An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems

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Abstract—A general class of linear time invariant systems with time delay is studied. Recently, they attracted considerable interest in the sys-tems and control community. The complexity arises due to the exponen-tial type transcendental terms in their characteristic equation. The tran-scendentality brings infinitely many characteristic roots, which are cum-bersome to elaborate as evident from the literature. A number of method-ologies have been suggested with limited ability to assess the stability in the parametric domain of time delay. This study offers an exact, structured and robust methodology to bring a closure to the question at hand. Ultimately we present a unique explicit analytical expression in terms of the system parameters which not only reveals the stability regions (pockets) in the do-main of time delay, but it also declares the number of unstable character-istic roots at any given pocket. The method starts with the determination of all possible purely imaginary (resonant) characteristic roots for any pos-itive time delay. To achieve this a simplifying substitution is used for the transcendental terms in the characteristic equation. It is proven that the number of such resonant roots for a given dynamics is finite. Each one of these roots is created by infinitely many time delays, which are periodi-cally distributed. An interesting property is also claimed next, that the root crossing directions at these locations are invariant with respect to the delay and dependent only on the crossing frequency. These two unique findings facilitate a simple and practical stability method, which is the highlight of the work.

Keywords—Commensurate time delay, linear time invariant (LTI), re-tarded systems, stability.

I. Introduction and the Problem Statement

A general class of time delayed linear time invariant (LTI) systems is considered

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}(t-\tau) \quad \mathbf{x}(n\times 1),\ \mathbf{A},\mathbf{B} \in \Re(n\times n),\ \ \tau \in \Re^{+}. \quad (1)$$

Clearly, the delay injects exponential transcendentality to the characteristic equation. This results in infinitely many *finite* characteristic roots, which make the stability outlook very complex. As many recent investigations declare [2], [8], [11], the stability question of this class of systems is not fully resolved up to now. We propose a novel treatment in this text yielding a practical and structured methodology to bring a closure to the question.

The characteristic equation of the system in (1) is

$$\det(s\mathbf{I} - \mathbf{A} - \mathbf{B}e^{-\tau s}) = 0 \text{ with } \tau > 0 \quad (2)$$

which imparts a generic form of

$$\begin{aligned} \mathrm{CE}(s,\tau) =& a_n(s)e^{-n\tau s} + a_{n-1}(s)e^{-(n-1)\tau s} + \cdots + a_0(s) \\ =& \sum_{k=0}^{n} a_k(s)e^{-k\tau s} = 0 \end{aligned} \quad (3)$$

where $a_k(s)$ are polynomials of degree $n-k$ in s with real coefficients. This system is "retarded," that is, $a_o(s)$ is the only system containing the highest degree of s, which is n, and no delay term accompanying it. Equation (3) represents n-toppled commensurate time delay (i.e., delays of integer multiple of τ).

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The linear system in (1) is asymptotically stable if and only if all the characteristic roots of the transcendental equation (3) are on the left half of the complex "s" plane. Since there are infinitely many roots to be examined, to assure this is a complex task.

To resolve this difficulty we deploy a procedure known as the D-Subdivision method (or the "continuity argument"), [6]. It simply states, that in one-dimensional (1-D) parameter space, τ, there are regions (we call them 'pockets') where the number of unstable (or stable) roots of (3) is fixed. At the boundaries separating these regions the corresponding τ values engender at least one pair of purely imaginary roots. If all such boundary values of τ are determined the next question is to assess the tendency of imaginary roots (i.e., from stable to unstable or vice versa) at each boundary crossing. In the case of the former the transition causes an increase of unstable roots by two, and for the latter a decrease by two. When this D-Subdivision methodology is applied to all of the boundary values of τ, one obtains the complete picture of stable regions in τ space.

