Analysis of Degrees of Freedom of Wideband Random Multipath Fields Observed Over Time and Space Windows

Farhana Bashar, S.M. Akramus Salehin and Thushara D. Abhayapala Research School of Engineering, The Australian National University (ANU)

Canberra 0200 ACT, Australia

Abstract

In multipath systems, available degrees of freedom can be considered as a key performance indicator, since the channel capacity grows linearly with the available degrees of freedom. However, a fundamental question arises: given a size limitation on the observable region, what is the intrinsic number of degrees of freedom available in a wideband random multipath wavefield observed over a finite time interval? In this paper, we focus on answering this question by modelling the wavefield as a sum of orthogonal waveforms or spatial orders. We show that for each spatial order, (i) the observable wavefield is band limited within an effective bandwidth rather than the given bandwidth and (ii) the observation time varies from the given observation time. These findings show the strong coupling between space and time as well as space and bandwidth. In effect, for spatially diverse multipath wavefields, the classical degrees of freedom result of *time-bandwidth* product does not directly extend to *time-space-bandwidth* product.

Index Terms

Random multipath, degrees of freedom, spatial diversity, signal to noise ratio.

I. INTRODUCTION

In multipath wireless communication systems, the use of the spatial aspects of multipaths can ensure improved system performances [1]. The study of the spatial aspects of multipath fields, thus, becomes an important thread of research in wireless communications and signal processing, and has more recently been addressed by [2]–[6]. None of these approaches, however, provide clear view of the interrelationships between space, frequency and time that affects the degrees of freedom of multipath fields.

In this work, we study a band limited random multipath field observed over a limited source-free region of space over a finite time window. The observable multipath wavefield is considered to be farfield, and we study this from a physical wavefield perspective. In particular, the underlying physics of free space propagation is used

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to model the multipath field as a sum of orthogonal waveforms or spatial orders. This mathematical framework is similarly used in [7]. However, in comparison, our derived result is more accurate, since we have considered the effect of available spatial information on the observation time. Moreover, the results provided in [7], are derived by using a geometrical argument to extend the narrowband degrees of freedom result of [2] to a broadband scenario and resulted in a complicated formula. Further, it is unclear, how the usable (effective) bandwidth varies from the given frequency bandwidth for the different spatial orders. In this work, on the contrary, the degrees of freedom of wideband multipath fields is evaluated in a simple manner, and we derive that the wavefield is bandlimited within an effective frequency bandwidth for each spatial order.

The work [4] characterized multi-antenna systems in a wideband transmission regime, but how the coupling between space and time as well as space and frequency affects the information content of the waves was left as an open and important problem. We show that the effective frequency bandwidth of each spatial order is essentially related to the spatial dimension of the observable field and varies from the frequency bandwidth of the channel. Our results indicate that even though for lower spatial orders, effective bandwidth is equal to the given frequency bandwidth, for higher orders, effective bandwidth is less than the given frequency bandwidth. Moreover, we show that the effective observation time is independent of spatial order and is related to the finite size of the observation region. These findings clearly indicate the strong link between space and time as well as space and frequency in spatially diverse multipath fields. These findings also indicate that the classical degrees of freedom result of *time-bandwidth* product does not directly extend to *time-space-bandwidth* product as shown in [3], [6], rather the degrees of freedom of any particular spatial order n can be expressed as a product of *effective observation time* and *effective bandwidth of the* n^{th} order. If we denote the degrees of freedom of the nth order as D_n , the total degrees of freedom can be calculated by $\sum_n D_n$, for all possible values of n.

In recent times, the study of degrees of freedom of distributed MIMO communications (e.g., [5]) has gained a lot of attention. In distributed MIMO systems, the users cooperate in clusters to receive information. In such scenarios, our derived results indicate that the lower spatial orders (independent channels) can utilize the full bandwidth, whereas, the higher orders can utilize only a fraction of the given bandwidth. We envisage that our results will be useful in characterizing the degrees of freedom of distributed or large scale MIMO systems.

