Fundamental Design Limitations of the General Control Configuration

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***Abstract*—The theory of fundamental design limitations is well understood for the case that the performance variable is measured for feedback. In the present paper, we extend the theory to systems for which the performance variable is not measured. We consider only the special case for which the performance and measured outputs and the control and exogenous inputs are all scalar signals. The results of the paper depend on the control architecture, specifically, on the location of the sensor relative to the performance output, and the actuator relative to the exogenous input. We show that there may exist a tradeoff between disturbance attenuation and stability robustness that is in addition to the tradeoffs that exist when the performance output is measured. We also develop a set of interpolation constraints that must be satisfied by the disturbance response at certain closed right half plane poles and zeros, and translate these constraints into generalizations of the Bode and Poisson sensitivity integrals. In the absence of problematic interpolation constraints we show that there exists a stabilizing control law that achieves arbitrarily small disturbance response. Depending on the system architecture, this control law will either be high gain feedback or a finite gain controller that depends explicitly on the plant model. We illustrate the results of this paper with the problem of active noise control in an acoustic duct.**

***Index Terms*—Disturbance response, fundamental design limitations, nonminimum phase zeros, sensitivity.**

I. INTRODUCTION

THERE exists an extensive theory of fundamental design limitations applicable to linear time invariant feedback systems with a single input and a single output [1]. Much of this theory is based on the Bode *sensitivity function* [2]. As is well known, the sensitivity function describes the response of the system output to disturbances and provides a measure of stability robustness, in that its inverse is a measure of the distance from the Nyquist plot to the critical point. In practice, the sensitivity function must satisfy the Bode sensitivity integral and, thus, exhibit a design tradeoff termed the "waterbed effect." In words, this tradeoff states that as sensitivity is made small over one frequency range, it necessarily becomes large at other frequencies. A meaningful tradeoff is obtained only in the presence of bandwidth constraints, such as those required to avoid exciting unmodeled high frequency dynamics (cf. [1, Cor. 3.1.6]). The presence of open loop nonminimum phase zeros implies a related waterbed effect that is described by the Poisson sensitivity integral [3]. A thorough review of results on fundamental design limitations is found in [1].

An assumption implicit in most work on fundamental design limitations is that the system output measured for feedback is also the performance variable. In many engineering applications, this assumption is not satisfied. Examples include the military vehicle in [4] and the acoustic duct in [5]. The research described in this paper is directly motivated by the experience of the authors with these and other applications.

Suppose that the performance output differs from the measured output. Then, as we shall see, the *sensitivity function*

$$S \triangleq (1 - G_{yu}K)^{-1}$$

describes only the response of the *measured* output to exogenous disturbances. This fact implies that the existing theory of design limitations, which is based on the sensitivity function, cannot be directly applied to study tradeoffs that must be satisfied by the performance output.[1] Furthermore, it may happen that a controller designed to minimize the response of the performance output to disturbances will possess poor stability robustness, as quantified by the proximity of the Nyquist plot to the critical point. In fact, there may exist tradeoffs between disturbance response and stability robustness that have no counterpart in those cases for which the performance output is measured for feedback. We study such problems by considering systems in the *general control configuration* depicted in Fig. 1, where the performance output is denoted by z, the measured output by y, the control input by u, and the exogenous input by w. We shall assume throughout the paper that w, z, y, and u are scalar; this assumption allows us to focus on essential concepts without introducing additional notation required to describe multivariable systems.

If we partition the system G as

$$G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}$$

Manuscript received May 28, 2002; revised December 22, 2002 and April 7, 2003. Recommended by Guest Editor J. Chen. This work was supported by the National Science Foundation under Grant ECS-9810242 and by the Centre for Integrated Dynamics and Control, a Commonwealth of Australia Special Research Centre.

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Digital Object Identifier 10.1109/TAC.2003.815017

[1]The authors thank Dr. V. Marcopoli of General Dynamics Land Systems, Sterling Heights, MI, for emphasizing this point and encouraging us to extend the theory.

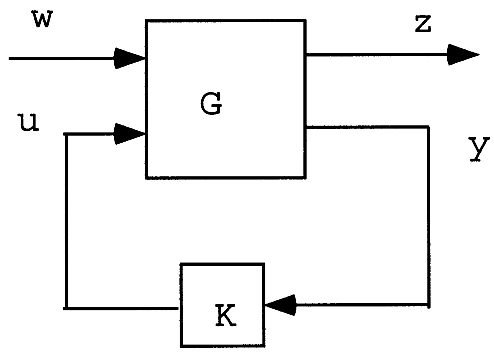

Fig. 1. General control configuration.

then the response of z to w is given by the transfer function

$$T_{zw} = G_{zw} + G_{zu}K(1 - G_{yu}K)^{-1}G_{yw}. \tag{1}$$

As described in [6] and [7], any linear control problem can be placed into the general control configuration, and various optimization procedures can be applied to minimize T_{zw}. Unless G_{yu} is identically zero, the system in Fig. 1 will contain a feedback loop and, thus, stability and stability robustness issues must be addressed. *For ease of reference we shall refer to T_{zw} as the* closed loop disturbance response, *even in those cases where w is an exogenous input other than a disturbance, and the feedback loop is not present*. In the case, that G_{zw} is not identically zero, it is useful to compare the closed-loop to the open-loop response using the *disturbance response ratio*

$$R_{zw} \triangleq T_{zw}/G_{zw}. \tag{2}$$

Our goal in this paper is to develop a theory of fundamental design limitations applicable to the general disturbance attenuation problem of Fig. 1 under the assumption that all signals are scalar. Following [8] and [9], these limitations will be classified as "algebraic" if they involve tradeoffs between system properties at the same frequency, or "analytic" if they involve tradeoffs between properties in different frequency ranges. We shall investigate whether the disturbance response can be made arbitrarily small, subject to the restriction that the controller is stabilizing.

Definition I.1: The *Ideal Disturbance Attenuation Problem* is solvable if, for each $\epsilon > 0$, there exists a stabilizing controller such that

$$|R_{zw}(j\omega)| < \epsilon \qquad \forall\omega.$$

A controller that achieves ideal disturbance attenuation may not be proper, and thus cannot be implemented. In this case, we ask whether it is possible to achieve arbitrarily small disturbance response over an arbitrarily wide frequency interval, and arbitrarily small disturbance amplification outside this interval.

Definition I.2: The *Proper Disturbance Attenuation Problem* is solvable if, for each $\epsilon > 0, \alpha > 0$, and $\omega_c > 0$, there exists a proper stabilizing controller such that

$$\begin{aligned} |R_{zw}(j\omega)| &< \epsilon \qquad \forall\omega < \omega_c \\ |R_{zw}(j\omega)| &< 1 + \alpha \qquad \forall\omega \geq \omega_c. \end{aligned}$$

Solutions to the previous problems are available for single loop feedback systems, and may be found in the seminal work on sensitivity minimization by Zames *et al.* [10]–[12].

The results of this paper are outlined as follows. In Section II, we define terminology and state a list of standing assumptions. In Section III, we define those feedback systems whose disturbance response is governed by the sensitivity function to be "reducible to a feedback loop". We characterize such systems in Section III-A, and note that the property of reducibility depends on the *control architecture*, by which we mean the location of the sensor relative to the performance output, and the actuator relative to the exogenous input. We next consider systems that are not reducible, and show in Sections III-B and III-C that they face an algebraic tradeoff between disturbance response and the feedback properties of stability robustness and sensitivity to parameter variations. *Although the existence of this tradeoff is easy to demonstrate, it does not appear to be widely known*. In Section III-D, we show that if the system reduces to a feedback loop, then the control law used to achieve disturbance attenuation will consist of high gain feedback. Disturbance attenuation for systems that do not reduce to a feedback loop is achieved using a finite gain controller that depends explicitly on the plant model. In Section IV we provide necessary and sufficient conditions for solvability of the ideal and proper disturbance attenuation problems stated in Definitions I.1 and I.2. We show in Sections IV-A and IV-B that T_{zw} must satisfy *interpolation constraints* at certain closed right-half plane zeros of G_{zu} and G_{yw}, and at certain closed right-half plane poles of G. These interpolation constraints imply that the value of T_{zw} is fixed, independently of any stabilizing controller, at these poles and zeros. We characterize those interpolation constraints that prevent the disturbance response from being made arbitrarily small, and thus show that a necessary condition for solvability of the ideal disturbance attenuation problem is that no such interpolation constraints exist. In Section IV-C, we show that this condition is also sufficient for the solvability of the ideal disturbance response problem. The controller that does so will generally be improper and thus, in Section IV-D, we present an approximation that solves the proper disturbance attenuation problem. In Section V, we translate the interpolation constraints derived in Sections IV-A and IV-B into integral relations that impose analytic design tradeoffs upon the disturbance response. We show in Section V-A that R_{zw} must satisfy a generalized version of the Bode sensitivity integral, and use this fact in Section V-B to demonstrate the existence of an *analytic* tradeoff between disturbance response and feedback properties. In Sections V-C and V-D we show that T_{zw} and R_{zw} must satisfy Poisson integrals for each pole and zero that is responsible for a problematic interpolation constraint. It is not generally possible to characterize the zeros of T_{zw} in the closed right-half plane, as they may depend on the compensator, and thus we present compensator-independent lower bounds on the various integrals. We illustrate the results of the paper in Section VI by applying them to the problem of active noise control in an acoustic duct.

Design limitations due to nonminimum phase zeros for systems whose performance output is not measurable were studied in [4], and applied to the problem of stabilizing the elevation axis of a military tank. A partial version of our Proposition IV.6

is found in [13], which also discusses the impact of sensor and actuator placement upon the existence of design tradeoffs. We carefully compare our results to those of [13] in Section VI-B.

Additional examples and details are found in the technical report [14], which is a longer version of this paper. Proofs of several results that are straightforward have been omitted from the present paper, and may also be found in [14].