The main theme of the above methodology is not new and has been implemented for time delayed systems in the past [4], [5], [10], [13]–[15]. The contributions of this study are to recognize some critical and enabling features of the time-delayed systems and to introduce a systematic implementation of the methodology. To achieve this, two critical properties are claimed for the systems represented by (1), lack of which forced limitations to the investigations up to now. We wish to briefly state these properties at this point in order to better prepare the reader and provide their proofs in the later parts of the text.

a) This class of systems exhibits only a *finite number* of possible imaginary characteristic roots for all $\tau \in \Re^{+}$ at given frequencies. And the method must detect all of them. Let us call this set

$$\{\omega_c\} = \{\omega_{c1}, \omega_{c2}, \ldots, \omega_{cm}\} \quad (4)$$

where subscript c refers to "crossing" the imaginary axis. This finite number, m, is influenced not only by n, but also the numerical formation of $\mathbf{A}$ and $\mathbf{B}$ matrices. Furthermore to each ω_{ck}, $k = 1,\ldots,m$ correspond infinitely many, *periodically* spaced τ values. Call this set

$$\{\tau_k\} = \{\tau_{k1}, \tau_{k2}, \ldots, \tau_{k\infty}\} \quad k = 1,\ldots,m \quad (5)$$

where $\tau_{k,\ell+1} - \tau_{k,\ell} = 2\pi/\omega_k$ is the apparent period of repetition.

b) The *root sensitivities* associated with each purely imaginary characteristic root, $\omega_{ck}i$ with respect to τ is defined as

$$S_{\tau}^{s}\Big|_{s=\omega_{ck}i} = \frac{ds}{d\tau}\Big|_{s=\omega_{ck}i}, \quad i = \sqrt{-1} \quad (6)$$

and it displays an unexpected feature: a quantity defined as

$$\text{Root Tendency (RT)} = \mathrm{RT}|_{s=\omega_{ck}i} = \mathrm{sgn}\left[\mathrm{Re}\left(S_{\tau}^{s}|_{s=\omega_{ck}i}\right)\right] \quad (7)$$

is invariant with respect to τ. Notice that RT represents the direction of transition of the roots at $\omega_{ck}i$ as τ increases from $\tau_{k\ell} - \varepsilon$, to $\tau_{k\ell} + \varepsilon$, $0 < \varepsilon \ll 1$. That is, the root $\omega_{ck}i$ crosses the imaginary axis either to the unstable right half plane (for $\mathrm{RT} = +1$) or to the stable left half plane $(\mathrm{RT} = -1)$ independent of $\tau_{k\ell}$, $\ell = 1,2,\ldots,\infty$. This finding is presented as the key and enabling contribution of the note.

The method offers an alternative practical and simple approach to those introduced in [2], [8], and [11]. In [2], the crossing frequencies are determined using Hermite matrices [1], however the approach cannot treat the pockets of stable regions in τ. As a reference book

[8] contains a good cross-section of the state-of-the-art on the stability problem at hand. Most accepted methods therein introduce a complex integral transformation for (1), which imparts some undesired additional dynamics. This feature alone causes substantial limitations in determining the stability margin in τ space. Needless to say, these techniques also fall short in detecting the various stability pockets in τ. It is also clear [7], [16] that the conventional Nyquist stability method yields exact limits of first stable pocket of τ. Its graphical nature, however, is prohibitive for casual applications. Not only that, but also the tedious search for the secondary stable pockets using Nyquist is insurmountably difficult. The method presented here is exact, but it avoids the hurdles of the Nyquist approach. We can perform numerical examples that have much higher dimensions than those offered in the literature. This is an indication of simplicity of the new procedure.

A number of investigators studied the same question in the past, one of which [12] proposed a substitution in place of $e^{-\tau s}$. This procedure converts the transcendental characteristic equation (3) into an algebraic polynomial, which is then analyzed relatively easily for the cases with purely imaginary roots. Most investigations following this proposition [4], [13], [14] were confined to small dimensions (generally $n \leq 2$) because of the lack of recognizing the feature (b).

Another study [15] introduces a strategy, which reduces all the commensurate delay terms to one. The correspondence between this final form and the original system is only at the coincidence of their imaginary roots. The stability features, however, are different and as such the method fails to offer a practical tool.

The strategy we pursue here starts with the directions suggested by Rekasius [12], and further investigated in [4] and [13]. This new approach is explained in Section II with proofs. Section III offers an example case, for $n = 3$. As we routinely demonstrate, there is no structural limitation to n for the new method.

