup \left\{ \bar{\sigma}(\tilde{G}(j\omega)) \middle|\ \omega \in \mathbb{R} \right\},$$

where $\bar{\sigma}(\cdot)$ is the function taking a matrix to its maximal singular value.

Theorem 6 (Theorem 9.1 in [2]): The feedback system (7) has finite L^2 gain (with 0 bias) if

$$||G_1||_\infty ||G_2||_\infty < 1.$$

In conclusion, we note that there are numerous versions of the small gain theorem. For instance, a specialization of Theorem 4 to the systems represented in the state space model can be found in [20].

VI. Conclusion

The paper gave the overview of three methods for analyzing robustness in control. All the methods discussed used much the same approach of defining a convenient space of maps and subsequently studying small perturbations in that space. Three approaches were addressed: robustness of a gradient-like vector fields in the space of C^r vector fields on a closed manifold, robustness of locally asymptotically stable vector fields, and robustness of the input-output maps in the extended L^p-space.

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