Unified asymptotic distribution of subspace projectors in complex elliptically symmetric models

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***Abstract*—The statistical performance of subspace-based algorithms depends on the deterministic and stochastic statistical model of the noisy linear mixture of the data, the estimate of the projector, and the algorithm that estimates the parameters from the projector. This paper presents different circular and non-circular complex elliptically symmetric (CES) models of the data and different associated non-robust and robust covariance estimators whose asymptotic distributions are derived. This allows us to unify and complement the asymptotic distribution of subspace projectors adapted to these models and to prove several invariance properties that have impacts on the parameters to be estimated in CES data models.**

***Index Terms*—Asymptotic distribution of subspace projector, complex elliptically symmetric distribution**

I. Introduction

Subspace-based algorithms that exploit the orthogonality between a sample subspace and a parameter-dependent subspace have proved very useful in many applications in signal processing. These algorithms have been intensely studied in the literature in the circular complex Gaussian framework (see e.g., [1]–[8] and references therein). But this framework is often insufficient for non-Gaussian heavy-tailed distributed data that are well modeled by circular CES (C-CES) or non-circular CES (NC-CES) distributions.

The aim of this paper is to unify and complement different deterministic and stochastic CES models of the data and asymptotic distributions of the associated projectors derived from different estimate of the covariance matrix of the parametric noisy linear mixture data presented in the literature. The asymptotic distribution (w.r.t. the number of measurements) of the sample covariance matrix (SCM), maximum likelihood (ML), robust M, Tyler's M and sample sign covariance matrix (SSCM) estimate of the covariance are considered. This allows us to derive the asymptotic distribution of the associated projectors and to prove several invariance properties.

The following notations are used in this paper. $\otimes$ is the Kronecker product of matrices, $\mathrm{vec}(\cdot)$ is the vectorization operator that turns a matrix into a vector by stacking the columns of the matrix one below another. $\mathbf{K}$ is the vec-permutation matrix (i.e., $\mathrm{vec}(\mathbf{C}^T) = \mathbf{K}\mathrm{vec}(\mathbf{C})$) and $\mathbf{J}$ is the block exchange matrix $\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$ of appropriate dimension. $\mathrm{RES}_m(\boldsymbol{\mu}, \mathbf{R}, g)$, C-$\mathrm{CES}_m(\boldsymbol{\mu}, \mathbf{R}, g)$, NC-$\mathrm{CES}_m(\boldsymbol{\mu}, \mathbf{R}, \mathbf{C}, g)$ and $\mathbb{C}N_m(\boldsymbol{\mu}, \mathbf{R}, \mathbf{C})$ denote the real (RES), circular and non-circular complex valued elliptically symmetric, and Gaussian distributions of dimension m with finite 2nd order moments, respectively where $\boldsymbol{\mu}$, $\mathbf{R}$ and $\mathbf{C}$ are the mean, the covariance and complementary covariance matrices, respectively, and g the density generator.

II. Stochastic and deterministic CES data model

A. Noisy linear mixture model

Consider the following general noisy linear mixture model

$$\mathbf{x}_i = \mathbf{A}\mathbf{s}_i + \mathbf{n}_i \in \mathbb{C}^m, \quad i = 1, ..., n, \tag{1}$$

where $(\mathbf{x}_i)_{i=1}^n$ are independent observations, $\mathbf{s}_i$ and $\mathbf{n}_i$ represent a signal of interest and an additive measurement noise, respectively, which are assumed to be zero-mean mutually uncorrelated. $\mathbf{n}_i$ is assumed to be complex circular spatially uncorrelated with $\mathrm{E}(\mathbf{n}_i\mathbf{n}_i^H) = \sigma_n^2\mathbf{I}$ and $\mathrm{E}(\mathbf{n}_i\mathbf{n}_i^T) = \mathbf{0}$.

Deterministic and stochastic parametric data models have been commonly used to model the distribution of $(\mathbf{s}_i, \mathbf{n}_i)$, where $\mathbf{n}_i$ is complex circular Gaussian distributed [2]. These two statistical data models are extended here within the framework of CES distributions.

B. Deterministic CES data model

In the conditional or deterministic model, $(\mathbf{s}_i)_{i=1,..,n}$ is conditioned from an independent zero-mean Gaussian process. As explained in [2], the sequence $(\mathbf{s}_i)_{i=1,..,n}$ is frozen here in all the realizations of the random data $(\mathbf{x}_i)_{i=1,..,n}$. For complex-valued $\mathbf{s}_i$ with arbitrary circularity, we assume that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{s}_i\mathbf{s}_i^H = \mathbf{R}_{s,\infty}$ exists and is also positive definite. The law of large numbers then implies that

$$\mathbf{R}_{x,\infty} \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^H = \mathbf{A}\mathbf{R}_{s,\infty}\mathbf{A}^H + \sigma_n^2\mathbf{I}. \tag{2}$$