II. Methodology

The method evolves as follows.

A) Root Crossings For the characteristic (3), we first evaluate the complete root crossing structure $[\omega_{ck}, \{\tau_k\}]$, $k = 1, \ldots, m$ for $\tau_k \in \Re^+$. A systematic procedure is given for this operation:

A1) For an obvious simplification in the characteristic equation, we deploy a substitution by Rekasius [12], which is given by

$$e^{-\tau s} = \frac{1 - Ts}{1 + Ts} \quad \tau \in \Re^+, \ T \in \Re \tag{8}$$

and defined only for $s = \omega i$, $\omega \in \Re$. This is an exact substitution, not an approximation, with the obvious mapping condition of

$$\tau = \frac{2}{\omega}\left[\tan^{-1}(\omega T) \mp \ell\pi\right] \quad \ell = 0, 1, 2, \ldots \tag{9}$$

This equation describes an asymmetric mapping in which one T is mapped into infinitely many τ's for a given ω. Inversely for the same ω, one particular τ corresponds to one T only.

The substitution of (8) into (3) results in a rational polynomial

$$\sum_{k=0}^{n} a_k(s)\left(\frac{1 - Ts}{1 + Ts}\right)^k = 0 \tag{10}$$

or recasting it into a simpler form (by multiplying with $(1 + Ts)^n$)

$$\sum_{k=0}^{n} a_k(s)(1 + Ts)^{n-k}(1 - Ts)^k = 0. \tag{11}$$

Sorting the terms in power of s, this equation becomes

$$\sum_{k=0}^{2n} b_k s^k = 0 \tag{12}$$

where $b_k = b_k\ (T, a_{\text{ij}}, b_{\text{ij}})$, $a_{\text{ij}}, b_{\text{ij}}$, $1 \leq i, j \leq n$ being the elements of $\mathbf{A}$ and $\mathbf{B}$ matrices. Assuming $\mathbf{A}$ and $\mathbf{B}$ are given constant matrices b_k's are parameterized in T only. Note that $T \in \Re$, thus it *can also be negative*. Please take note that nth degree transcendental characteristic equation with delay (3) is now converted into $2n$-degree polynomial without transcendentality (12) and its purely imaginary characteristic roots coincide with those of (3) *exactly*.

A2) These coincident imaginary roots are determined next. For this, the Routh–Hurwitz criterion is applied on the simpler characteristic equation (12), which reveals

$$\text{Number of unstable roots } = \textbf{N}\text{umber of }\textbf{S}\text{ign changes in the first column elements} \tag{NS}$$

Note that NS is parameterized in T as well, i.e., $\mathrm{NS}(T)$, because the Routh's array contains T as the only free variable within the terms b_k. As $b_k(T)$ are polynomials of T, the Routh's array forms a first column which consists of rational functions of T, and their numerators and denominators are simple polynomials of T. The real roots of these polynomials can be (exhaustively) calculated. It is clear that the sign change in the first column elements will only come at these values of T's. Their number is finite.

A3) Those values of T are determined next, where a change in the NS takes place. Let's say there are m such points, which we call $\{\mathbf{T}_c\} = T_{c1}, T_{c2}, \ldots, T_{cm}$. At each T_{ck} and only at these T_{ck} values, the characteristic equation (12) possesses one pair of imaginary roots $(\mp\omega_{ck}i)$, as well as (3). There is a one-to-one mapping between $\{\mathbf{T}_c\}$ and $\{\omega_c\}$ sets, as per Routh's Criterion. This can be confirmed by the difference between the successive NS values, which should be 2 as T varies crossing any one of the $\{\mathbf{T}_c\}$ values.

A noteworthy and interesting feature is observed at $T = 0$. As T varies $-\varepsilon \to 0 \to +\varepsilon(\varepsilon \ll 1)$ the NS value drops by n although for this transition there are no imaginary axis crossings of the characteristic roots. This feature is due to the padding of the characteristic equation from n degrees to $2n$ during the Rekasius substitution. Therefore, this point $(T = 0)$ should be overlooked when the set $\{\mathbf{T}_c\}$ is formed, as long as the variation of NS is "n". The proof of this claim and the treatment of degenerate cases when the variation of NS is not "n" are left for another publication due to length restrictions here.