II. RANDOM 2D Multipath Fields

We consider a wireless multipath wavefield band limited to $[F_0 - W, F_0 + W]$ and observed within a 2D disk region of radius R over a finite time interval [0,T]. The observable multipath wavefield within this channel is assumed to be farfield and is generated by a source or distribution of sources and scatterers that exist outside the region of interest.

A. Multipath Plane Wave Representation

Let $\Psi(\mathbf{x}, \omega)$ denote a finite complex-valued wideband multipath field in the region of interest where $\mathbf{x} \equiv (r, \phi_x)$ represents a position vector within the 2D observation region, $r = \|\mathbf{x}\| \le R$ denotes the Euclidean distance of \mathbf{x}

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from the origin, $\phi_x \in [0, 2\pi)$ is the azimuth angle of vector x and ω is the angular frequency. Note that a standard multipath model involves modeling every distinct path explicitly as a plane wave. Hence, in this model, the multipath field is generated by the superposition of plane waves as

$$\Psi(\mathbf{x},\omega) = \int_0^{2\pi} a(\phi,\omega) e^{ik\mathbf{x}\cdot\hat{\mathbf{y}}} d\phi$$
(1)

where $\hat{y} \equiv (1, \phi)$, $k = \omega/c$ is the scalar wavenumber, *c* is the wave velocity and $a(\phi, \omega)$ is the complex-valued gain of scatterers as a function of direction of arrival $\phi \in [0, 2\pi)$ and angular frequency ω .

B. Orthogonal Basis Expansion

We consider that the multipath field is generated by sources external to the region of interest. Hence, we can use Jacobi-Anger expansion [8, p. 67] to represent the plane waves in (1) as

$$e^{ik\mathbf{x}\cdot\hat{\mathbf{y}}} = \sum_{n=-\infty}^{\infty} i^n J_n(\frac{\omega}{c}r) e^{in(\phi_x - \phi)}$$
(2)

where $J_n(\cdot)$ is the Bessel function of the first kind of integer order n, and we can identify a countable set of orthogonal basis functions over the 2D disk since

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$$\int_{0}^{2\pi} \Phi_n(\mathbf{x}) \Phi_m^*(\mathbf{x}) d\phi_x = \begin{cases} 2\pi, & n = m \\ 0, & \text{otherwise} \end{cases}$$
(3)

where $\Phi_n(\mathbf{x}) = e^{in\phi_x}$ and $(\cdot)^*$ is the complex conjugate operator. Observe that by substituting (2) into (1), we obtain

$$\Psi(\mathbf{x},\omega) = \sum_{n=-\infty}^{\infty} i^n \alpha_n(\omega) J_n(\frac{\omega}{c}r) e^{in\phi_x}$$
(4)

where $\alpha_n(\omega)$ is the n^{th} frequency dependent coefficient and using (1) can be defined as

$$\alpha_n(\omega) = \int_0^{2\pi} a(\phi, \omega) e^{-in\phi} d\phi.$$
⁽⁵⁾

However, the information available about scatterers that generate the multipath field $\Psi(\mathbf{x},\omega)$ is usually limited. Thus, it is reasonable to represent the multipath field as a random process. Referring to (1), the scattering gain $a(\phi,\omega)$ is random and so is $\alpha_n(\omega)$ in (5). For mathematical simplicity of the analysis, we assume uncorrelated scattering. As a result, the random gains $a(\phi,\omega)$ and $a(\phi',\omega)$ at two distinct incident angles and different frequencies are uncorrelated from each other. Hence, using (5), the uncorrelated scattering assumption, and following a few intermediate steps, we find that

$$E\{|\alpha_n(\omega)|^2\} = \int_0^{2\pi} E\{a(\phi,\omega)a^*(\phi,\omega)\}d\phi$$
(6)

where $E\{\cdot\}$ represents the expectation operation. Therefore, our observable field (4) is a random multipath field and can be represented by an infinite but countable set of orthogonal basis functions.