For strictly non-circular complex (also called rectilinear) valued $\mathbf{s}_i$, i.e. whose entries satisfy the condition

$$s_{i,k} = r_{i,k}e^{i\phi_k}, k = 1, .., p \text{ where } r_{i,k} \text{ are real-valued,} \tag{3}$$

in which $\mathbf{s}_i = \boldsymbol{\Delta}\mathbf{r}_i$ where $\boldsymbol{\Delta} \stackrel{\text{def}}{=} \mathrm{Diag}(e^{i\phi_1}, ..., e^{i\phi_p})$ and $\mathbf{r}_i \stackrel{\text{def}}{=} (r_{i,1}, ..., r_{i,p})^T$ with $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{r}_i\mathbf{r}_i^T = \mathbf{R}_{r,\infty}$ exists and is also positive definite. The phases ϕ_k associated with different propagation delays are assumed fixed. To take into account this property (3) of the signals $s_{i,k}$, we consider the extended observation $\widetilde{\mathbf{x}}_i \stackrel{\text{def}}{=} [\mathbf{x}_i^T, \mathbf{x}_i^H]^T$ which leads as in (2) to

$$\mathbf{R}_{\tilde{x},\infty} \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \widetilde{\mathbf{x}}_i \widetilde{\mathbf{x}}_i^H = \widetilde{\mathbf{A}}_r \mathbf{R}_{r,\infty} \widetilde{\mathbf{A}}_r^H + \sigma_n^2 \mathbf{I}, \tag{4}$$

where $\widetilde{\mathbf{A}}_r \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A}\boldsymbol{\Delta} \\ \mathbf{A}^*\boldsymbol{\Delta}^* \end{bmatrix}$. In this deterministic model $(\mathbf{s}_i)_{i=1,..,n}$ or $(\mathbf{r}_i)_{i=1,..,n}$ and $(\phi_1, ..., \phi_p)$ are unknown deterministic parameters. However, the noise $\mathbf{n}_i$ is assumed C-CES distributed. Consequently, the distribution of the observed data $\mathbf{x}_i$ is either C-CES$_m(\mathbf{A}\mathbf{s}_i, \sigma_n^2\mathbf{I}, g_n)$ or C-CES$_m(\mathbf{A}\boldsymbol{\Delta}\mathbf{r}_i, \sigma_n^2\mathbf{I}, g_n)$ distributed, for complex-valued $\mathbf{s}_i$ with arbitrary circularity or rectilinear, respectively.

C. Stochastic CES data model

In the unconditional or stochastic model, a first extension consists in modeling the independent signals $\mathbf{s}_i$ and $\mathbf{n}_i$ by CES distributions to take into account possible heavy-tailed (with respect to the Gaussian one) signals. The noise $\mathbf{n}_i$ is always C-CES$_m(\mathbf{0}, \sigma_n^2\mathbf{I}, g_n)$ distributed. As for $\mathbf{s}_i$, when it is circular, it is C-CES$_p(\mathbf{0}, \mathbf{R}_s, g_s)$ distributed with

$$\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_i\mathbf{x}_i^H) = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}, \tag{5}$$

where $\mathbf{R}_s \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_i\mathbf{s}_i^H)$ is positive definite. In the complex rectilinear case, the signals $s_{i,k}$, $k = 1, .., p$, satisfy constraint (3). In this case, $\mathbf{r}_i$ is RES$_p(\mathbf{0}, \mathbf{R}_r, g_s)$ distributed, which is equivalent to $\mathbf{s}_i$ being NC-CES$_p(\mathbf{0}, \mathbf{R}_s, \mathbf{C}_s, g_s)$, where $\mathbf{R}_s = \boldsymbol{\Delta}\mathbf{R}_r\boldsymbol{\Delta}^*$ and $\mathbf{C}_s = \boldsymbol{\Delta}\mathbf{R}_r\boldsymbol{\Delta}$ with $\mathbf{R}_r \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{r}_i\mathbf{r}_i^T)$ is positive definite, and the covariance of $\widetilde{\mathbf{x}}_i$ is given by

$$\mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \mathrm{E}(\widetilde{\mathbf{x}}_i\widetilde{\mathbf{x}}_i^H) = \widetilde{\mathbf{A}}_r\mathbf{R}_r\widetilde{\mathbf{A}}_r^H + \sigma_n^2\mathbf{I}. \tag{6}$$

For arbitrary non-circular $\mathbf{s}_i$, $\mathbf{s}_i$ is also NC-CES$_p(\mathbf{0}, \mathbf{R}_s, \mathbf{C}_s, g_s)$ distributed, where $\mathbf{R}_{\tilde{x}}$ is given by

$$\mathbf{R}_{\tilde{x}} = \widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H + \sigma_n^2\mathbf{I}, \tag{7}$$

with $\mathbf{R}_{\tilde{s}} \stackrel{\text{def}}{=} \mathrm{E}(\widetilde{\mathbf{s}}_i\widetilde{\mathbf{s}}_i^H) = \begin{pmatrix} \mathbf{R}_s & \mathbf{C}_s \\ \mathbf{C}_s^* & \mathbf{R}_s^* \end{pmatrix}$, where $\widetilde{\mathbf{s}}_i \stackrel{\text{def}}{=} [\mathbf{s}_i^T, \mathbf{s}_i^H]^T$ and $\widetilde{\mathbf{A}}_c \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^* \end{pmatrix}$. It is worth noting here that in this stochastic data model $\mathbf{x}_i$ is not CES distributed (except for Gaussian distributions) because this family of distributions is not closed under summations.