Before progressing further we state the two propositions mentioned earlier and their proofs:

Proposition I: A given time delayed system (1) can exhibit *only a finite number* (m) of purely imaginary characteristic roots $\pm\omega_{ck}i$, $k = 1, \ldots, m$ for all possible $\tau \in \Re^+$.

Proof: It is evident that the coefficients of (12) are polynomials in T, and consequently the elements of the first column, $\{\mathbf{FC}\}$, of Routh's array are rational polynomials of T. The number of *real* zeros and poles of this set, $\{\mathbf{FC}\}$, is therefore finite for varying $-\infty < T < +\infty$. Let us call this finite set $\{\overline{\mathbf{T}_c}\}$. A subset of $\{\overline{\mathbf{T}_c}\}$ called $\{\mathbf{T}_c\}$ contains all the values of T for which NS registers a change (i.e., increases or decreases); $\{\mathbf{T}_c\} \in \{\overline{\mathbf{T}_c}\}$. At each one of the m elements of $\{\mathbf{T}_c\}$

$$\{\mathbf{T}_c\} = T_{c1}, T_{c2}, \ldots, T_{cm} \tag{13}$$

there is one pair of imaginary roots $(\mp\omega_{ck}i)$ which crosses to the other half of the complex plane. These ω_{ck} values are also determined out of the Routh's array. For the typical changes of NS, which is 2 at a T_{ck}, it is clear that there is only one pair of imaginary roots at hand, ω_{ck}. Thus, the set $\{\omega_c\}$ is of dimension m and it is finite. For the exceptional and degenerate cases, when successive NS values differ by 4, 6, or higher valued even numbers, we find the respective $2, 3, \ldots, \omega_{ck}$'s again through Routh's Criterion. In such cases the array $\{\mathbf{T}_c\}$ is also augmented by $2, 3, \ldots$ repeated T_c values. This situation does not change the proposition nor its utilization. ■

From (9) for each T_{ck} (or the corresponding ω_{ck}), there are infinitely many time delays $\{\tau_k\}$. In summary, the dynamics given in (1) can exhibit only finite number of purely imaginary roots $\{\omega_c i\} = \{\omega_{c1} i, \omega_{c2} i, \ldots, \omega_{cm} i\}$ for infinitely many (but not dense) values of time delays. Schematically, the correspondence is as follows:

$$\begin{aligned} &T_{ck} \xrightarrow{\text{generates}} \omega_{ck} \xrightarrow{\text{generates}} \tau_{k\ell} \\ &k = 1, 2, \ldots, m \quad \ell = 0, 1, 2, \ldots, \infty. \end{aligned} \tag{14}$$

Proposition II: As τ reaches one of the infinitely many values of $\tau_{k\ell}$, $k = 1, \ldots, m$, $\ell = 0, 1, \ldots, \infty$, the system in (1) possesses a pair of imaginary characteristic roots at $s = \mp\omega_{ck} i$. The root tendency (RT) at these locations as defined in (7) is:

$$\begin{aligned} \mathrm{RT}\Big|_{\substack{\omega=\omega_{ck} \\ \tau=\tau_{k\ell}}} &= \mathrm{sgn}\left[\mathrm{Re}\left(\frac{ds}{d\tau}\Big|_{\substack{s=\omega_{ck} i \\ \tau=\tau_{k\ell}}}\right)\right] \\ k &= 1, \ldots, m \quad \ell = 0, 1, \ldots, \infty \end{aligned} \tag{15}$$

is invariant for a given ω_{ck}, *independent of* $\tau_{k\ell}$.