II. PRELIMINARIES

Denote the open and closed left and right halves of the complex plane by OLHP, CLHP, ORHP, and CRHP, respectively. We shall assume that all transfer functions are rational with real coefficients. Such a transfer function is *stable* if all its poles lie in the OLHP. A rational transfer function f has *relative degree* equal to r if f has precisely r more poles than zeros, and we denote the relative degree of f by $\delta(f)$. A matrix of rational functions is said to be *proper* if each element of the matrix has relative degree at least zero, and *strictly proper* if each element has relative degree at least one. Two polynomials are *coprime* if they have no common zeros. Given a set of complex numbers $\{z_i : i = 1, \ldots, N_z\}$, where each $z_i \in \mathrm{ORHP}$ and may have multiplicity greater than one, we denote the complex conjugate of z_i by $\bar{z}_i$, and define [1] the Blaschke product $B_z(s) \triangleq \prod_{i=1}^{N_z}(z_i - s)/(\bar{z}_i + s)$. We denote a rational function f that is identically zero by $f \equiv 0$. A square transfer function matrix M is *nonsingular* or *invertible* if $\det M \not\equiv 0$, and *singular* otherwise. A stable rational function f has H^∞ norm $\|f\|_\infty \triangleq \sup_\omega |f(j\omega)|$.

A. Standing Assumptions

We invoke the following list of *standing assumptions* throughout the paper to simplify the exposition and to avoid trivial situations.

- The system G is stabilizable by feedback from y to u. See Section II-C for discussion of this obviously necessary hypothesis.
- The transfer functions G_{zu} and G_{yw} are not identically zero. Otherwise, $T_{zw} \equiv G_{zw}$, and no controller can influence the disturbance response.
- The signals z, w, y, and u are scalar valued. This assumption simplifies the derivation of interpolation constraints and integral relations.
- Whenever the disturbance response ratio R_{zw} (2) is discussed, we assume that $G_{zw} \not\equiv 0$.

B. Transmission Zeros

Consider a $p \times p$ transfer function matrix, P, and let (A, B, C, D) denote a minimal realization of P with degree equal to n. The *characteristic polynomial* of P is given by $\phi_P(s) = \det(sI - A)$, and the multiplicity of a given pole of P is equal to its multiplicity as a zero of ϕ_P. If P has full normal rank [7], then we say that ζ is a *transmission zero* of P if the rank of the system matrix [7] evaluated at ζ is less than $n + p$. Define the *zero polynomial* of P by $N_P \triangleq \phi_P \det P$. If P has less than full normal rank, then $N_P \equiv 0$. Otherwise, the transmission zeros of P are equal to the zeros of N_P. If P has at least one transmission zero in the ORHP then P is said to be *nonminimum phase* (NMP), and the zero is termed a NMP zero. Otherwise, P is said to be *minimum phase*.

C. Stabilizability and Stability

We define G to be *stabilizable* [7] if there exists a proper controller K that internally stabilizes[2] the system in Fig. 1. It follows from Lemma 12.1 of [7] that G is stabilizable if and only if all CRHP poles of G are poles, with the same multiplicity, of G_{yu}. Under the assumption of stabilizability [7], K internally stabilizes G if and only if K internally stabilizes G_{yu}. Recall the sensitivity function S, and define the *complementary sensitivity function* $T \triangleq 1 - S$. It may be shown [7, Lemma 5.3] that K internally stabilizes G_{yu} if and only if the four transfer functions S, T, SK, and $G_{yu}S$ are stable. When we say that the system in Fig. 1 is stable, we mean that these four transfer functions have no poles in the CRHP. The feedback system is *well-posed* if these four transfer functions are proper. Denote *coprime polynomial factorizations* of the individual transfer functions in G by $G_{\alpha\beta} = N_{\alpha\beta}/D_{\alpha\beta}, \alpha = z, y, \beta = w, u$. Suppose that we factor the controller as $K = N_K/D_K$, where N_K and D_K are coprime polynomials. Then the four transfer functions S, T, SK, and $G_{yu}S$ are stable if and only if the *closed-loop characteristic polynomial*

$$\phi_T \triangleq D_K D_{yu} - N_K N_{yu} \tag{3}$$

has no CRHP zeros. We shall adopt the following notation.

Definition II.1: Consider a complex scalar σ. If $G_{\alpha\beta} \not\equiv 0$, then let $m_{\alpha\beta}(\sigma)$ and $\gamma_{\alpha\beta}(\sigma)$ denote the multiplicity of σ as a zero and pole of $G_{\alpha\beta}$. If $K \not\equiv 0$, let $m_K(\sigma)$ and $\gamma_K(\sigma)$ denote the multiplicity of σ as a zero and pole of K. Let $\gamma_G(\sigma)$ denote the multiplicity of σ as a pole of G. If $\det G \not\equiv 0$, let $m_G(\sigma)$ denote the multiplicity of σ as a transmission zero of G.

III. DISTURBANCE ATTENUATION VERSUS FEEDBACK PROPERTIES

A useful measure of robustness in a feedback system is the *stability radius*, defined to be the minimum distance from the critical point -1 to the Nyquist plot of $-G_{yu}K$. The stability radius is equal to the reciprocal of the peak in the Bode sensitivity function, and thus any system for which S has a large peak will possess a poor stability margin. In this section, we show that there may exist a tradeoff between the disturbance response and the stability radius.

A. Systems Reducible to a Feedback Loop

The potential existence of a tradeoff between disturbance response and stability robustness depends on the *control architecture*.

Proposition III.1: Assume that $\det G \equiv 0$ and that $G_{yu} \not\equiv 0$. Then (1) and (2) reduce to $T_{zw} = G_{zw}S$ and $R_{zw} = S$.

We say that the system in Fig. 1 is "reducible to a feedback loop" if $\det G \equiv 0$. For such systems, there is no conflict

[2]By internal stability, we mean that for given stabilizable and detectable state space realizations of G and K, the associated state equations for the system of Fig. 1 have an "A" matrix with no eigenvalues in the CRHP.

between making both the disturbance response and the sensitivity function small, as they are governed by the same transfer function. Two important classes of systems have the sensor or actuator located so that they reduce to a feedback loop. First suppose that the performance output is measured for feedback. Then $G_{yw} = G_{zw}$ and $G_{zu} = G_{yu}$. Alternately, suppose that the control and disturbance actuate the system identically. Then, $G_{zw} = G_{zu}$ and $G_{yw} = G_{yu}$. In either case, $\det G \equiv 0$.

B. An Algebraic Tradeoff

The disturbance response of a system for which $\det G \not\equiv 0$ is no longer given by the sensitivity function, but by (1)–(2). Hence, making the disturbance response small is no longer equivalent to making the stability radius large. In fact, we now show that these goals may be mutually exclusive, in that there exists a tradeoff between the size of S and that of R_{zw}. The severity of this tradeoff is determined by the *dimensionless* quantity

$$\Gamma \triangleq \frac{G_{zu}G_{yw}}{G_{zw}G_{yu}}.$$

The following result is an immediate consequence of the important identity $R_{zw} + \Gamma T = 1$.

Proposition III.2: Consider the sensitivity function S associated with the feedback loop in Fig. 1, and the disturbance response ratio R_{zw} (2), defined whenever $G_{zw} \not\equiv 0$.

a) Given ω, in the limit as $R_{zw}(j\omega) \to 0$

$$S(j\omega) \to 1 - 1/\Gamma(j\omega), \qquad T(j\omega) \to 1/\Gamma(j\omega). \tag{4}$$

b) Given ω, in the limit as $S(j\omega) \to 0$, the disturbance response ratio satisfies

$$R_{zw}(j\omega) \to 1 - \Gamma(j\omega). \tag{5}$$

If $\Gamma \equiv 1$, then $\det G \equiv 0$ and Proposition III.1 implies there is no tradeoff between disturbance attenuation and feedback properties. Such a tradeoff does exist if $\det G \not\equiv 0$, and will be severe at any frequency for which $|\Gamma(j\omega)|$ is either very large or very small.

C. Differential Sensitivity

In order to compute sensitivity to uncertainty, we must distinguish between systems for which $\det G \equiv 0$ only at the nominal value of G, and those for which this property holds robustly.

Definition III.3: Suppose that the true value of G is uncertain, but known to lie in a set $G \in \mathcal{G}$. If $\det G \equiv 0, \forall G \in \mathcal{G}$, then we say that the system in Fig. 1 *is robustly reducible to a feedback loop*.

The architecture of the systems discussed at the close of Section III-A guarantees that each is robustly reducible to a feedback loop, and thus Proposition III.1 will hold despite uncertainty in the transfer functions G_{yu} and G_{zw}.

To study differential sensitivity, we decompose the disturbance response ratio (2) as $R_{zw} = 1 + H_{zw}$, where

$$H_{zw} \triangleq G_{zw}^{-1}G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \tag{6}$$

and compute the sensitivity of H_{zw} to plant and controller uncertainty. Our approach is thus directly analogous to that followed in standard textbooks [15], wherein the differential sensitivity of T with respect to plant and controller uncertainty is shown to be equal to S. Indeed, for systems that are robustly reducible to a feedback loop $H_{zw} = -T$.

Proposition III.4:

a) Assume that the system is robustly reducible to a feedback loop. Then the relative sensitivities of H_{zw} with to uncertainty in G_{yu} and K satisfy

$$\frac{G_{yu}}{H_{zw}}\frac{\partial H_{zw}}{\partial G_{yu}} = \frac{K}{H_{zw}}\frac{\partial H_{zw}}{\partial K} = S.$$

b) Assume that the system *is not* robustly reducible to a feedback loop. Then, the relative sensitivities of H_{zw} with respect to uncertainty in G_{yu} and K satisfy

$$\frac{G_{yu}}{H_{zw}}\frac{\partial H_{zw}}{\partial G_{yu}} = -T, \qquad \frac{K}{H_{zw}}\frac{\partial H_{zw}}{\partial K} = S.$$

For systems that satisfy robustly reduce to a feedback loop, sensitivity to uncertainty in both G_{yu} and K can be reduced by requiring the sensitivity function to be small. Otherwise, the identity $S + T = 1$ implies that the sensitivity to G_{yu} and the sensitivity to K cannot both be small at the same frequency.