To take advantage of robust covariance matrix estimators available in the context of CES distributions, the CES distribution has been preferred over the Gaussian distribution to model the data $\mathbf{x}_i$ in many DOA finding and beamforming processing (see e.g., [9]–[12]). In this case, the distributions of $\mathbf{s}_i$ and $\mathbf{n}_i$ are generally not specified, but only their second-order statistics are imposed by fixing the structured covariance in (5) or extended covariance matrices in (6) and (7). In addition, the complex compound Gaussian distribution which is a subclass of the CES distributions which was used to model the clutter in Radar [13] was also used in DOA estimation [14] in the form of the model $\mathbf{x}_i = \mathbf{A}\mathbf{s}_i + \mathbf{n}_i \stackrel{\text{def}}{=} \sqrt{\tau_i}(\mathbf{A}\mathbf{s}'_i + \mathbf{n}'_i)$, where $\tau_i > 0$ (with $\mathrm{E}(\tau_i) = 1$) is independent of $(\mathbf{s}'_i, \mathbf{n}'_i)$ which are complex Gaussian distributed. More specifically, in the case of circular complex and non-circular complex signals $\mathbf{s}_i$, the observations $\mathbf{x}_i$ are C-CES$_m(\mathbf{0}, \mathbf{R}_x, g_x)$ and NC-CES$_m(\mathbf{0}, \mathbf{R}_x, \mathbf{C}_x, g_x)$ distributed, respectively.

D. Parameterized mixing matrix

Since the complex-valued signals $\mathbf{s}_i$, can be either circular, rectilinear, or non-circular and non-rectilinear signals, together with the dependence of (1) on $m \times p$ mixing matrix $\mathbf{A}$ and on the parameter of interest $\boldsymbol{\theta}$, leads us to distinguish the following two parameterized cases:

(a) For circular, and non-circular and non-rectilinear complex-valued signals $\mathbf{s}_i$, $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank matrix $\mathbf{A}$ with $p < m$. We will use the parameterizations $\mathbf{B}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \mathbf{A}$ in the circular case and $\mathbf{B}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \widetilde{\mathbf{A}}_c$, in the non-circular and non-rectilinear case.

(b) For rectilinear complex-valued signals $\mathbf{s}_i$, $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank $2m \times p$ extended mixing matrix $\mathbf{B}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \widetilde{\mathbf{A}}_r$ with $p < 2m$.

"This low-rank signal in full-rank noise data model (1) encompasses many far or near-field, narrow or wide-band DOA models with scalar or vector-sensors for an arbitrary number of parameters per source $s_{i,k}$ (with $\mathbf{s}_i \stackrel{\text{def}}{=} (s_{i,1}, ..s_{i,k}, .., s_{i,p})^T$) and many other models as the bandlimited SISO, SIMO [3] and MIMO [5] channel models. For example, parametrization $\widetilde{\mathbf{A}}_r$ can be applied for DOA estimation modeling with rectilinear or strictly second-order sources and for SIMO channels estimation modeling with BPSK or MSK symbols [15] where $\boldsymbol{\theta}$ represents both the localization parameters (azimuth, elevation, range) and the phase of the sources, and the real and imaginary parts of channel impulse response coefficients, respectively. Whereas, parametrization $\widetilde{\mathbf{A}}_c$ is used for DOA modeling with generally non-circular and non-rectilinear complex sources."

III. Subspace-based estimation approaches

Since the parameter of interest $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank matrices $\mathbf{A}$, $\widetilde{\mathbf{A}}_c$ or $\widetilde{\mathbf{A}}_r$, a simple way to get rid of the nuisance parameters, is to consider subspace-based algorithms as the following mapping:

$$(\mathbf{x}_1, .., \mathbf{x}_i, ..., \mathbf{x}_n) \longmapsto \widehat{\mathbf{R}} \longmapsto \widehat{\boldsymbol{\Pi}} \stackrel{\text{alg}}{\longmapsto} \widehat{\boldsymbol{\theta}} = \mathrm{alg}(\widehat{\boldsymbol{\Pi}}) \tag{8}$$

where $\widehat{\mathbf{R}}$ can be either any estimator $\widehat{\mathbf{R}}_x$ of $\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_i\mathbf{x}_i^H)$ or any estimator $\widehat{\mathbf{R}}_{\tilde{x}}$ of $\mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{x}}_i\tilde{\mathbf{x}}_i^H)$, and $\widehat{\boldsymbol{\Pi}}$ denotes the orthogonal projection matrix $\widehat{\boldsymbol{\Pi}}_x$ [resp., $\widehat{\boldsymbol{\Pi}}_{\tilde{x}}$] associated with the so-called noise subspace of $\widehat{\mathbf{R}}_x$ derived from its SVD [resp., $\widehat{\mathbf{R}}_{\tilde{x}}$]. The functional dependence $\widehat{\boldsymbol{\theta}} = \mathrm{alg}(\widehat{\boldsymbol{\Pi}})$ constitutes an extension of the mapping