Proof: From (3), the following is obtained by differentiation:

$$\frac{ds}{d\tau}\Big|_{s=\omega_{ck} i} = \frac{\sum_{j=0}^{n} a_j j s e^{-j\tau s}}{\sum_{j=0}^{n} \left[\frac{da_j}{ds} - j a_j \tau\right] e^{-j\tau s}}\Bigg|_{\substack{s=\omega_{ck} i \\ \tau=\tau_{k\ell}, \ell=0,1,2,\ldots,\infty}} \tag{16}$$

for any one of $\omega_{ck}, k = 1, 2, \ldots, m$. Notice in this expression $a_k(s)$ is only a function of ω_{ck}. So is

$$\begin{aligned} e^{-j\tau s}\Big|_{\substack{s=\omega_{ck} i \\ \tau=\tau_{k\ell}}} &= \left(\frac{1-T_s}{1+T_s}\Big|_{\substack{s=\omega_{ck} i \\ T=T_{ck}}}\right)^j \\ k &= 1, \ldots, m \quad \ell = 0, 1, \ldots, \infty \end{aligned} \tag{17}$$

which eliminates the dependence on τ. Therefore all the terms in (16) but the τ in the denominator is independent of the particular time delay involved.

We rewrite (16) dividing the numerator and the denominator by the coefficient of τ.

$$\frac{ds}{d\tau}\Big|_{s=\omega_{ck} i} = \frac{s}{\frac{\sum_{j=0}^{n} a_j' e^{-j\tau s}}{\sum_{j=0}^{n} j a_j e^{-j\tau s}} - \tau}\Bigg|_{\substack{s=\omega_{ck} i \\ \tau=\tau_{k\ell}}}. \tag{18}$$

Consequently

$$\begin{aligned} \mathrm{RT}\Big|_{\substack{\omega=\omega_{ck} \\ \tau=\tau_{k\ell}}} &= \mathrm{sgn}\left[\mathrm{Re}\left(\frac{ds}{d\tau}\Big|_{s=\omega_{ck} i}\right)\right] \\ &= \mathrm{sgn}\left[\mathrm{Im}\left(\frac{\sum_{j=0}^{n} a_j' e^{-j\tau s}}{\sum_{j=0}^{n} j a_j e^{-j\tau s}}\Bigg|_{\substack{s=\omega_{ck} i \\ \tau=\tau_{k\ell}}}\right)\right]. \end{aligned} \tag{19}$$

Using (17) and the ensuing arguments, it is clear that (19) is *invariant* with respect to $\tau_{k\ell}$. It only depends on T_{cj} or the corresponding ω_{cj}. ■

In short the characteristic roots of (1) cross the imaginary axis at m locations, $\omega_{ck} i$, $k = 1, \ldots, m$ for infinitely many delays, $\tau_{k\ell}$, $k = 1, \ldots, m$, $\ell = 0, 1, \ldots, \infty$. The root tendencies (or the root crossing directions) at these points are independent of the corresponding delay, $\tau_{k\ell}$. We will show that this feature offers a very convenient tool in determining the stability regions in τ. It was also observed by Cooke and van den Driessche [3] for scalar systems $(n = 1)$ following an analysis which is prohibitive to deploy for $n > 1$. To the knowledge of the authors, investigations to date failed to recognize this property for higher dimensional dynamics.

B) Having established the $[\omega_{ck}, \{\tau_k\}]$, $k = 1, \ldots, m$ sequence, we now return to the main problem, and present structured steps of the methodology (i.e., the deployment of D-Subdivision method).

- Form a table of $\tau_{k\ell}$, $k = 0, 1, \ldots, m$, $\ell = 1, \ldots, \infty$ and $\mathrm{RT}|_{\tau_{k\ell}} = \mathrm{RT}|_k$ in ascending order of $\tau_{k\ell}$. Notice the slight breach of notation that we are suppressing $s = \omega_{ck} i$ to simply k.
- Consider $\tau = 0$ for which the number of unstable roots is known from Routh's Criterion.
- Go to the smallest $\tau_{k\ell} > 0$, assess the number of unstable roots as $\tau = \tau_{k\ell} + \varepsilon$, $0 < \varepsilon \ll 1$ using the $\mathrm{RT}|_k$. Let us call this number NU, which is obviously a function of τ. If $\mathrm{RT} = +1$ NU increases by 2, if $\mathrm{RT} = -1$ it decreases by 2. This step is where we use the D-Subdivision method precisely.
- Repeat the previous step for the next $\tau_{k\ell}$. Continue completing the analysis until the target value of τ is reached.
- Identify those regions in τ, where $\mathrm{NU}(\tau) = 0$ as *stable* and others as *unstable*.

