

PREVIOUS INVESTIGATION OF REALIZATION OF AN ARBITRARY  
SWITCHING FUNCTION WITH A NETWORK OF THRESHOLD  
AND PARITY ELEMENTS

Keith W. Henderson

ABSTRACT

Attention is called to previous research on realization of an arbitrary switching function by a network of threshold gates (or threshold elements) and modulo - 2 gates (or parity elements), establishment of greatest lower bounds on the number of gates needed, artifices that lead to further network reduction in special cases, and systematic minimization of the number of modulo - 2 gates required. Although not widely published, a report describing the research in detail is readily available.

(Submitted to Correspondence Section. IEEE Trans. on Computers)

Index terms: arbitrary switching function, linearly separable function, matrix partitioning, minimal network, modulo - 2 gate, optimal realization, parity element, threshold element, threshold gate, Walsh matrix, Walsh series.

(Proposed revision of IEEE TC L2421)

Ostapko and Yau have recently published a short note on the realization of an arbitrary switching function by a two-level network of threshold and parity elements [1]. Nearly a decade ago this writer and others also investigated the realization of an arbitrary switching function by a more general network of these same elements, but followed a very different approach and sought a different form of optimization [2]<sup>1</sup>. Unfortunately it was not feasible then to publish our results more widely, even though they were not classified. However, they are readily available at a nominal price.

It is now well known that any m-ary function (not merely binary functions) of n binary variables and  $2^n$  arguments can be expanded into a finite Walsh series. The search for linearly separable threshold functions is simply a search for switching functions that are, or can be, represented by the constant and linear terms of their Walsh expansion. Section III of [1] is simply a redevelopment of the widely known Walsh matrix and its binary equivalent [3], [4].

In view of the great decrease in the fraction of switching functions that are linearly separable as n is increased, the writer, like Ostapko and Yau, became aware that the search for switching functions realizable by threshold devices alone promised very limited rewards, and that a more fruitful search might be one for optimal means of realizing any switching function by a combination of threshold gates (corresponding to the threshold elements of [1]) for the constant and linear terms in the Walsh expansion, and modulo-2 gates (corresponding to the parity elements of [1]) for the nonlinear terms.

Instead of endeavoring to optimize the realization by programming methods, the writer applied the well known method of partitioning matrices to the recursive Walsh matrix derived by Lechner [3], [4], and showed that any switching function could be realized by a three-level network, the first consisting of at most

$2^{n-p} - (n-p) - 1$  modulo-2 gates (where  $p$  is the number of partitionings), the second consisting of at most  $2^p$  threshold gates, and the third consisting of a single conventional OR gate [5]. The OR gate is, of course, a special form of threshold element. If there is only one threshold element in the second level, the OR gate is superfluous, and the network becomes identical to that in Fig. 1 of [1]. Thus, the network in [1] is a special case of that in [2].

Each partitioning eliminates one variable from the modulo-2 sums in the Walsh expansion, and, of course, reduces the number of modulo-2 gates needed. The variables eliminated are then used as control signals to cause each threshold gate to be responsive to a particular subset of  $2^{n-p}$  arguments, and inhibited for the remaining  $2^n - 2^{n-p}$  arguments.

Assuming each modulo-2 sum in the Walsh expansion to require a separate modulo-2 gate, and assuming all gates of all three types (modulo-2, threshold, and conventional OR) to be of equal cost, regardless of number of inputs and values of weights, the writer showed that the total number of gates can be optimized with respect to the number of partitionings. This result has been tabulated for  $1 \leq n \leq 12$  and  $0 \leq p \leq 11$  [6]. Although this optimization does not necessarily yield a minimal network with respect to number of gates, it does furnish a greatest lower bound for further investigation. It is perfectly general, and applicable to totally symmetrical functions as well as less complicated unate and pure threshold functions.

A considerable portion of [5] is devoted to artifices that can be used in various special cases, to cause some of the coefficients of the Walsh expansion to vanish identically (by forming a permissible  $m$ -ary function from a given binary switching function and its duals with respect to its variables), to reduce the order of certain modulo-2 sums in the Walsh expansion, and to combine certain modulo-2 sums in

the expansion to form "nearly constant" functions (whose values are identical for all arguments except 0, 0, . . . 0 or 1, 1, . . . 1 or both, which may be "don't care" conditions). Where applicable, the first of these techniques yields smaller greatest lower bounds (hence still more economical networks), which have also been tabulated for the same values of n and p [7].<sup>2</sup>

The writer's colleague M. Tannenbaum showed how the number of modulo-2 sums, hence the number of gates, can always be minimized by a systematic search for a minimal set of acceptable covers [8].

Keith W. Henderson  
Stanford Linear Accelerator Center<sup>3</sup>  
Stanford University  
Stanford, California 94305

#### REFERENCES

- [1] D. L. Ostapko and S. S. Yau, "Realization of an arbitrary switching function with a two-level network of threshold and parity elements," *IEEE Trans. Comput. (Short Notes)*, Vol. C-19, pp. 262-269, March 1970.
- [2] Lockheed Missiles and Space Co., Palo Alto, Calif., Research on Automatic Computer Electronics - Vol II, Logical Design Research, Technical Documentary Rept. No. RTD-TDR-63-4173, Vol II, February 1964 (unclassified). Available from Clearinghouse for Federal Scientific and Technical Information, Defense Documentation Center, Alexandria, Va., AD No. 436349 (price \$3.00).
- [3] R. Lechner, "A transform theory for functions of binary variables," in Theory of Switching, Harvard Computation Lab., Cambridge, Mass., Progress Rept. BL-30, Sec. X, pp. 1-37, November 1961.
- [4] K. W. Henderson, "Some notes on the Walsh functions," *IEEE Trans. Electron. Comput. (Correspondence)*, Vol EC-13, pp. 50-52, February 1964.

- [5] Lockheed Missiles and Space Co., loc. cit, Sec. B.3, "Logical design by means of orthogonal functions," pp. B-8 to B-34.
- [6] Ibid., p. B-30.
- [7] Ibid., pp. B-31 and B-32.
- [8] Ibid., pp. B-34 to B-66.

#### FOOTNOTES

1. The research was performed under the direction of R. I. Tanaka and R. D. Merrill.
2. W. Y. Dere also contributed to this portion of the research.
3. Operated under contract with the U. S. Atomic Energy Commission.