III. OBSERVATION TIME OF THE SPATIAL ORDERS

If the wavefield is generated by a single point/ source transmitting a time domain signal, then observing the resulting travelling wavefield over a time window [0,T] within a 2D disk of radius R captures information content of the time domain signal over a time interval T + 2R/c. We formalize this statement for the n^{th} order time domain signal $a_n(t)$ producing the n^{th} order space-time wavefield $\psi_n(r,t)$ in the following theorem.

Theorem 1 (Observation time of the spatial orders): Given that the spatial orders n are separated, observing a random wireless multipath wavefield over a 2D disk of radius R for a time interval T is equivalent to observing the information content of the underlying n^{th} order time domain signal $a_n(t)$ over an effective time interval

$$T_{eff} = T + \frac{2R}{c}.$$
(7)

Further, this effective time interval T_{eff} is not order dependent and increases with the size of the observation region.

Proof: We can define the n^{th} order signal spectrum over space from (4) as

$$\Psi_n(r,\omega) \triangleq \alpha_n(\omega) J_n(\frac{\omega}{c}r).$$
(8)

Let $\psi_n(r,t)$ be the inverse Fourier transform of $\Psi_n(r,\omega)$. Then, by taking the inverse Fourier transform of (8), we obtain

$$\psi_n(r,t) = a_n(t) * U_n(\frac{tc}{r}) \tag{9}$$

where the time domain coefficient $a_n(t)$ is the inverse Fourier transform of $\alpha_n(\omega)$ which represents the n^{th} order time domain signal and the Chebyshev Polynomial of the first kind $U_n(tc/r)$ is the inverse Fourier transform of $J_n(\omega r/c)$.

Observe that in (9), the n^{th} order signal over space $\psi_n(r,t)$ is a convolution between the n^{th} order time domain signal $a_n(t)$ and the Chebyshev Polynomial $U_n(tc/r)$. Hence, any information content in the n^{th} order signal over space $\psi_n(r,t)$ is contained in the n^{th} order time domain signal $a_n(t)$.

We observe the n^{th} order signal over space $\psi_n(r,t)$ over a time window [0,T] within a 2D disk region of radius R. Moreover, the Chebyshev Polynomial $U_n(z)$ is defined only for $-1 \le z \le 1$, as illustrated in Fig. 1. Hence, $U_n(tc/r)$ is defined only for $-r/c \le t \le r/c$. As a result, if we consider that the n^{th} order signal over space $\psi_n(r,t)$ is observed within a disk of radius R over the time window [0,T], it is possible to capture information about the n^{th} order time domain signal $a_n(t)$ over the time window [-R/c, T + R/c]. This is equivalent to observing the n^{th} order time domain signal $a_n(t)$ over a maximum time window of T + 2R/c.

IV. Effective Bandwidth of The n^{th} Order

Ideally, if it is possible to measure signals with infinite precision without noise, each spatial order n would have an effective bandwidth equal to the frequency bandwidth available, i.e., from $F_0 - W$ to $F_0 + W$. However, in practical communication systems, noise is present. As a result, it is not possible to detect signals within the band of frequencies where the signal power to noise ratio (SNR) drops below a certain threshold γ . This threshold is dependent on the sensor sensitivity or the robustness of the signal processing method to noise.

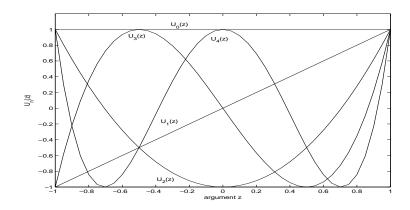


Fig. 1. Chebyshev Polynomial of the first kind $U_n(z)$ for n=0,1,2,3,4.

Lets consider $\eta_R(\phi_x, \omega)$ as the white Gaussian noise on the circle (at radius R) associated with antenna/ sensor at an angle ϕ_x . Hence, the received signal on the circle is given by

$$\Psi(R,\phi_x,\omega) = \sum_{n=-\infty}^{\infty} i^n \alpha_n(\omega) J_n(\frac{\omega}{c}R) e^{in\phi_x} + \eta_R(\phi_x,\omega).$$
(10)

The following theorem proves that the white Gaussian noise power remains the same at all frequencies in the modal expansion (4).