D. Strategies for Disturbance Attenuation

A major difference between those systems that reduce to a feedback loop $(\det G \equiv 0)$, and those that do not, lies in the control strategy required to achieve disturbance attenuation. Suppose that $G_{yu}(j\omega) \neq 0$. It follows easily from (1) that

$$\lim_{|K(j\omega)|\to\infty} T_{zw}(j\omega) = \frac{\det G(j\omega)}{G_{yu}(j\omega)}$$

and thus high-gain feedback may be used to achieve disturbance attenuation only at frequencies for which $\det G(j\omega) = 0$. Suppose that $\det G \equiv 0$. Then Proposition III.1 shows that $R_{zw} = S$ and it follows from results in [10]–[12], [16], [17] that if G_{yu} has no CRHP zeros then disturbance attenuation may be achieved through high gain feedback.

A disadvantage of feedback control is that it introduces stability robustness issues. Alternately, suppose that the disturbance were directly measurable, so that $G_{yw} = 1$ and $G_{yu} = 0$. Then $T_{zw} = G_{zw} + G_{zu}K$ and, if G is stable and G_{zu} has a stable inverse, the ideal disturbance attenuation problem may be solved using feedforward control: $K = -G_{zu}^{-1}G_{zw}$. A disadvantage of this strategy is that it requires a perfect plant model. As we shall see, the solution to the disturbance attenuation problem for the general system shown in Fig. 1 suffers from the disadvantages of both feedforward and feedback control.

We now state conditions required for the existence of a controller that sets the closed loop disturbance response in Fig. 1 identically to zero, thus providing a solution to the ideal disturbance attenuation problem posed in Definition I.1.

Lemma III.5: Assume that: i) $G_{zu} \not\equiv 0$, ii) $G_{yw} \not\equiv 0$, and iii) $G_{zw}G_{yu} - G_{zu}G_{yw} \not\equiv 0$. Then, the controller

$$K^C \triangleq \frac{G_{zw}}{G_{zw}G_{yu} - G_{zu}G_{yw}} \tag{7}$$

yields $T_{zw} \equiv 0$. Furthermore, suppose that $G_{zw} \not\equiv 0$. Then, if any of the hypotheses i)–iii) is violated, it is impossible to find a finite gain controller that achieves $T_{zw} \equiv 0$.

Setting $K = K^C$ in Fig. 1 results in the uncompensated path from w to z being exactly cancelled by the path from w to z through the compensator, and hence we refer to (7) as a "cancellation controller." Such controllers have previously been discussed in the literature [13], [18]. Conditions under which K^C is stabilizing are given in Section IV-C, and a proper approximation to K^C is presented in Section IV-D.

The controller K^C depends explicitly on the plant model, and thus the resulting system will be sensitive to model uncertainty. Furthermore, unless $G_{yu} \equiv 0$, the compensated system will contain a feedback loop with attendant stability robustness issues. Hence, use of such a controller incurs the potential drawbacks of both feedback and feedforward control. Furthermore, the sensitivity and complementary functions resulting from K^C must satisfy

$$S^C \triangleq 1 - 1/\Gamma, \qquad T^C \triangleq 1/\Gamma. \tag{8}$$

It follows that the cancellation controller, which nominally solves the ideal disturbance attenuation problem, will possess both poor stability robustness and poor differential sensitivity at frequencies for which the ratio Γ is small. Note finally that if $T_{zw} \to 0$, then necessarily $K(j\omega) \to K^C(j\omega)$. Hence any control design that forces $T_{zw}(j\omega)$ to be small over some frequency range will require a controller that approximates $K^C(j\omega)$ at these frequencies, and result in sensitivity and complementary sensitivity functions that approximate (8).

IV. Arbitrarily Small Disturbance Response

The requirement of internal stability implies that T_{zw} and R_{zw} must satisfy *interpolation constraints* at certain points of the CRHP. By an interpolation constraint, we mean that the values of T_{zw} and R_{zw} are fixed independently of the choice of stabilizing controller. The points at which interpolation constraints must be satisfied are located at a subset of the CRHP zeros of G_{zu} and G_{yw} and a subset of the CRHP poles of G.

A. CRHP Zeros of G_{zu} and G_{yw}

Suppose that ζ is a CRHP zero of G_{zu} or G_{yw} that is not a pole of G. We shall state conditions under which the presence of ζ prevents T_{zw} from being made arbitrarily close to zero.

Proposition IV.1: Suppose that the system in Fig. 1 is stable. Let ζ be a CRHP zero of G_{zu} or G_{yw}, and assume that ζ is not a pole of G.

a) Under these conditions $T_{zw}(\zeta) = G_{zw}(\zeta)$. It follows that $T_{zw}(\zeta) = 0$ if and only if $G_{zw}(\zeta) = 0$.

b) Assume in addition that $G_{zw} \not\equiv 0$ and that the multiplicities of ζ as a zero of G_{zw}, G_{zu}, and G_{yw} satisfy the bound

$$m_{zw}(\zeta) < m_{zu}(\zeta) + m_{yw}(\zeta). \tag{9}$$

Then, we may factor $T_{zw}(s) = T^1_{zw}(s)(s-\zeta)^{m_{zw}}$, and $G_{zw}(s) = G^1_{zw}(s)(s-\zeta)^{m_{zw}}$, where T^1_{zw} and G^1_{zw} have no poles at ζ, and $T^1_{zw}(\zeta) = G^1_{zw}(\zeta) \neq 0$.

c) Assume that inequality (9) holds. Then the disturbance response ratio satisfies $\lim_{s\to\zeta} R_{zw}(s) = 1$.

The interpolation constraint at a NMP zero may be used to obtain a nonzero lower bound on the achievable level of disturbance attenuation.

Corollary IV.2: Assume that ζ is a NMP zero that satisfies the hypotheses of Proposition IV.1 (b). Factor $G_{zw} = G^0_{zw}B_\zeta$, where B_ζ is a Blaschke product with $m_{zw}(\zeta)$ zeros at ζ and, if ζ is complex, at its complex conjugate. Then $\|T_{zw}\|_\infty \geq |G^0_{zw}(\zeta)| > 0$.

It is well known that a CRHP zero of G_{yu} constrains the sensitivity function [1]. Corollary IV.2 shows that such a zero constrains the disturbance response only if Proposition III.1 is applicable, so that the system is reducible to a feedback loop. We shall illustrate this point with the acoustic duct example in Section VI.

B. Unstable Poles of G

The requirement that the system be stabilizable implies that interpolation constraints due to unstable controller poles are more complicated to analyze than are those due to CRHP zeros. Thus, we begin our analysis by considering the simpler case of an unstable pole of the controller.

Proposition IV.3: Suppose that the system in Fig. 1 is stable. Let p be a CRHP pole of K that is not a pole of G. Then

$$T_{zw}(p) = \det G(p)/G_{yu}(p)$$

and thus $T_{zw}(p) = 0$ if and only if $\det G(p) = 0$.

Next, we consider a CRHP pole, p, of G in the special case that Proposition III.1 is applicable. If $\gamma_G(p) = \gamma_{zw}(p)$, then it follows immediately that the value of $T_{zw}(p)$ depends on the controller. If $\gamma_G(p) > \gamma_{zw}(p)$, then $T_{zw}(p) = 0$.

We now state the general result, which is applicable to an unstable pole of G that may also be a zero of G_{zu} or G_{yw}.

Proposition IV.4: Assume that the system in Fig. 1 is stable, and that $\det G \not\equiv 0$. Let p denote a CRHP pole of G with multiplicity $\gamma_G(p) \geq 1$.

a) Suppose that $\gamma_G(p) = \gamma_{zw}(p)$. Then, the value of $T_{zw}(p)$ depends on the controller K.

b) Suppose that $\gamma_G(p) > \gamma_{zw}(p)$. Then

$$T_{zw}(p) = \lim_{s\to p} \frac{\det G(s)}{G_{yu}(s)}. \tag{10}$$

Furthermore, $T_{zw}(p) = 0$ if and only if p is a transmission zero of G.

c) Suppose that $\gamma_G(p) > \gamma_{zw}(p)$ and that p is a transmission zero of G with multiplicity $m_G(p) > 0$. Assume that either $G_{zw} \equiv 0$, or that

$$m_G(p) < m_{zw}(p) + \gamma_G(p) - \gamma_{zw}(p). \tag{11}$$

Then we may factor $T_{zw}(s) = T^1_{zw}(s)(s-p)^{m_G(p)}$, where T^1_{zw} has no pole at $s = p$, and $T^1_{zw}(p) = \lim_{s\to p} \det G(s)/G_{yu}(s)(s-p)^{m_G(p)} \neq 0$.

d) Suppose that p is a CRHP pole of G. Then $R_{zw}(p) = \lim_{s\to p} \det G(s)/G_{yu}(s)G_{zw}(s)$. If p is also either: i) a transmission zero of G with $m_G(p) > m_{zw}(p)$, or ii) a pole of G_{zw}, then $R_{zw}(p) = 0$.

Proof: See Appendix A. ■

The interpolation constraint at an ORHP pole may be used to obtain a nonzero lower bound on the achievable level of disturbance attenuation. The following is a counterpart to Corollary IV.2.

Corollary IV.5: Assume that p is an ORHP pole of G that satisfies the hypotheses of Proposition IV.4 (c). Let B_p denote a Blaschke product with $m_G(p)$ zeros at p and, if p is complex, at its complex conjugate. Then $\|T_{zw}\|_\infty \geq \lim_{s\to p}\left|\det G(s)B_p^{-1}(s)/G_{yu}(s)\right|$, where the limit is finite and nonzero.

C. Ideal Disturbance Attenuation

The results of Sections IV-A and IV-B yield *necessary* conditions for the solvability of the ideal disturbance attenuation problem: G_{zu} and G_{yw} can have no CRHP zeros that satisfy the inequality (9), and G can have no CRHP poles that satisfy the inequality (11). If either of these conditions is violated, then Corollary IV.2 or IV.5 shows that there is a nonzero lower bound on achievable disturbance attenuation. We now show that these conditions are also *sufficient* to guarantee solvability of the ideal disturbance attenuation problem. To do so, we show that in the absence of such zeros and poles the cancellation controller (7) stabilizes the system.