$$\boldsymbol{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{B}(\boldsymbol{\theta})[\mathbf{B}^H(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})]^{-1}\mathbf{B}^H(\boldsymbol{\theta}) \stackrel{\text{alg}}{\longmapsto} \boldsymbol{\theta}, \tag{9}$$

in the neighborhood of $\boldsymbol{\Pi}(\boldsymbol{\theta})$ with $\mathbf{B}(\boldsymbol{\theta})$ can either be $\mathbf{A}$, $\tilde{\mathbf{A}}_r$ or $\tilde{\mathbf{A}}_c$. Each extension $\mathrm{alg}(.)$ specifies a particular subspace-based algorithm. Conventional MUSIC algorithm [1] based on $\widehat{\mathbf{\Pi}}_x$ and non-circular MUSIC algorithms [16] based on $\widehat{\mathbf{\Pi}}_{\tilde{x}}$ for parametrization (6) can be seen as examples in DOA estimation. According to mapping (8), the statistical properties of the estimator $\widehat{\boldsymbol{\theta}}$ depends on both the choice of the covariance estimator $\widehat{\mathbf{R}}$ and that of the subspace-based algorithm "alg".

IV. Asymptotic distributions of covariance estimators

Deriving the asymptotic distribution of the estimated projectors $\widehat{\mathbf{\Pi}}$ requires determining the asymptotic distribution of different covariance estimators $\widehat{\mathbf{R}}$ adapted to the different data models presented in Section II, which have not all been previously addressed in the authors' work.

A. Deterministic data model

We only consider in this model the SCM estimators $\widehat{\mathbf{R}} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^H$ for complex-valued of arbitrary circularity signals $\mathbf{s}_i$ and extended SCM estimators $\widehat{\mathbf{R}} = \frac{1}{n}\sum_{i=1}^n \widetilde{\mathbf{x}}_i\widetilde{\mathbf{x}}_i^H$ for complex rectilinear-valued signals $\mathbf{s}_i$.

Under finite fourth-order moments of $\mathbf{n}_i$, using the Liapounov central limit theorem (CLT) for independent non identically distributed r.v. $\mathbf{x}_i^*\otimes\mathbf{x}_i$ (see e.g., [17, Th. 2.7.1]) and the Slutsky theorem (see e.g., [17, Th. 5.1.6]), we get the following convergences in distribution for complex-valued of arbitrary circularity signals $\mathbf{s}_i$ [18], [19]:

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{R}})-\mathrm{vec}(\mathbf{R})) \to_d \mathbb{C}N_{m^2}(\mathbf{0},\mathbf{R}_{r_x},\mathbf{R}_{r_x}\mathbf{K}), \quad (10)$$

with $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_{x,\infty}$ defined in (2) and

$$\begin{aligned}\mathbf{R}_{r_x} &= \mathbf{A}^*\mathbf{R}_{s,\infty}\mathbf{A}^T\otimes\sigma_n^2\mathbf{I}+\sigma_n^2\mathbf{I}\otimes\mathbf{A}\mathbf{R}_{s,\infty}\mathbf{A}^H\\ &+ \sigma_n^4[(1+\kappa_n)\mathbf{I}+\kappa_n\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})], \quad (11)\end{aligned}$$

where κ_n is the kurtosis parameter of $\mathbf{n}_i$. Similarly, we obtain for complex rectilinear-valued $\mathbf{s}_i$ that:

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{R}})-\mathrm{vec}(\mathbf{R})) \to_d \mathbb{C}N_{4m^2}(\mathbf{0},\mathbf{R}_{r_{\tilde{x}}},\mathbf{R}_{r_{\tilde{x}}}\mathbf{K}) \quad (12)$$

with $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_{\tilde{x},\infty}$ defined in (4) and

$$\begin{aligned}\mathbf{R}_{r_{\tilde{x}}} &= [\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})][(\widetilde{\mathbf{A}}_r^*\mathbf{R}_{r,\infty}\widetilde{\mathbf{A}}_r^T\otimes\sigma_n^2\mathbf{I})+(\sigma_n^2\mathbf{I}\otimes\\ &\widetilde{\mathbf{A}}_r\mathbf{R}_{r,\infty}\widetilde{\mathbf{A}}_r^H)+\sigma_n^4(1+\kappa_n)\mathbf{I}]+\sigma_n^4\kappa_n\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}). \quad (13)\end{aligned}$$

B. Stochastic data model

We consider here two cases:

1) SCM estimators for both $\mathbf{s}_i$ and $\mathbf{n}_i$ CES distributed with finite fourth-order moments: By applying the classic CLT to the r.v. $\mathbf{x}_i^*\otimes\mathbf{x}_i$ and $\widetilde{\mathbf{x}}_i^*\otimes\widetilde{\mathbf{x}}_i$, we get, for circular and non-circular $\mathbf{s}_i$, respectively, the following asymptotic distribution of $\widehat{\mathbf{R}}$ which did not appear in the literature (see [20]).