This completes the procedural steps of the new method.

These steps could be even further simplified. Notice that $\tau_{k\ell}$ sequence (9) for a fixed k is periodic, with the period of $\Delta\tau_k = 2\pi/\omega_k$. The $\mathrm{RT}|_k$ is the same for all of these points as per the Proposition II. Then the table previously explained is achieved only with the knowledge of

$$\tau_{k\ell}, \omega_{ck}, \mathrm{RT}\Big|_k \quad k = 1, \ldots, m, \;\; \ell = 0.$$

The stability regions of τ can then be declared for arbitrarily large range of time delays. This conclusion is possible due to the highlight contribution of this work, Proposition II.

We can now express the number of unstable roots $NU(\tau)$ as an explicit function of τ

$$NU(\tau) = NU(0) + \sum_{k=1}^{m} \Gamma\left(\frac{\tau - \tau_{k0}}{\Delta\tau_k}\right) \cdot U(\tau, \tau_{k0}) \cdot \mathrm{RT}_k \tag{20}$$

where $NU(0)$ is the number of unstable roots when $\tau = 0$, $U(\tau, \tau_{k0})$ = step function in τ with the step taking place at τ_{k0}

$$U(\tau, \tau_{k0}) = \begin{cases} 0 & 0 < \tau < \tau_{k0} \\ 1 \text{ for} & \tau \geq \tau_{k0}, \omega_{ck} = 0 \\ 2 & \tau \geq \tau_{k0}, \omega_{ck} \neq 0 \end{cases}$$

$\Gamma(x)$ = Ceiling function of x, Γ returns the smallest integer greater than or equal to x. This expression $NU(\tau)$ requires the knowledge of four things

i) $NU(0)$;
ii) τ_{k1}, $k = 1, \ldots, m$, the smallest τ value corresponding to each one of the T_{ck}'s;
iii) $\Delta\tau_k$, $k = 1, \ldots, m$;
iv) $\mathrm{RT}\Big|_k$, $k = 1, \ldots, m$.

All of these quantities are calculated as described in the above text. And these calculations can be completed within a single MAPLE file, the output of which is the stability outlook of the system in various regions of time delay.

III. Example Case Study

Let us take $\mathbf{A}$ and $\mathbf{B}$ of (1) as

$$A = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix} \quad B = \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}. \quad (21)$$

It can be shown that the system is stable for $\tau = 0$. The characteristic roots are $-2 \mp 2i$ and -2.9. The corresponding characteristic equation to (3) is

$$a_3 e^{-3\tau s} + a_2 e^{-2\tau s} + a_1 e^{-\tau s} + a_0 = 0 \quad (22)$$

where

$$\begin{aligned} a_3(s) &= 119.4; \\ a_2(s) &= 90.9s - 185.1; \\ a_1(s) &= 0.9s^2 - 116.8s - 22.1; \\ a_0(s) &= s^3 + 6s^2 + 45.5s + 111.0 \end{aligned}$$

The Rekasius substitution converts this into [corresponding to (12)]:

$$b_6(T)s^6 + b_5(T)s^5 + b_4(T)s^4 + b_3(T)s^3 + b_2(T)s^2 + b_1(T)s + b_0(T) = 0$$

where

$$\begin{aligned} b_6(T) &= T^3; \\ b_5(T) &= 5.1T^3 + 3T^2; \\ b_4(T) &= 253.2T^3 + 17.1T^2 + 3T; \\ b_3(T) &= -171.4T^3 + 162.4T^2 + 18.9T + 1; \\ b_2(T) &= 898.4T^2 - 71.2T + 6.9; \\ b_1(T) &= 137.8T + 19.6; \\ b_0(T) &= 23.2. \end{aligned}$$

The classical Routh's array is formed for this polynomial. *The number of sign changes* in the first column elements (NS) is depicted in Fig. 1. NS indicates the number of right hand side (unstable) roots. Notice that the T values at five crossings are real, $T_{ck} \in \Re$, $k = 1, \dots, 5$, some of which are negative.