Theorem 2 (White Gaussian Noise in L^2): Given a zero mean white Gaussian noise with variance σ_0^2 in $L^2(\mathbb{S}^1)$ represented by a random variable $\eta_R(\phi)$ where $\phi \in \mathbb{S}^1$, such that for any function $\psi_i(\phi) \in L^2(\mathbb{S}^1)$ the complex scalar ν_i

$$u_i \triangleq \int_{\mathbb{S}^1} \eta_R(\phi) \psi_i^*(\phi) d\phi = \langle \eta_R(\phi), \psi_i(\phi) \rangle$$

is also a zero mean Gaussian random variable with variance $E\{|\nu_i|^2\} = \sigma_0^2 \int_{\mathbb{S}^1} |\psi_i(\phi)|^2 d\phi = \sigma_0^2 (\|\psi_i(\phi)\|_{L^2})^2$. [9, eqn 8.1.35]

Definition 1: By taking $\psi_i(\phi)$ to be the orthogonal basis functions $e^{in\phi_x}$, the spatial Fourier coefficients for the noise is

$$\nu_n(\omega) = \int_{\mathbb{S}^1} \eta_R(\phi_x, \omega) e^{-in\phi_x} d\phi_x \tag{11}$$

and applying Theorem 2, $\nu_n(\omega)$ are also zero mean Gaussian random variables with variance σ_0^2 .

Based on Definition 1, we can define the n^{th} order received signal at radius R as

$$\Psi_n(R,\omega) = \alpha_n(\omega) J_n(\frac{\omega}{c}R) + \nu_n(\omega), \qquad (12)$$

and we assume that the noise and the signal are independent of each other.

Note that we can consider $\alpha_n(\cdot)$ as the n^{th} order signal spectrum that is defined only over the range $[F_0 - W, F_0 + W]$. Also note that for a fixed value of the radius, $J_n(\cdot)$ can be treated as a function of frequency. However, it is evident from Fig. 2 that except for the 0^{th} order, Bessel functions start small before increasing monotonically to their maximum. Further, the Bessel functions start more slowly as the order n increases. Thus, for any particular

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order |n|(>0) and for frequencies less than a critical frequency F_n , the magnitude of the Bessel functions $|J_n(\cdot)|$ is negligible.

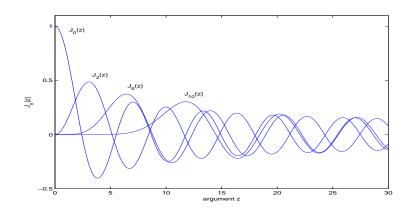


Fig. 2. Bessel functions of first kind $J_n(z)$ vs. argument z for different values of n.

We note that our observable signal spectrum (12) is a product of $\alpha_n(\cdot)$ and $J_n(\cdot)$. Thus, for a fixed value of radius, at each order |n| > 0, the SNR is less than the threshold γ for frequencies less than a critical frequency F_n . In effect, we can not detect the signal spectrum for frequencies less than F_n . From Fig. 2, $J_0(\cdot)$ is active within the frequency range $[0, \infty)$, hence, the effective bandwidth of the 0^{th} order signal spectrum is 2W.

Theorem 3 (Effective Bandwidth of the n^{th} Order): Given any wireless random multipath wavefield band limited to $[F_0 - W, F_0 + W]$ and observed within a 2D disk region of radius R, such that the wavefield is encoded in finite number of orders $n < N_u$, the effective frequency bandwidth of the n^{th} order signal spectrum is given by

$$W_{n} = \begin{cases} 2W, & n = 0\\ F_{0}+W - \max\{F_{0}-W, F_{n}\}, & |n| < N_{u}\\ 0, & \text{otherwise} \end{cases}$$
(13)

where N_u is the lowest order for which the critical frequency $F_n > F_0 + W$ and

$$F_n \ge \frac{nc}{e\pi R} + \frac{c}{2e\pi R} \log\left(\frac{\gamma}{(SNR)_{\max}}\right). \tag{14}$$

with the threshold γ depicting the ability of the system to detect signals buried in noise and assuming that the power of the spectrum $\alpha_n(\omega)$ is finite and bounded for all frequencies ω and orders n, i.e., $E\{|\alpha_n(\omega)|^2\} \leq P_{\max}$, the maximum SNR for any order n is

$$(SNR)_{\max} = \frac{P_{\max}}{\sigma_0^2}.$$
 (15)