The expression for K^C given by (7) may contain CRHP pole zero cancellations that must be removed before assessing closed-loop stability. Hence, we factor

$$K^C = \bar{N}_K^C / \bar{D}_K^C \tag{12}$$

where $\bar{N}_K^C$ and $\bar{D}_K^C$ have no common CRHP zeros. Denote the resulting closed-loop characteristic polynomial (3) by

$$\phi_T = \bar{D}_K^C D_{yu} - \bar{N}_K^C N_{yu}. \tag{13}$$

Proposition IV.6: Assume that $\det G \not\equiv 0$, that a) each CRHP zero ζ of G_{zu} or G_{yw} that is not a pole of G satisfies the bound

$$m_{zw}(\zeta) \geq m_{zu}(\zeta) + m_{yw}(\zeta) \tag{14}$$

and that b) each CRHP pole p of G satisfies the bound

$$m_G(p) \geq m_{zw}(p) + \gamma_G(p) - \gamma_{zw}(p). \tag{15}$$

Then the closed-loop characteristic polynomial (13) has no zeros in the CRHP and (12) is stabilizing.

Proof: See Appendix B. ∎

It follows immediately that if the hypotheses of Proposition IV.6 are satisfied, then the ideal disturbance attenuation problem is solvable. The controller (12) both stabilizes the system and sets the closed-loop disturbance response identically equal to zero.

D. Proper Disturbance Attenuation

In general, K^C need not be proper, and the resulting feedback system need not be well posed. In such cases, the controller cannot be implemented. We now show how to find a controller that solves the proper disturbance attenuation problem described in Definition I.2. In fact, our procedure will guarantee that K^C is strictly proper. Note that if G_{yu} is proper, then a sufficient condition for K^C to be strictly proper is that $G_{zw}/G_{zu}G_{yw}$ is strictly proper.

Proposition IV.7: Assume that the hypotheses of Proposition IV.6 are satisfied, that G_{yu} is proper, and that K^C is not strictly proper. For given values of $\epsilon > 0, \alpha > 0$, and $\omega_c > 0$, choose a stable rational function B such that

a) $G_{zw}B/G_{zu}G_{yw}$ is strictly proper;
b) $|1 - B(j\omega)| < \epsilon, \forall\omega < \omega_c$;
c) $|1 - B(j\omega)| < 1 + \alpha, \forall\omega \geq \omega_c$;
d) $1 - B$ has CRHP zeros precisely at the CRHP poles of G, including multiplicities.

Let K_p^C denote the cancellation controller obtained by replacing G_{zw} with $G_{zw}B$:

$$K_p^C = \frac{G_{zw}B}{G_{zw}BG_{yu} - G_{zu}G_{yw}}. \tag{16}$$

Then K_p^C is strictly proper and stabilizing, $T_{zw} = G_{zw}(1-B)$, and $R_{zw} = 1 - B$.

Proof: See Appendix C. ∎

To construct the function B required for Proposition IV.7, let P be a rational function such that P has no CRHP zeros, P has CRHP poles precisely at the CRHP poles of G, including multiplicities, and $G_{zw}P/G_{zu}G_{yw}$ is strictly proper. Let us view P as a plant to be stabilized with a controller C, and denote the resulting sensitivity function by $S_{PC} \triangleq 1/(1 - PC)$. Then, [10, Th. 2] may be used to show that, for any specified ϵ, α, and ω_c, there exists a C that is stable, proper, and stabilizing, and that yields $|S_{PC}(j\omega)| < \epsilon, \forall\omega < \omega_c$, and $|S_{PC}(j\omega)| < 1+\alpha, \forall\omega \geq \omega_c$. Furthermore, the fact that C is stabilizing implies that C has no zeros at the CRHP poles of P and, thus, S_{PC} has zeros at these poles. Finally, since C is proper the relative degree of PC will be at least that of P. It follows that $B \triangleq 1 - S_{PC}$ satisfies requirements a)–d) of Proposition IV.7.

Example IV.8: Let

$$G(s) = \begin{bmatrix} \frac{0.5(s+1)}{s-1} & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s-1} \end{bmatrix}.$$

It is easy to see that $m_G(1) = 0, m_{zw}(1) = 0, \gamma_G(1) = 1$, and $\gamma_{zw}(1) = 1$ and, thus, (15) is satisfied. The cancellation controller that solves the ideal disturbance attenuation problem is improper and the resulting feedback system is not well posed: $K^C(s) = s + 1$ and $S = 0.5(1 - s)$. It is easy to verify that simply adding filtering to K^C does not result in closed loop stability: $K_p^C = (s+1)/(\tau s + 1)(\sigma s + 1)$ fails to stabilize for any values of $\tau > 0$ and $\sigma > 0$.

We now apply the procedure outlined following Proposition IV.7. Choose $P(s) = 1/(s-1)(s+1)^2$, and $C(s) = -k(s+1)^2/(\tau s + 1)^2$, where $k > 0, \tau > 0$. Then it is not hard to show that $B = -PC/(1 - PC)$ satisfies properties a)–d) of Proposition IV.7 for sufficiently large values of k and $1/\tau$. It is interesting to note that the resulting controller (16) is itself unstable.

V. Integral Relations

We now state several integral relations that describe design tradeoffs between disturbance response properties in different frequency ranges. To do so requires some technical lemmas concerning the CRHP zeros of T_{zw}.

Definition V.1: Denote the set of all NMP zeros of T_{zw} by $\{\xi_i\colon i = 1,\ldots,N_\xi\}$, and separate these into a set $\{\beta_i\colon i = 1,\ldots,N_\beta\}$ of zeros that are shared with G_{zw} and a set of additional zeros $\{\gamma_i\colon i = 1,\ldots,N_\gamma\}$. Then, we may factor $T_{zw} = \tilde{T}_{zw}B_\xi = \tilde{T}_{zw}B_\beta B_\gamma$, where B_ξ, B_β, and B_γ are Blaschke products. Denote the set of NMP zeros of G_{zw} that are not shared with T_{zw} by $\{\alpha_i\colon i=1,\ldots,N_\alpha\}$, and the set of all ORHP poles of G_{zw} by $\{\rho_i\colon i=1,\ldots,N_\rho\}$. Then G_{zw} may be factored as $G_{zw}=\tilde{G}_{zw}B_\alpha B_\beta B_\rho^{-1}$, where B_α and B_ρ are Blaschke products.

Except in special cases, such as those noted in Lemma V.2 below, it is not possible to characterize the CRHP zeros of T_{zw}, and their existence and location must be determined after the compensator is designed (cf. [4, Ex. 2.3]).

Lemma V.2:

i) Suppose that $\det G\equiv 0$. Then, the CRHP zeros of T_{zw} consist of the union of the CRHP zeros of G_{zw}, the CRHP poles of G_{yu} that are not shared with G_{zw}, and the CRHP poles of K.

ii) Suppose that the controller is given by K_p^C (16). Then the CRHP zeros of T_{zw} consist of the union of the CRHP zeros of G_{zw}, and the CRHP poles of G_{yu} that are not shared with G_{zw}.

In those cases for which it is not possible to characterize the zeros of T_{zw}, we will state compensator-independent lower bounds on the various integral relations. To do so requires, as in Definition V.1, that we separate the NMP zeros of G_{zw} into those that are shared with T_{zw} and those that are not.

Lemma V.3: Assume that the closed loop system is stable, let η be a NMP zero of G_{zw} with multiplicity $m_{zw}(\eta)$, and define $m_1(\eta)$ to be the multiplicity of η as a zero of $G_{zu}KSG_{yw}$.

i) Suppose that η is not a pole of G. Then

$$m_1(\eta)=m_{zu}(\eta)+m_{yw}(\eta)+m_K(\eta).$$

ii) Suppose that η is a pole of G. Then

$$m_1(\eta)=m_{zu}(\eta)-\gamma_{zu}(\eta)+m_{yw}(\eta)-\gamma_{yw}(\eta)+\gamma_{yu}(\eta).$$

In either case, if $m_{zw}(\eta)>m_1(\eta)$, then $\eta\in\{\beta_i\}$, with multiplicity $m_1(\eta)$, and $\eta\in\{\alpha_i\}$, with multiplicity $m_{zw}(\eta)-m_1(\eta)$. If $m_{zw}(\eta)\le m_1(\eta)$, then $\eta\in\{\beta_i\}$, with multiplicity $m_{zw}(\eta)$.

A. Generalized Bode Sensitivity Integral

We now show that the disturbance response ratio R_{zw} must satisfy an integral constraint analogous to the Bode sensitivity integral [1]. Write the disturbance response ratio (2) as $R_{zw}=1+H_{zw}$, where H_{zw} is given by (6).

Proposition V.4: Suppose that the system in Fig. 1 is stable, and assume that the relative degree of H_{zw} satisfies $\delta(H_{zw})>1$. Then

$$\int_0^\infty \log|R_{zw}(j\omega)|\,d\omega=\pi\sum_{i=1}^{N_\gamma}\mathrm{Re}\,\gamma_i+\pi\sum_{i=1}^{N_\rho}\mathrm{Re}\,\rho_i-\pi\sum_{i=1}^{N_\alpha}\mathrm{Re}\,\alpha_i. \quad (17)$$

Suppose first that G_{zw} has no ORHP poles and that T_{zw} and G_{zw} have no NMP zeros. Then, (17) evaluates to zero and the area of disturbance attenuation ($|R_{zw}(j\omega)|<1$) must necessarily be balanced by an equal area of disturbance amplification ($|R_{zw}(j\omega)|>1$). This tradeoff is precisely the same as that described by the usual Bode sensitivity integral in the case that the plant and controller have no ORHP poles [1], [3]. Indeed, for systems that are reducible to a feedback loop, Lemma V.2 1) may be used to derive the following corollary to Proposition V.4, which shows that the integral (17) reduces to the Bode sensitivity integral.