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{R}})-\mathrm{vec}(\mathbf{R})) \to_d \mathbb{C}N_{m^2}(\mathbf{0},\mathbf{R}_{r_x},\mathbf{R}_{r_x}\mathbf{K}) \quad (14)$$

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{R}})-\mathrm{vec}(\mathbf{R})) \to_d \mathbb{C}N_{4m^2}(\mathbf{0},\mathbf{R}_{r_{\tilde{x}}},\mathbf{R}_{r_{\tilde{x}}}\mathbf{K}) \quad (15)$$

with $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_x$ defined in (5) for circular $\mathbf{s}_i$ and $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_{\tilde{x}}$ defined in (6) and (7) for non-circular $\mathbf{s}_i$, respectively, and

$$\begin{aligned}\mathbf{R}_{r_x} &= (\mathbf{R}_x^*\otimes\mathbf{R}_x)+\kappa_s[(\mathbf{A}^*\mathbf{R}_s^*\mathbf{A}^T)\otimes(\mathbf{A}\mathbf{R}_s\mathbf{A}^H)\\ &+\mathrm{vec}(\mathbf{A}\mathbf{R}_s\mathbf{A}^H)\mathrm{vec}^H(\mathbf{A}\mathbf{R}_s\mathbf{A}^H)]+\sigma_n^4\kappa_n[\mathbf{I}+\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})], \quad (16)\\ \mathbf{R}_{r_{\tilde{x}}} &= [\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})][(\mathbf{R}_{\tilde{x}}^*\otimes\mathbf{R}_{\tilde{x}})+\kappa_s(\widetilde{\mathbf{A}}_c^*\mathbf{R}_{\tilde{s}}^*\widetilde{\mathbf{A}}_c^T)\\ &\otimes (\widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H)+\kappa_s\mathrm{vec}(\widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H)\mathrm{vec}^H(\widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H)\\ &+ \sigma_n^4\kappa_n\mathbf{I}]+\sigma_n^4\kappa_n\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}), \quad (17)\end{aligned}$$

where κ_s is the kurtosis parameter of $\mathbf{s}_i$. Note that (15) and (17) remain valid for complex rectilinear signals $\mathbf{s}_i$ if κ_s is replaced by κ_r. Furthermore, in this case $\widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H$ reduces to $\widetilde{\mathbf{A}}_r\mathbf{R}_r\widetilde{\mathbf{A}}_r^H$.

2) Covariance estimators for $\mathbf{x}_i$ CES distributed: For these distributions, many covariance estimators have been proposed in the literature. We consider here (a) the SCM estimator, (b) the ML estimator which is often considered as the reference estimator but can be drastically affected by the presence of outliers or when the data distribution deviates slightly from the CES distribution of the model, (c) a class of M estimators, introduced by Maronna [21] for RES distributions, then extended to C-CES and NC-CES distributed data in [9] and [19], respectively, and later studied and used in various signal processing application (see [12] and references therein), (d) Tyler's [22] and (e) the SSCM estimators [23]–[29] that are both distribution-free.

The asymptotic distributions of all these estimators are also given by (14) and (15) for circular and non-circular $\mathbf{x}_i$, respectively, where $\mathbf{R}$ is defined under (a) and (b) by $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_x$ defined in (5) and $\mathbf{R}\stackrel{\text{def}}{=}\mathbf{R}_{\tilde{x}}$ defined in (6) and (7), under (c) by $\mathbf{R}\stackrel{\text{def}}{=}c^{-1}\mathbf{R}_x$ in the circular case and $\mathbf{R}\stackrel{\text{def}}{=}c^{-1}\mathbf{R}_{\tilde{x}}$ in the non-circular case with c is given by [12, rel. (46)], under (d) by $\mathbf{R}\stackrel{\text{def}}{=}\frac{m}{\mathrm{Tr}(\mathbf{R}_x)}\mathbf{R}_x$ in the circular case and $\mathbf{R}\stackrel{\text{def}}{=}\frac{m}{\mathrm{Tr}(\mathbf{R}_x)}\mathbf{R}_{\tilde{x}}$ in the non-circular case, and under (e) by $\mathbf{R}\stackrel{\text{def}}{=}\sum_{k=1}^m\chi_k\mathbf{v}_k\mathbf{v}_k^H$ and $\mathbf{R}\stackrel{\text{def}}{=}\sum_{k=1}^{2m}\widetilde{\chi}_k\widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H$, respectively, where $\sum_{k=1}^m\lambda_k\mathbf{v}_k\mathbf{v}_k^H$ and $\sum_{k=1}^{2m}\widetilde{\lambda}_k\widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H$ denote respectively the EVD of $\mathbf{R}_x$ and $\mathbf{R}_{\tilde{x}}$, and where closed-form expressions of the eigenvalues χ_k and $\widetilde{\chi}_k$ are given by [30, rel. (11), (12)].