Proof: From (4) and (12), the observable random multipath field can be represented by an infinite but countable set of orthogonal basis functions as follows

$$\Psi(\mathbf{x},\omega) = \sum_{n=-\infty}^{\infty} i^n [\alpha_n(\omega) J_n(\frac{\omega}{c}R) + \nu_n(\omega)] e^{in\phi_x}$$
(16)

$$\frac{1}{2\pi} \int_0^{2\pi} E\{|\Psi(\mathbf{x},\omega)|^2\} d\phi_x = \sum_{n=-\infty}^\infty E\{|\alpha_n(\omega)|^2\} |J_n(\frac{\omega}{c}R)|^2 + \sigma_0^2.$$
(17)

The SNR at the n^{th} order over the frequency band $[0, \omega_n]$ with $\omega_n = 2\pi F_n$ is

$$(SNR)_n = \frac{\int_0^{\omega_n} E\{|\alpha_n(\omega)|^2\}|J_n(\frac{\omega}{c}R)|^2 d\omega}{\int_0^{\omega_n} \sigma_0^2 d\omega}.$$
(18)

Note that we consider white noise and is independent of frequency. Hence, using (15), (18) can be rewritten as

$$(SNR)_n \le (SNR)_{\max} \frac{(R/c)^{2n}}{\omega_n 2^{2n} [\Gamma(n+1)]^2} \int_0^{\omega_n} \omega^{2n} d\omega.$$
⁽¹⁹⁾

This result is obtained based on the fact that for large order n, the Bessel functions can be approximated as [10, eqn 9.1.7]

$$J_n(z) \sim (\frac{1}{2}z)^n / \Gamma(n+1), \quad n \ge 0$$
 (20)

where $\Gamma(\cdot)$ is the Gamma function. Now, we use the Stirling lower bound on the Gamma functions, $\Gamma(n+1) > \sqrt{2\pi n n^n e^{-n}}$, to write (19) as

$$(SNR)_{n} < (SNR)_{\max} \frac{1}{2\pi n(2n+1)} \left(\frac{e\omega_{n}R/c}{2n}\right)^{2n}$$
$$< (SNR)_{\max} e^{-(2n-2\pi eF_{n}R/c)}$$
(21)

since $\beta = 1/(2\pi n(2n+1)) < 1$ and using the exponential inequality, $(1 + x/n)^n \le e^x$ for $n \ne 0$.

Note that for the n^{th} order, the $(SNR)_n$ must be larger than the threshold γ , in effect,

$$(SNR)_{\max}e^{-(2n-2\pi eF_nR/c)} \ge \gamma \tag{22}$$

which results in (14). This means that for order n, signals below frequency F_n are not detectable since (22) will not be satisfied. Observe that for any particular order |n|(>0), if $F_n > F_0 - W$, the effective bandwidth of that order is $F_0 + W - F_n$. In addition, if $F_n > F_0 + W$, the effective bandwidth of this order and orders above this is zero and we can truncate the infinite series in (16) to $|n| < N_u$. These arguments can be written mathematically as (13).