Corollary V.5: Assume that $\det G\equiv 0$. Then, $\{\alpha_i\}=\emptyset$, $\{\beta_i\}$ consists of all NMP zeros of G_{zw}, and $\{\gamma_i\}$ consists of those ORHP poles of G_{yu} that are not shared with G_{zw}, plus the ORHP poles of K. Furthermore, $\delta(H_{zw})>1$ if and only if $\delta(G_{yu}K)>1$.

By Lemma V.3, it is possible to characterize those NMP zeros of T_{zw} that are shared with G_{zw}, and thus to determine the set $\{\alpha_i\}$. The following bound, which is a corollary to Proposition V.4, imposes a waterbed tradeoff upon the closed-loop disturbance response that will only be worsened by the presence of additional NMP zeros of T_{zw}:

$$\int_0^\infty \log|R_{zw}(j\omega)|\,d\omega\ge\pi\sum_{i=1}^{N_\rho}\mathrm{Re}\,\rho_i-\pi\sum_{i=1}^{N_\alpha}\mathrm{Re}\,\alpha_i. \quad (18)$$

Without additional information, it is a mistake to suppose that the NMP zeros of G_{zw} that are not shared with T_{zw} will significantly lessen the design tradeoff imposed by the Bode integral. The fact that $\lim_{K\to 0}T_{zw}=G_{zw}$ implies that if G_{zw} has a NMP zero outside the control bandwidth, then T_{zw} will tend to have a nearby NMP zero. Hence, the contributions of these zeros to the first and third terms on the right-hand side of (17) will approximately cancel.

B. Analytic Tradeoff Between Disturbance Response and Stability Robustness

We now use the generalized Bode sensitivity integral (17) to show that an analytic tradeoff also exists between disturbance response and feedback properties. The requirement of stability robustness against unmodeled high frequency dynamics and the need to limit the size of the control signal will require that the complementary sensitivity function must satisfy a bandwidth constraint of the form $|T(j\omega)|<M_T(\omega)$, where $M_T(\omega)\to 0$ at high frequencies. It is bandwidth constraints of this sort that preclude solvability of the proper disturbance attenuation problem for single loop feedback systems (cf. [1, Sec. 3.1.3]). Our next result shows that a similar limitation applies to systems that do not reduce to a feedback loop.

Proposition V.6: Assume that the hypotheses of Proposition V.4 are satisfied, and that

$$\begin{aligned}|R_{zw}(j\omega)|&\le M_R(\omega) &&\forall\omega\le\omega_0\\ |T(j\omega)|&\le M_T(\omega) &&\forall\omega\ge\omega_1>\omega_0.\end{aligned}$$

Then, necessarily

$$\begin{aligned}\sup_{\omega\in(\omega_0,\omega_1)}&(\omega_1-\omega_0)\log|R_{zw}(j\omega)|\ge\\ &-\int_0^{\omega_0}\log M_R(\omega)\,d\omega\\ &-\int_{\omega_1}^\infty\log(1+|\Gamma(j\omega)|M_T(\omega))\,d\omega\\ &+\pi\sum_{i=1}^{N_\gamma}\mathrm{Re}\,\gamma_i+\pi\sum_{i=1}^{N_\rho}\mathrm{Re}\,\rho_i-\pi\sum_{i=1}^{N_\alpha}\mathrm{Re}\,\alpha_i. \quad (19)\end{aligned}$$

The analytic tradeoff implied by Proposition V.6 states that requiring low frequency disturbance attenuation together with a high frequency bandwidth constraint implies that a peak in disturbance response will exist at intermediate frequencies. If the system reduces to a feedback loop, then this peak also corresponds to a small stability margin.

C. Poisson Integral for NMP Zeros of G_{zu} and G_{yw}

The interpolation constraints due to CRHP zeros of G_{zu} and G_{yw} that were derived in Section IV-A will now be used to state Poisson integral relations that must be satisfied by T_{zw} and R_{zw}. The Poisson integral for T_{zw} was used in [4] to analyze the problem of elevation control for a military tank. It was shown that the problems of command tracking, pitch disturbance attenuation, and heave disturbance attenuation face different design limitations due to the presence or absence of NMP zeros in different elements of G.

Proposition V.7: Assume that the system in Fig. 1 is stable. Let $\zeta = x + jy$ denote a NMP zero of G_{zu} or G_{yw} that is not also a pole of G.

a) Assume that $G_{zw}(\zeta) \neq 0$. Then

$$\int_0^\infty \log|T_{zw}(j\omega)| W(\zeta,\omega)\, d\omega = \pi \log\left|G_{zw}(\zeta) B_\xi^{-1}(\zeta)\right|$$

and

$$\int_0^\infty \log|R_{zw}(j\omega)| W(\zeta,\omega)\, d\omega = \pi \log\left|B_\alpha(\zeta) B_\gamma^{-1}(\zeta) B_\rho^{-1}(\zeta)\right|$$

where $W(\zeta,\omega) \triangleq x/(x^2+(y-\omega)^2)+x/(x^2+(y+\omega)^2)$.

b) Assume that $G_{zw} \not\equiv 0$, that $G_{zw}(\zeta) = 0$, and that the inequality (9) is satisfied. Then the integrals in (a) hold, where the limit

$$G_{zw}(\zeta) B_\xi^{-1}(\zeta) \triangleq \lim_{s\to\zeta} G_{zw}(s) B_\xi^{-1}(s)$$

is finite and nonzero.

Proof: See Appendix D. ∎

The values of the Poisson integrals in Proposition V.7 depend upon the ORHP zeros of T_{zw}. In general, it is not possible to characterize these zeros because their existence and location depend upon the compensator. Nevertheless, it is possible to state *lower bounds* on the Poisson integral that may be evaluated without knowing all the NMP zeros of T_{zw}.

Corollary V.8: Assume that the hypotheses of Proposition V.7 are satisfied. Then

$$\int_0^\infty \log|T_{zw}(j\omega)| W(\zeta,\omega)\, d\omega \geq \pi \log\left|\tilde{G}_{zw}(\zeta) B_\alpha(\zeta) B_\rho^{-1}(\zeta)\right|$$

$$\int_0^\infty \log|R_{zw}(j\omega)| W(\zeta,\omega)\, d\omega \geq \pi \log\left|B_\alpha(\zeta) B_\rho^{-1}(\zeta)\right|$$

where $\tilde{G}_{zw}, B_\alpha$, and B_ρ are defined in Definition V.1.

To apply the bounds in Corollary V.8, one uses the characterization of the set $\{\alpha_i\}$ from Lemma V.3.

D. Poisson Integral for ORHP Poles of G

In Proposition IV.4 we saw that T_{zw} will satisfy nonzero interpolation constraints at certain ORHP poles of G. We now use these constraints to state Poisson integral relations that must be satisfied by T_{zw} and R_{zw}.

Proposition V.9: Assume that $\det G \not\equiv 0$ and that the system in Fig. 1 is stable. Let $p = x + jy$ denote an ORHP pole of G.

a) Assume that $\gamma_G(p) > \gamma_{zw}(p)$, and that p is not a transmission zero of G. Then

$$\int_0^\infty \log|T_{zw}(j\omega)| W(p,\omega)\, d\omega = \pi \log\left|T_{zw}(p) B_\xi^{-1}(p)\right|$$

where $T_{zw}(p)$ is the *nonzero compensator-independent* limit given by (10).

b) Assume that $\gamma_G(p) > \gamma_{zw}(p)$, that p is a transmission zero of G with multiplicity $m_G(p) > 0$, and that inequality (11) is satisfied. Then, the integrals in a) hold, where the limit

$$T_{zw}(p) B_\xi^{-1}(p) \triangleq \lim_{s\to p} \frac{\det G(s) B_\xi^{-1}(s)}{G_{yu}(s)}$$

is finite and nonzero.

c) Assume that i) if p is a transmission zero of G, then its multiplicity as a zero of G is strictly greater than its multiplicity as a zero of G_{zw}, and that ii) p is not a pole of G_{zw}. Then

$$\int_0^\infty \log|R_{zw}(j\omega)| W(p,\omega)\, d\omega = \pi \log\left|R_{zw}(p) B_\alpha(p) B_\gamma^{-1}(p) B_\rho^{-1}(p)\right|$$

where $R_{zw}(p)$ is the *nonzero compensator-independent* limit given Proposition IV.4 d).

The bounds in Proposition V.9 depend on all the zeros of T_{zw}. To obtain integral inequalities analogous to those in Corollary V.8, one may replace B_ξ with B_β in a) and b), and remove the term due to B_γ in c).

E. Feedback Properties With a Cancellation Controller

It is well known that S and T must satisfy interpolation constraints at the CRHP zeros and poles of G_{yu} and K [1], [3]. We now characterize the CRHP zeros and poles of the cancellation controller K^C (7). A weaker version of the next result, applicable to stable systems, is found in [18].

Proposition V.10: Assume that $G_{zw} \not\equiv 0$ and that K^C is stabilizing.

a) The cancellation controller K^C has a CRHP pole p if and only if G has a transmission zero p that satisfies the inequality

$$m_G(p) + \gamma_{zw}(p) > m_{zw}(p) + \gamma_G(p) \tag{20}$$

and $G_{yu}(p) \neq 0$.

b) The cancellation controller K^C has a CRHP zero z if and only if G_{zw} has a transmission zero z that satisfies the inequality

$$m_{zw}(z) + \gamma_G(z) > m_G(z) + \gamma_{zw}(z). \tag{21}$$

Proof: See Appendix E. ∎

Using Proposition V.10, we have the following catalog of interpolation constraints for the sensitivity and complementary sensitivity functions S^C and T^C (8) that result from use of a cancellation controller (7).