The matrices $\mathbf{R}_{r_x}$ in (14) and $\mathbf{R}_{r_{\tilde{x}}}$ in (15) are all given for the SCM, ML, M and Tyler's estimators by

$$\mathbf{R}_{r_x}=\sigma_1(\mathbf{R}_x^*\otimes\mathbf{R}_x)+\sigma_2\mathrm{vec}(\mathbf{R}_x)\mathrm{vec}^H(\mathbf{R}_x), \quad (18)$$

$$\mathbf{R}_{r_{\tilde{x}}}=\sigma_1[\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})](\mathbf{R}_{\tilde{x}}^*\otimes\mathbf{R}_{\tilde{x}})+\sigma_2\mathrm{vec}(\mathbf{R}_{\tilde{x}})\mathrm{vec}^H(\mathbf{R}_{\tilde{x}}), \quad (19)$$

where under (a) $\sigma_1=1+\kappa_x$ and $\sigma_2=\kappa_x$ (where κ_x is the kurtosis parameter of $\mathbf{x}_i$), under (b):

$$\sigma_1=\frac{m(m+1)}{\mathrm{E}[\mathcal{Q}_x^2\phi_x^2(\mathcal{Q}_x)]} \text{ and } \sigma_2=-\frac{2\sigma_1(1-\sigma_1)}{1+2m(1-\sigma_1)}. \quad (20)$$

where $\mathcal{Q}_x$ is the 2nd-order modular variate of $\mathbf{x}_i$, under (c) σ_1 and σ_2 are given by [12, rel (48)(49)] and under (d)

$$\sigma_1=-\frac{m^2}{[\mathrm{Tr}(\mathbf{R}_x)]^2}(1+\frac{1}{m}) \text{ and } \sigma_2=-\frac{m^2}{[\mathrm{Tr}(\mathbf{R}_x)]^2}(1+\frac{1}{m}), \quad (21)$$

Finally, under (e), $\mathbf{R}_{r_x}$ and $\mathbf{R}_{r_{\tilde{x}}}$ are no longer given by (18) and (19), respectively, but share the same eigenvectors as those of (18) and (19), respectively, but with different eigenvalues given by [30, rel. (17-20)].

V. Asymptotic distributions of projector estimators

From the asymptotic distributions (10), (12) and (14) of the different estimators $\widehat{\mathbf{R}}$ adapted to the different models presented in Section II, we note that $\widehat{\mathbf{R}}$ converge in probability to the matrices $\mathbf{R}$. All these matrices are structured as

$$\mathbf{R} = \mathbf{S} + \sigma^2\mathbf{I}, \tag{22}$$

where $\mathrm{Span}(\mathbf{S}) = \mathrm{Span}(\mathbf{B}(\boldsymbol{\theta}))$, where $\mathbf{B}(\boldsymbol{\theta})$ denotes the mixing matrices $\mathbf{A}$, $\widetilde{\mathbf{A}}_r$ and $\widetilde{\mathbf{A}}_c$ for circular, rectilinear, and non-rectilinear and non-circular complex-valued $\mathbf{s}_i$, respectively. Then, using the standard perturbation result associated with the mapping (8)

$$\widehat{\mathbf{R}} = \mathbf{R} + \delta(\mathbf{R}) \mapsto \widehat{\mathbf{\Pi}} = \mathbf{\Pi}(\boldsymbol{\theta}) + \delta(\mathbf{\Pi}), \tag{23}$$

for orthogonal projectors [31] (see also the operator approach in [32]) applied to $\mathbf{\Pi}(\boldsymbol{\theta})$ associated with the noise subspace of $\mathbf{R}$,

$$\delta(\mathbf{\Pi}) = -\mathbf{\Pi}(\boldsymbol{\theta})\delta(\mathbf{R})\mathbf{S}^{\#} - \mathbf{S}^{\#}\delta(\mathbf{R})\mathbf{\Pi}(\boldsymbol{\theta}) + o(\delta(\mathbf{R}), \tag{24}$$

the asymptotic behaviors of $\widehat{\mathbf{\Pi}}$ and $\widehat{\mathbf{R}}$ are directly related. The standard theorem of continuity (see e.g., [33, p. 122]) on regular functions of asymptotically Gaussian statistics applies and we get

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{\Pi}}) - \mathrm{vec}(\mathbf{\Pi}(\boldsymbol{\theta}))) \to_d \mathbb{C}N_{m^2}(\mathbf{0}, \mathbf{R}_{\pi_x}, \mathbf{R}_{\pi_x}\mathbf{K}) \tag{25}$$

$$\sqrt{n}(\mathrm{vec}(\widehat{\mathbf{\Pi}}) - \mathrm{vec}(\mathbf{\Pi}(\boldsymbol{\theta}))) \to_d \mathbb{C}N_{4m^2}(\mathbf{0}, \mathbf{R}_{\pi_{\tilde{x}}}, \mathbf{R}_{\pi_{\tilde{x}}}\mathbf{K}), \tag{26}$$

for circular and non-circular complex-valued $\mathbf{s}_i$, respectively, where $\mathbf{\Pi}(\boldsymbol{\theta})$ is given by (9) with its associated $\mathbf{B}(\boldsymbol{\theta})$ and where