V. DEGREES OF FREEDOM OF 2D MULTIPATH FIELDS

In this section, we derive an expression to estimate the degrees of freedom available in a wideband multipath field observable over finite time and space windows. Note that so far we showed that any wireless multipath wavefield band limited to $[F_0 - W, F_0 + W]$ and observed within a 2D disk region of radius R over a finite time interval [0,T] can be represented as a series of orthogonal basis functions encoded in a finite number of orders n. A simple observation based on this representation is that for each order n, the observation time over space T_{eff} is fixed (7). Whereas, for each order n, the effective frequency bandwidth W_n is different (13). Note that the work of Shannon [11] provided a theorem to determine the degrees of freedom available in a wideband channel observed over a finite time interval for point to point communications. We can think of Shannon's model as a wavefield encoded in only one spatial order. Hence, following the classical degrees of freedom result of *time-bandwidth product*+ 1, the available degrees of freedom for each order is $W_n T_{eff}$ + 1. Therefore, we can evaluate the total degrees of freedom available in our observable 2D multipath field as

$$D = \sum_{|n| < N_u} (W_n T_{eff} + 1) = \sum_{|n| < N_u} \left[W_n \left(T + \frac{2R}{c} \right) + 1 \right]$$
(23)

where W_n is given by (13). Note that the degrees of freedom result in (23) does not agree with the well established result of evaluating degrees of freedom of spatially diverse wideband wavefields as a product of *space-timebandwidth* [3], [6]. However, in the propagation of waves even though space, time and frequency are separate entities, in spatially diverse wideband wavefields space and time as well as space and frequency are strongly coupled, the results of [3], [6] fail to show those coupling relationships. On the contrary, our derived result clearly indicates the coupling relationships between space and time as well as space and frequency.

VI. CONCLUSION

In this paper, we express any band limited wireless multipath wavefield observed within a 2D disk region of finite radius over a finite time interval as a series of orthogonal basis functions encoded in a finite number of spatial orders. Our analysis shows that (i) the effective observation time varies from given observation time and is not spatial order dependent, and (ii) the lower spatial orders can utilize the full frequency bandwidth, whereas, the higher orders can utilize only a fraction of the given bandwidth. These findings portray the strong coupling relations between space and time as well as space and frequency. Thus, our derived degrees of freedom result based on these findings clearly indicates how the coupling relations impact the available degrees of freedom of any 2D wideband multipath field observed over finite time and space windows. We also show that the degrees of freedom is affected by the acceptable SNR in each spatial order.

REFERENCES

- G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.
- [2] R. A. Kennedy, T. D. Abhayapala, and H. M. Jones, "Bounds on the spatial richness of multipath," in Australian Communications Theory Workshop, AusCTW, Canberra, Austraia, Feb. 4-5 2002, pp. 76–80.
- [3] A. S. Y. Poon, R. W. Brodersen, and D. N. C. Tse, "Degrees of freedom in multiple-antenna channels: A signal space approach," *IEEE Transactions on Information Theory*, vol. 51, no. 2, pp. 523–536, Feb. 2005.
- [4] M. Franceschetti and K. Chakraborty, "Space-time duality in multiple antenna channels," *IEEE Transactions on Wireless Communications*, vol. 8, no. 4, pp. 1733–1743, April 2009.
- [5] A. Ozgur, O. Leveque, and D. Tse, "Spatial degrees of freedom of large distributed MIMO systems and wireless ad hoc networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 2, pp. 202–214, Feb. 2013.
- [6] M. Franceschetti, "On the information content of scattered waves," in *IEEE International Symposium on Information Theory Proceedings*, 2012, pp. 1523–1527.
- [7] L. W. Hanlen and T. D. Abhayapala, "Space-time-frequency degrees of freedom: Fundamental limits for spatial information," in *IEEE Intl.Symp. on Information Theory 2007, ISIT 2007*, Nice, France, June 24-29 2007, pp. 701–705.

- [8] D. Colton and R. Kress, Inverse acoustic and electromagnetic scattering theory, Applied Mathematical Sciences, vol. 93, 2nd edition, 1998.
- [9] R. G. Gallager, Information Theory and Reliable Communication, John Wiley & Sons, New York, USA, 1968.
- [10] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions*, vol. 55, National Bureau of Standards, US Government Printing Office, Washington, DC, 10th edition, 1972.
- [11] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the IRE*, vol. 37, no. 1, pp. 10–21, Jan. 1949, Reprinted in Proceedings of the IEEE, vol. 86, no. 2, pp. 447-457, Feb. 1998.