Proposition V.11: Assume that $G_{zw} \not\equiv 0$ and that the hypotheses of Proposition IV.6 are satisfied.

a) Suppose that p is either: i) a CRHP pole of G_{yu} or ii) a CRHP transmission zero of G that satisfies inequality (20). Then $S^C(p) = 0$ and $T^C(p) = 1$.
b) Suppose that z is either: i) a CRHP zero of G_{yu} or ii) a CRHP zero of G_{zw} that satisfies (21). Then, $S^C(z) = 1$ and $T^C(z) = 0$.

The interpolation constraints due to the CRHP poles and zeros of G_{yu} will be present for any stabilizing controller [1], [3]. Those due to the CRHP transmission zeros of G and the CRHP zeros of G_{zw} are present due to the cancellation controller.

The results of [1], [3], together with the interpolation constraints from Proposition V.11, yield Bode and Poisson integrals that must be satisfied by S^C and T^C. We refer to the following integral in Section VI.

Corollary V.12: Assume that the hypotheses of Proposition V.11 are satisfied. Let $\{p_i : i = 1, \ldots, N_p\}$ denote the union of the sets of ORHP poles of G_{yu} and the NMP transmission zeros of G that satisfy (20), and let B_p denote the associated Blaschke product. Then, if z is either a NMP zero of G_{yu} or a NMP zero of G_{zw} that satisfies inequality (21), the sensitivity function (8) must satisfy

$$\int_0^\infty \log\left|S^C(j\omega)\right| W(z,\omega)\, d\omega = \pi \log\left|B_p^{-1}(z)\right|. \qquad (22)$$

VI. Active Noise Control in an Acoustic Duct

We illustrate the theory developed in this paper by applying it to the problem of active noise control in an acoustic duct, cautioning the reader that our results are not intended to be a thorough study of such problems. In Section VI-A we show that the closed loop disturbance response must satisfy the generalized Bode sensitivity integral (17), and thus exhibits "waterbed effect" design tradeoffs. We also explain why the cancellation controller (7) violates the hypotheses of Proposition V.10 and, thus, does not stabilize the noise control system. We instead propose an approximation to (7) that is stabilizing. The resulting sensitivity function exhibits large peaks that are due to the limiting behavior (4) and the lightly damped zeros of the plant. In Section VI-B, we relate our conclusions to those found in [13] and [19], which also study design limitations for the active noise control problem.

A. Design Limitations for an Acoustic Duct

We consider a finite-dimensional model of the acoustic duct shown in Fig. 2. The design goal is to use the control speaker, u, and the measurement microphone, y, to attenuate the effect of the disturbance (or noise) speaker, w, upon the performance microphone, z. For the duct dynamics we assume a one-dimensional wave equation description, such as the one developed in [20], that is valid for small diameter-to-length ratios and open-ended terminations. We consider a 0.85 meter long duct with speaker-microphone pairs located 0.15 meters from the ends and model the speaker dynamics as in [20] with a 67 Hz low-frequency cutoff. To obtain a finite-dimensional approximation to these dynamics we truncate the modal expansion of this wave equation at its fifth modal frequency[3]. Symmetry implies that $G_{zw} = G_{yu}$ and $G_{zu} = G_{yw}$. These transfer functions possess identical poles, but the lightly damped zeros of G_{zw} and G_{yu} differ from those of G_{zu} and G_{yw}. In addition, the transfer functions G_{zw} and G_{yu} possess three nonminimum phase zeros. The model of the acoustic duct includes speaker dynamics that introduce two zeros at the origin into all four transfer functions, and all four transfer functions have relative degree equal to two.

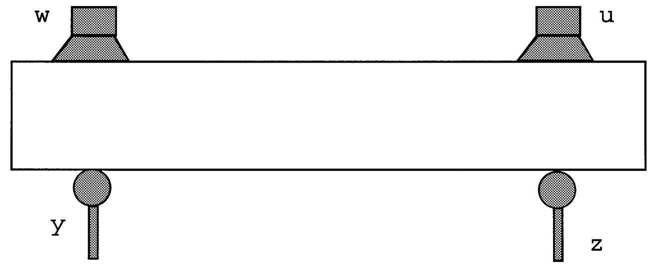

Fig. 2. Acoustic duct for active noise control.

Let us evaluate the tradeoff between disturbance attenuation and stability robustness described by Proposition III.2. As $T_{zw}(j\omega) \to 0, S(j\omega)$ and $T(j\omega)$ converge to the limits (4). These limits are plotted in Fig. 3, and reveal that S and T will have large peaks located at the dips in Γ that are due to the lightly damped zeros in G_{zu} and G_{yw} not shared with G_{zw} and G_{yu}. Any system that achieves disturbance attenuation in the vicinity of these dips will exhibit poor sensitivity and robustness.

It is important to note that *the NMP zeros of G_{yu} do not cause the large peaks in sensitivity that appear in* Fig. 3. Although Corollary V.12 shows that these zeros do prevent sensitivity from being made arbitrarily small, it is possible to obtain a sensitivity function with a smaller peak by using a controller that does not force the disturbance response to be small.

We now consider the proper disturbance attenuation problem, which is solvable if the cancellation controller is stabilizing. The zeros at the origin introduced by the speaker dynamics imply that $m_{zw}(0) < m_{zu}(0) + m_{yw}(0)$, and thus condition (14) of Proposition IV.6 is violated and the cancellation controller (7) does not stabilize the system. (It is easy to show that (7) will posses two integrators that cancel the two zeros at the origin of G_{yu}.) We thus select an approximation $\hat{K}^C$ to the cancellation controller that is stabilizing. To do so, we modify the duct model by shifting the zeros at the origin slightly into the OLHP (at $s = -10^{-4}$ Hz), and let $\hat{K}^C$ be the cancellation controller for the modified duct. We further modify $\hat{K}^C$ to obtain a strictly proper controller. Since G is stable, the construction of Proposition IV.7 is unnecessary, and we simply add filtering to obtain $\hat{K}_p^C = \hat{K}^C/(\tau s + 1)^n, n = 3$ and $\tau = 10^{-5}$ sec. Bode plots of (7) and of $\hat{K}_p^C$ show that the approximation is very good

[3]Details of our model are found in [14].

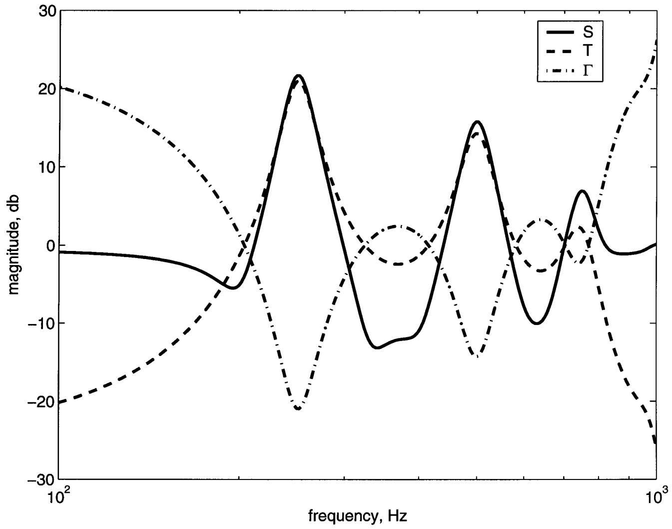

Fig. 3. Sensitivity and complementary sensitivity in the limit as $T_{zw}(j\omega) \rightarrow 0$.

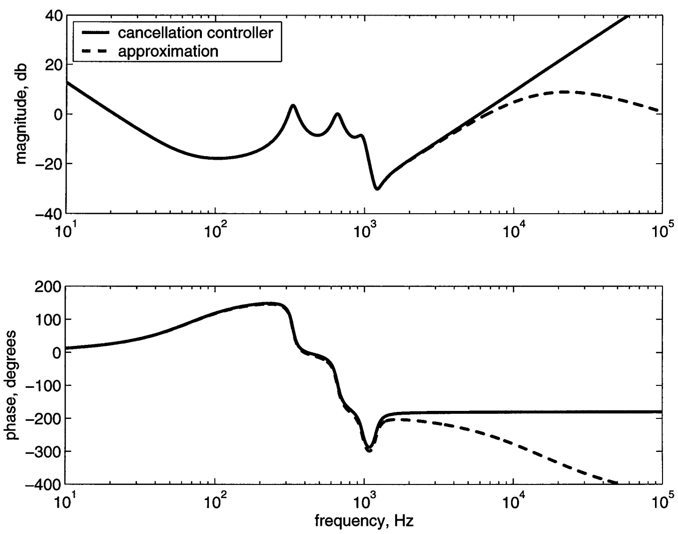

Fig. 4. Ideal cancellation controller (7) and the stabilizing and strictly proper approximation $\hat{K}_p^C$.

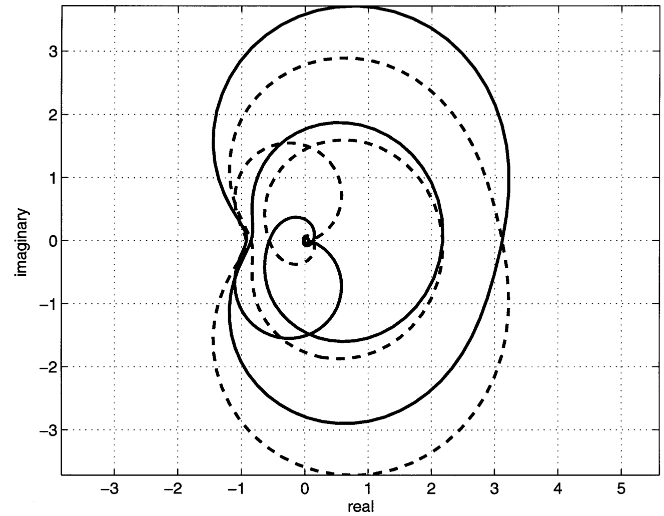

Fig. 5. Nyquist plot with the stabilizing and proper approximation $\hat{K}_p^C$ to the cancellation controller.