$$\begin{aligned}\mathbf{R}_{\pi_x} &= [(\mathbf{S}_x^{T\#} \otimes \mathbf{\Pi}_x) + (\mathbf{\Pi}_x^T \otimes \mathbf{S}_x^{\#})]\mathbf{R}_{r_x} \\ &\quad [(\mathbf{S}_x^{T\#} \otimes \mathbf{\Pi}_x) + (\mathbf{\Pi}_x^T \otimes \mathbf{S}_x^{\#})], \qquad (27)\\ \mathbf{R}_{\pi_{\tilde{x}}} &= [(\mathbf{S}_{\tilde{x}}^{T\#} \otimes \mathbf{\Pi}_{\tilde{x}}) + (\mathbf{\Pi}_{\tilde{x}}^T \otimes \mathbf{S}_{\tilde{x}}^{\#})]\mathbf{R}_{r_{\tilde{x}}} \\ &\quad [(\mathbf{S}_{\tilde{x}}^{T\#} \otimes \mathbf{\Pi}_{\tilde{x}}) + (\mathbf{\Pi}_{\tilde{x}}^T \otimes \mathbf{S}_{\tilde{x}}^{\#})], \qquad (28)\end{aligned}$$

where $\mathbf{R}_{r_x}$ and $\mathbf{R}_{r_{\tilde{x}}}$ are given by (11), (16), (18), and by (13), (17), (19), respectively, and where each of the two matrices ($\mathbf{S}_x$, $\mathbf{\Pi}_x$) and ($\mathbf{S}_{\tilde{x}}$, $\mathbf{\Pi}_{\tilde{x}}$) are the matrices $\mathbf{S}$ (of (22)) and $\mathbf{\Pi}(\boldsymbol{\theta})$ associated with the cases circular and non-circular (rectilinear and non-rectilinear) complex-valued $\mathbf{s}_i$, respectively.

Then plugging the expressions (11), (16), (18) of $\mathbf{R}_{r_x}$ and (13), (17), (19) of $\mathbf{R}_{r_{\tilde{x}}}$ into (27) and (28), and using $\mathbf{\Pi}_x\mathbf{S}_x = \mathbf{0}$ and $\mathbf{\Pi}_{\tilde{x}}\mathbf{S}_{\tilde{x}} = \mathbf{0}$, the following result which extends [34, Th. IV.1] and [30, Th. 3]:

Result 1: The covariance matrices $\mathbf{R}_{\pi_x}$ and $\mathbf{R}_{\pi_{\tilde{x}}}$ of the asymptotic distribution (25) and (26) of the different projector estimators $\widehat{\mathbf{\Pi}}$ have an unified structure given by

$$\mathbf{R}_{\pi_x} = (\mathbf{U}^T \otimes \mathbf{\Pi}(\boldsymbol{\theta})) + (\mathbf{\Pi}^T(\boldsymbol{\theta}) \otimes \mathbf{U}), \tag{29}$$

$$\mathbf{R}_{\pi_{\tilde{x}}} = [\mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J})][(\mathbf{U}^T \otimes \mathbf{\Pi}(\boldsymbol{\theta})) + (\mathbf{\Pi}^T(\boldsymbol{\theta}) \otimes \mathbf{U})], \tag{30}$$

where $\mathbf{\Pi}(\boldsymbol{\theta})$ are the projection matrices $\sum_{k=p+1}^{m} \mathbf{v}_k\mathbf{v}_k^H$, $\sum_{k=p+1}^{2m} \widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H$ and $\sum_{k=2p+1}^{2m} \widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H$ on the noise subspace (i.e., on the orthogonal complement of the range of $\mathbf{A}$, $\widetilde{\mathbf{A}}_r$ and $\widetilde{\mathbf{A}}_c$), associated with circular, rectilinear, and non-rectilinear and non-circular complex-valued $\mathbf{s}_i$, respectively. On the other hand, the matrices $\mathbf{U}$ depend on the covariance estimators $\widehat{\mathbf{R}}$ studied in Section IV adapted to the different data models presented in Section II.

For the deterministic model and the stochastic model where both $\mathbf{s}_i$ and $\mathbf{n}_i$ are CES distributed, $\mathbf{U}$ takes the common form

$$\mathbf{U} = \sigma_n^2\mathbf{S}^{\#}\mathbf{R}\mathbf{S}^{\#} + \kappa_n\sigma_n^4(\mathbf{S}^{\#})^2 = \sum_{k=1}^{p} \frac{\sigma_n^2(\lambda_k + \kappa_n\sigma_n^2)}{(\lambda_k - \sigma_n^2)^2}\mathbf{v}_k\mathbf{v}_k^H, \tag{31}$$