For $T = 0$ there is a change in NS by 3 (which is equal to n). We ignore this point as stated earlier. It is critical to observe that at each crossing (i.e., at each T_{ck}) the NS value changes by 2, which implies the presence of *one* pair of imaginary roots at each ω_{ck}. The points at which the roots cross the imaginary axis are given in Table I. In short, the example system can have *purely imaginary* roots only at the five frequencies given on the table. No values of τ can cause any other purely imaginary characteristic root for this system. This is the consequence of Proposition I. Therefore, if the system comes to resonance (meaning the imaginary roots are the dominant ones) the respective *resonance frequency* must be one of these five. This feature resembles the pole placement procedure followed for a recent active vibration absorption methodology [5], [9], [10]. In these studies, the absorber is sensitized using the time delay as one of the control parameters.

Next, we expand the table by adding the corresponding τ values from (9) and RT (root tendency) from (19). Invariance property of Proposition II is obvious from this table. Periodicity of τ can also be observed, for instance in the case of $T = -0.4269$, $\Delta\tau = 2\pi/15.5032 = 0.4052$. As Proposition II claims the RTs are all $+1$, i.e., for increasing τ the characteristic roots move to the unstable right-half plane (RHP) through this crossing point $\omega_{c1} i = \mp 15.5032i$, creating an increase in NU by 2 each time.

The stability posture of the system is given in the third column (Stable: S, Unstable: U). The number of roots causing the instability is also stated on the same column. This table completes the analysis. It reveals that there are two stable regions of τ (shaded)

$$0 < \tau < 0.1624$$
$$0.1859 < \tau < 0.222$$

with zero unstable roots. After $\tau > 0.222$ stability never returns.

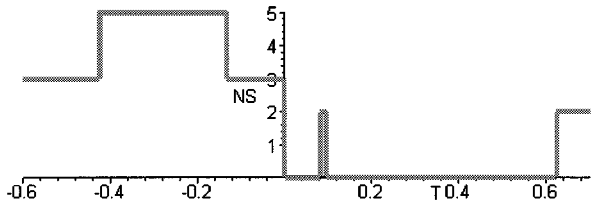

Fig. 1. The number of sign changes (NS).

TABLE I
Stability Regions (Shaded)

τ [sec]	RT	Stable / Unstable NU(τ)	ω [rad/sec]	T
		S, NU=0		
.1624	1		3.0347	.0829
		U, NU=2		
.1859	-1		2.9123	.0953
		S, NU=0		
.2219	1		15.5032	-.4269
		U, NU=2		
.6272	1		15.5032	-.4269
		U, NU=4		
.8725	1		2.1109	.6233
		U, NU=6		
⋮	⋮	⋮	⋮	⋮
		U, NU=42		
7.208	-1		0.8407	-0.1332
		U, NU=40		
⋮	⋮	⋮	⋮	⋮

IV. Conclusion

A structured method is presented for assessing the stability of linear time invariant systems with time delayed state feedback. The method starts with a substitution for the exponential terms in the characteristic equation just to facilitate the determination of the root crossing points over the imaginary axis, and the corresponding delays. Then the D-Subdivision method is deployed for the intervals of the delay, which renders stability outlook of the system.

There are two important contributory findings of the work.

a) This class of systems can have only *finite number of purely imaginary* characteristic roots and these roots are generated by infinitely many discrete (not dense) values of delays. The method solves all of these frequencies and the corresponding delays.

b) At each one of these finite number of imaginary roots, the RTs are invariant with respect to delay. That is, increasing the delay causes the same root crossing direction. They move either to unstable or stable half plane at the given frequency regardless of the value of τ which creates them.

Especially the second finding facilitates a very practical stability assessment tool. It is exact, considerably less involved in contrast to the alternative methods in the literature. More importantly, it results in an explicit expression for the number of unstable roots for any value of delay, τ.

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