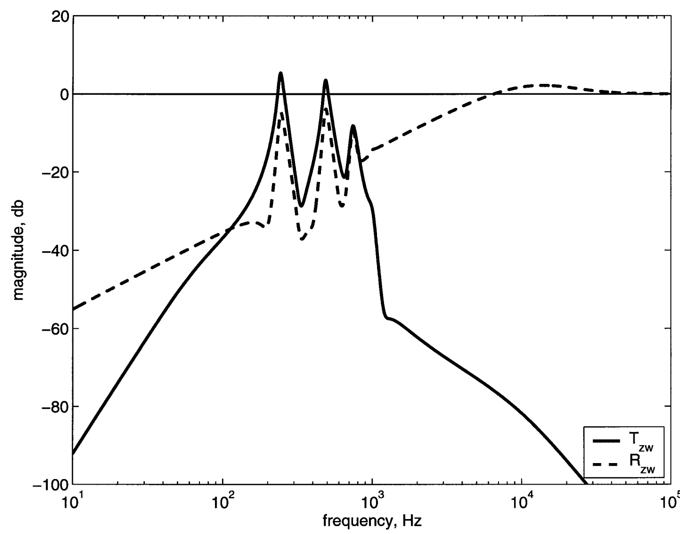

Fig. 6. Disturbance response with a stabilizing and proper approximation to the cancellation controller.

over a wide frequency range (Fig. 4). The Nyquist plot in Fig. 5 shows that the feedback system is nominally stable, albeit with poor stability margins. The resulting closed-loop disturbance response is plotted in Fig. 6. Note that T_{zw}, which should be identically zero with a cancellation controller, instead has peaks that exceed 0 db. This fact is consistent with the extreme sensitivity to the controller indicated by Proposition III.4 (b) and the peaks in S displayed in Fig. 3.

We close by discussing waterbed effect tradeoffs imposed on the disturbance response ratio. The plant G_{yu} has relative degree two, and thus with a proper controller the hypotheses of Proposition V.4 are satisfied and R_{zw} must satisfy the generalized Bode sensitivity integral (17) and the compensator independent lower bound (18). It follows immediately that the ideal disturbance attenuation problem is not solvable. Furthermore, with the strictly proper approximation $\hat{K}_p^C$ or (16), it follows from part (ii) of Lemma V.2 that the NMP zeros of G_{zw} will be shared with T_{zw} and, thus

$$\int_0^\infty \log |R_{zw}(j\omega)| \, d\omega \geq 0. \tag{23}$$

As predicted by (23), the plot of R_{zw} in Fig. 6 exhibits a peak exceeding 0 db. Although this peak is relatively small ($\approx$2 db), it occurs at a relatively high frequency ($\approx 10^4$ Hz). Realistic bandwidth limitations would require the controller gain to roll off at a lower frequency, resulting in a larger peak in R_{zw}, and precluding solvability of the proper disturbance attenuation problem. Additional insight into the severity of these tradeoffs may be obtained from bounds such as that in Proposition V.6.

B. *Discussion of Previous Work*

We now discuss a previous application of the theory of fundamental limitations to the problem of active noise control [13],

[19]. The results of these papers appear to be inconsistent with those of Section VI.A, and it is thus necessary to examine the reasons for the apparent discrepancy.

In [13] and [19], "spillover" is defined[4] to exist at any frequency for which $|R_{zw}(j\omega)| > 1$. Three different control architectures are considered. These include one for which the control speaker is collocated with the disturbance speaker ($u - w$ collocation), and one for which the measurement microphone and the performance microphone are collocated ($y - z$ collocation). The third geometry is depicted in Fig. 2, and referred to in [13] as the "standard feedforward setup." It is stated in [13] that "the geometric arrangement of speakers and microphones in the standard feedforward setup allows the control designer to effectively circumvent the inherent performance limitations of the Bode [sensitivity] integral". It is also stated in [13] that if both $u - w$ and $y - z$ collocation are avoided, as in Fig. 2, then "it is possible to overcome the Bode constraint in the sense that arbitrary attenuation of the open-loop transfer function can be achieved." It is these statements that appear to be inconsistent with our conclusion in Section VI-A.

The disturbance response ratio (2) of a system with either $u - w$ or $y - z$ collocation is shown in [13] to reduce to the sensitivity function, to satisfy the Bode sensitivity integral, and to exhibit spillover. These conclusions are consistent with ours, because such systems must satisfy $\det G \equiv 0$, and Corollary V.5 thus implies that (17) reduces to the usual Bode integral. In [13, Sec. III], it is noted that the area of disturbance amplification guaranteed to exist by the Bode integral can potentially be obtained by letting the sensitivity function exceed one by a very small amount spread over a very wide frequency range. It is then noted that "the ability to do this is subject to bandwidth and saturation limitations of the control actuator and electronics". In practice, bandwidth limitations would also be imposed by the need to avoid exciting higher frequency dynamics not included in the finite-dimensional plant model used for design.

Next discussed in [13] are systems, such as that in Fig. 2, which possess neither $u - w$ nor $y - z$ collocation. It is shown in [13] that the disturbance response of such a system can potentially be set equal to zero by using the cancellation controller (7), which is referred to in [13] as the "zero spillover controller" (ZSC). A procedure is also given for obtaining an "approximate zero spillover controller" (AZSC) that is strictly proper and, under appropriate hypotheses, stabilizing.

Although the AZSC can be made to approximate the ZSC arbitrarily closely, under the mild hypotheses of Proposition V.4 the disturbance response R_{zw} must satisfy the generalized Bode sensitivity integral (17) and the compensator independent lower bound (18). Hence, the ideal disturbance attenuation problem is not solvable and it is impossible to achieve "arbitrary attenuation" of the disturbance. Furthermore, it is easy to show that the AZSC will possess zeros at the zeros of G_{zw} [14] and thus that each NMP zero of G_{zw} must also be a zero of T_{zw}. It follows that the disturbance response ratio of a system with the AZSC must satisfy the integral inequality (23), and thus exhibit spillover as defined in [13]. This conclusion is inconsistent with

[4]Other definitions [21] state that spillover occurs when an actuator excites unmodeled plant dynamics, or when a sensor responds to such dynamics.

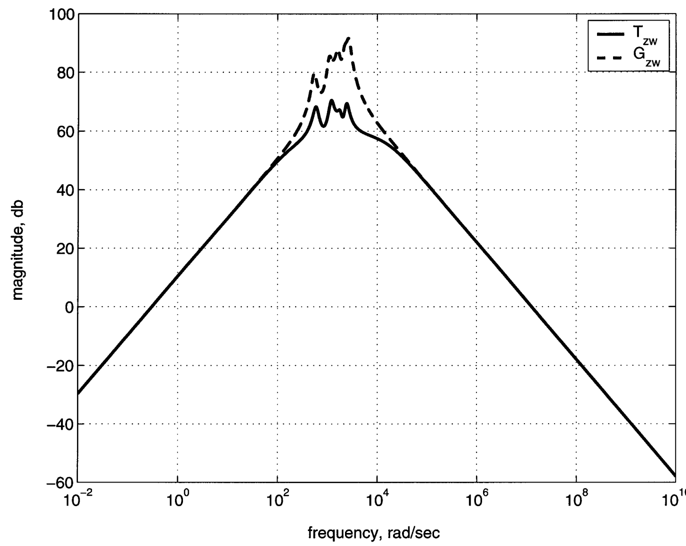

Fig. 7. Open- and closed-loop disturbance response for the example from [13].

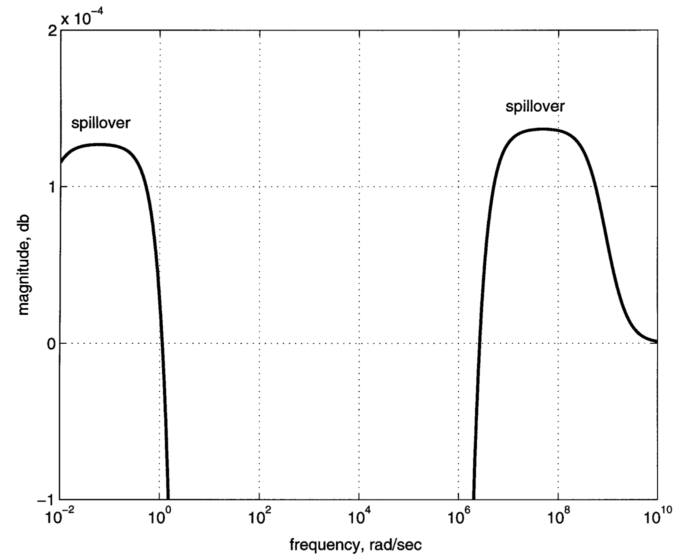

Fig. 8. Disturbance response ratio exhibits spillover [13] at low and high frequencies.

the statement in [19] that $y-z$ and/or $u-w$ collocation "causes spillover". Spillover will be present for all three control architectures considered in [13], provided only that reasonable bandwidth constraints are enforced. The amount of spillover will depend on the severity of the bandwidth constraint, which may indeed vary with the control architecture, and should be a matter of further investigation.

It is instructive to consider the duct model treated in [13], for which the AZSC (with parameter values $D_2 = 0.1$ and $E_2 = 2000$) is stabilizing. The Bode plots in Fig. 7 *appear* to show that the resulting closed loop disturbance response is never greater than that of the open loop. Closer inspection (Fig. 8) reveals the existence of two small peaks in R_{zw} that exceed unity, and which imply that spillover is present. In particular, R_{zw} has a peak of approximately 1.4×10^{-4} db at 5×10^7 rad/sec. This slight disturbance amplification is itself inconsequential; however, it does indicate control activity at a very high frequency.