with $\mathbf{R} \stackrel{\text{def}}{=} \mathbf{R}_{x,\infty} = \mathbf{R}_x = \sum_{k=1}^{m} \lambda_k\mathbf{v}_k\mathbf{v}_k^H$ and $\mathbf{S} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{R}_{s,\infty}\mathbf{A}^H = \mathbf{A}\mathbf{R}_s\mathbf{A}^H$ for $\mathbf{s}_i$ of arbitrary circularity. Similarly for rectilinear $\mathbf{s}_i$, (31) also applies where $\mathbf{R}_{x,\infty}$, $\mathbf{R}_x$, $\mathbf{A}\mathbf{R}_{s,\infty}\mathbf{A}^H$, $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$, λ_k and $\mathbf{v}_k$ are replaced by $\mathbf{R}_{\tilde{x},\infty}$, $\mathbf{R}_{\tilde{x}}$, $\widetilde{\mathbf{A}}_r\mathbf{R}_{r,\infty}\widetilde{\mathbf{A}}_r^H$, $\widetilde{\mathbf{A}}_r\mathbf{R}_r\widetilde{\mathbf{A}}_r^H$, $\widetilde{\lambda}_k$ and $\widetilde{\mathbf{v}}_k$ respectively. Furthermore, for non-circular and non-rectilinear stochastic $\mathbf{s}_i$ (31) still applies where $\mathbf{R} \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{x}}$, $\mathbf{S} \stackrel{\text{def}}{=} \widetilde{\mathbf{A}}_c\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}_c^H$ and p, λ_k and $\mathbf{v}_k$ are replaced, respectively, by $2p$, $\widetilde{\lambda}_k$ and $\widetilde{\mathbf{v}}_k$.

For the stochastic model where $\mathbf{x}_i$ is CES distributed, we get for SCM, ML, M and Tyler's estimator:

$$\mathbf{U} = \sigma_1\sigma_n^2\mathbf{S}_x^{\#}\mathbf{R}_x\mathbf{S}_x^{\#} = \vartheta\left(\sum_{k=1}^{p} \frac{\lambda_k\sigma_n^2}{(\lambda_k - \sigma_n^2)^2}\mathbf{v}_k\mathbf{v}_k^H\right) \tag{32}$$

for circular $\mathbf{s}_i$. For rectilinear, and non-circular and non-rectilinear $\mathbf{s}_i$, (32) still applies where $\mathbf{S}_x$, $\mathbf{R}_x$, λ_k and $\mathbf{v}_k$ are replaced by $\mathbf{S}_{\tilde{x}}$, $\mathbf{R}_{\tilde{x}}$, $\widetilde{\lambda}_k$ and $\widetilde{\mathbf{v}}_k$, respectively, and with p is replaced by $2p$ for non-circular and non-rectilinear $\mathbf{s}_i$. In (32), ϑ is given by:

$$\vartheta = 1 + \kappa_x \quad \text{for the SCM estimator,} \tag{33}$$

$$\vartheta = \sigma_1 \quad \text{for the ML estimator given by (20),} \tag{34}$$

$$\vartheta = c^2\sigma_1 \quad \text{for the } M\text{-estimator,} \tag{35}$$

$$\vartheta = 1 + m^{-1} \quad \text{for the Tyler's } M\text{-estimator,} \tag{36}$$

where for the M-estimator, c and σ_1 are given by [12, rel. (46)] and [12, rel. (48)], respectively. For the SSCM estimator, $\mathbf{U}$ is given by

$$\mathbf{U} = \sum_{k=1}^{p} \frac{\gamma_k}{(\chi_k - \chi)^2}\mathbf{v}_k\mathbf{v}_k^H \ \text{or}\ \mathbf{U} = \sum_{k=1}^{p} \frac{\widetilde{\gamma}_k}{(\widetilde{\chi}_k - \widetilde{\chi})^2}\widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H, \tag{37}$$

for circular where $\chi \stackrel{\text{def}}{=} \chi_{p+1} = \chi_{p+2} = ... = \chi_m$ and $\gamma_k \stackrel{\text{def}}{=} \gamma_{k,p+1} = \gamma_{k,p+2} = ... = \gamma_{k,m}$, or rectilinear where $\widetilde{\chi} \stackrel{\text{def}}{=} \widetilde{\chi}_{p+1} = \widetilde{\chi}_{p+2} = ... = \widetilde{\chi}_{2m}$ and $\widetilde{\gamma}_k \stackrel{\text{def}}{=} \widetilde{\gamma}_{k,p+1} = \widetilde{\gamma}_{k,p+2} = ... = \widetilde{\gamma}_{k,2m}$ are given by [30, rel. (17-20)]. Some remarks are in order from Result 1:

- The projector estimators have the same asymptotic distribution under both deterministic and stochastic CES distributed models for $\mathbf{s}_i$, with arbitrary circularity or rectilinear $\mathbf{s}_i$. This extends the results proved for many subspace-based DOA estimators [2] in the complex circular Gaussian noise framework.
- The asymptotic distributions of the projector estimators are invariant to the distribution of $\mathbf{s}_i$, whether $\mathbf{s}_i$ is circular or non-circular (rectilinear or non-rectilinear). This property extends the results proved for subspace-based estimators [35] in the complex circular Gaussian noise framework.
- For circular or non-circular CES distributed data $\mathbf{x}_i$, the coefficient ϑ (33)-(36) plays a major role as an index of efficiency for the estimation of the projector and consequently for the estimation of the parameter $\boldsymbol{\theta}$ which is deduced therefrom by all subspace-based algorithms 'alg' (9).

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