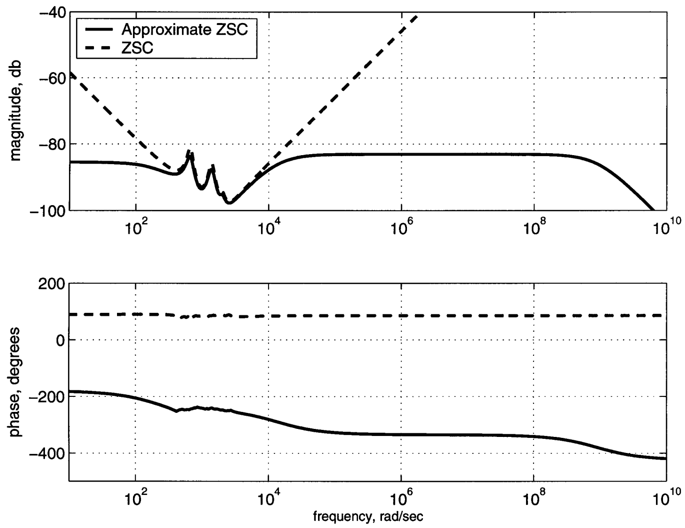

Fig. 9. Zero spillover and approximate zero spillover controllers.

Indeed, the Bode plots in Fig. 9 show that, although the AZSC is strictly proper, its gain does not begin to decrease until $\approx 10^9$ rad/s. Hence, we see that the design tradeoff imposed by the Bode sensitivity integral is accomplished by allowing R_{zw} to exceed one by a very small amount spread over a very wide frequency interval. This high frequency control activity may lead to robustness difficulties due to parameter uncertainty and unmodeled dynamics. In addition, as noted in the quote from [13] cited above, the ability to implement such a controller will be subject to actuator bandwidth and saturation limits.

To summarize, all the speaker/microphone configurations considered in [13] must satisfy the design limitations imposed by the Bode sensitivity integral, and will thus exhibit spillover as defined in [13]. In addition, the speaker and microphone configuration depicted in Fig. 2 will display the tradeoff between disturbance attenuation and feedback properties described by Proposition III.2. It is noted in [13] that the "poor form of the sensitivity" function resulting from use of the zero spillover controller is consistent with the fact that G_{yu} is nonminimum phase. Although the latter statement is correct, it misses the point made in Section VI-A, that the shape of the sensitivity function with the cancellation controller is determined by the limit (4), independently of whether or not G_{yu} has NMP zeros.

We close with a comparison of our Proposition IV.6 with [13, Prop. 4.1]. The latter presents a sufficient condition for the cancellation controller (or ZSC) to be stabilizing, and may be restated[5] as: "Assume that G has no CRHP poles, that G_{zu} and G_{yw} have no CRHP zeros, and that $\det G \not\equiv 0$. Then, the controller (7) results in the sensitivity function having no CRHP poles." Although internal stability is not explicitly considered in [13], it is straightforward to show that the hypotheses of [13] are sufficient to guarantee internal stability. Specifically, use of the cancellation controller (7) will result in the the closed-loop characteristic polynomial $\phi_T = -N_{zu}N_{yw}D_{zw}D_{yu}^2$, and it follows that these hypotheses guarantee internal stability. We note that the noise control example in [13] does not satisfy these hypotheses because G_{zu} and G_{yw} each possess a zero at $s = 0$, thus violating the condition that these transfer functions have no CRHP zeros. Although the sensitivity function resulting from the ZSC controller has no poles in the CRHP, it is easy to see that this controller will contain an integrator that cancels the zero of G_{yu} at the origin. As a result, the closed-loop transfer function KS will have a pole at $s = 0$, and is thus unstable.

[5]It may be inferred from their Proposition 4.1 that the term "nonminimum phase zero" is used in [13] to refer to a zero in the CRHP. In the present paper we use this term to refer to a zero in the ORHP.

VII. Conclusion

In this paper, we have developed a theory of fundamental design limitations for systems in the general feedback configuration of Fig. 1 under the assumption that all signals are scalar. We have shown that the nature of these limitations depends on the architecture of the control system. For those systems whose disturbance response is not described by the sensitivity function, there exists a potential tradeoff between disturbance response and feedback properties that tends to be severe for systems with lightly damped poles and zeros. We also derived interpolation constraints and integral relations that must be satisfied by the closed-loop disturbance response. The latter generalize the Bode and Poisson sensitivity integrals. We have used the problem of active noise control in an acoustic duct to illustrate the concepts of this paper. Additional work is required to determine the best choice of control architecture for a specific design problem, and our results should prove useful in assessing the limitations associated with a particular architecture. Theoretical research is needed to remove the assumption that the signals are scalar valued.

Appendix A
Proof of Proposition IV.4

Let $K = N_K/D_K$ denote a coprime polynomial factorization of K. Substituting this factorization and coprime factorizations for the $G_{\alpha\beta}$ into (1) yields

$$T_{zw} = \frac{(N_{zw}D_K D_{yu}\phi_G - N_K N_G D_{zw} D_{yu})}{\phi_T D_{zw}\phi_G} \tag{24}$$

where N_G and ϕ_T are the zero polynomial of G and the closed loop characteristic polynomial. Our assumption of closed loop stability implies that ϕ_T can have no CRHP zeros and, thus, $N_K(p)N_{yu}(p) \neq 0$. Note next that we can factor $\phi_G = \phi_G^* D_{zw} = \phi_G^\dagger D_{yu}$, where $\phi_G^\dagger(p) \neq 0$. Hence, we have

$$T_{zw} = \frac{(N_{zw}D_K\phi_G^* - N_K N_G)}{\phi_T\phi_G^\dagger} \tag{25}$$

and thus

$$T_{zw}(p) = \frac{(N_{zw}(p)D_K(p)\phi_G^*(p) - N_K(p)N_G(p))}{-N_K(p)N_{yu}(p)\phi_G^\dagger(p)}. \tag{26}$$

a) Suppose that the multiplicity of p as a pole of G is equal to its multiplicity as a zero of D_{zw}. Then, $N_{zw}(p)\phi_G^*(p) \neq 0$, and the value of (26) is compensator dependent.

b) Suppose that the multiplicity of p as a pole of G is strictly greater than its multiplicity as a zero of D_{zw}. Then (26) yields

$$T_{zw}(p) = \frac{N_G(p)}{N_{yu}(p)\phi_G^\dagger(p)}. \tag{27}$$

Hence, $T_{zw}(p) = 0$ if and only if p is also a zero of N_G. If $N_G(p) \neq 0$, then substituting the zero polynomial of G into (27), using the fact that $G_{yu}\phi_G = N_{yu}\phi_G^\dagger$, and rearranging shows that (10) holds.

c) Suppose first that $G_{zw} \equiv 0$. Then $N_{zw} \equiv 0$ and (25) reduces to $T_{zw} = -N_K N_G/\phi_T \phi_G^\dagger$. The assumption of internal stability precludes ϕ_T and N_K from having a zero at p, and thus T_{zw} has precisely m_G zeros at p. Hence T_{zw} as has the stated factorization, where T_{zw}^1 is stable, and the limit $T_{zw}^1(p)$ follows. Suppose next that $G_{zw} \not\equiv 0$. Then both terms in the numerator of (25) have a factor of $(s-p)^{m_G(p)}$, and T_{zw} has the stated factorization, where T_{zw}^1 is stable. The hypothesis (11) implies that $N_{zw}\phi_G^*$ has a factor of $(s-p)$ with multiplicity greater than $m_G(p)$, and the limit $T_{zw}^1(p)$ follows.

d) It follows from (24) that

$$R_{zw} = \frac{(N_{zw}D_K D_{yu}\phi_G - N_K N_G D_{zw} D_{yu})}{\phi_T N_{zw}\phi_G}. \tag{28}$$

If p is a pole of G, then

$$R_{zw}(p) = \frac{N_G(p)}{N_{yu}(p)} \frac{D_{zw}(p)}{N_{zw}(p)\phi_G^\dagger(p)}. \tag{29}$$

If either condition i) or ii) holds, then $R_{zw}(p) = 0$. If not, then rearranging shows that $R_{zw}(p)$ has the stated value.

Appendix B
Proof of Proposition IV.6

Using coprime polynomial factorizations for the $G_{\alpha\beta}$, the controller (7) may be written

$$K^C = \frac{N_K^C}{D_K^C} = \frac{N_{zw}D_{zu}D_{yw}D_{yu}}{N_{zw}N_{yu}D_{zu}D_{yw} - N_{zu}N_{yw}D_{zw}D_{yu}}. \tag{30}$$

Note that the factorization (N_K^C, D_K^C) defined in (30) need not be coprime. Hence, if N_K^C and D_K^C have a common CRHP zero, this zero will appear as a zero of the characteristic polynomial $\phi_T^C = D_K^C D_{yu} - N_K^C N_{yu}$. Hence, we work with the factorization (12), and assess stability using the characteristic polynomial (13).

We first consider the special case for which G is stable.

Lemma B.1: Assume that G is stable, and that the controller K^C is given by (12). Then, the system in Fig. 1 is internally stable if and only if the inequality (14) is satisfied for each CRHP zero ζ of G_{zu} or G_{yw}.

Proof: Let F_ζ denote a polynomial whose zeros consist precisely of the common CRHP zeros of G_{zw} and $G_{zu}G_{yw}$, including multiplicities, and factor $N_{zw} = \bar{N}_{zw}F_\zeta$ and $N_{zu}N_{yw} = \bar{N}_{zu}\bar{N}_{yw}F_\zeta$. Then, (30) may be written

$$K^C = \frac{\bar{N}_K^C}{\bar{D}_K^C} = \frac{\bar{N}_{zw}D_{zu}D_{yw}D_{yu}}{\bar{N}_{zw}N_{yu}D_{zu}D_{yw} - \bar{N}_{zu}\bar{N}_{yw}D_{zw}D_{yu}}. \tag{31}$$

To analyze nominal stability, it suffices to determine whether the characteristic polynomial (13) has any CRHP zeros. Substituting $\bar{N}_K^C$ and $\bar{D}_K^C$ defined in (31) into (13) and simplifying yields $\phi_T = \bar{N}_{zu}\bar{N}_{yw}D_{zw}D_{yu}^2$. Our assumption that G is stable implies that D_{zw} and D_{yu} can have no CRHP zeros. Hence, the system will be stable if and only if the polynomial $\bar{N}_{zu}\bar{N}_{yw}$ has no CRHP zeros. This condition will hold precisely when any CRHP zeros of $N_{zu}N_{yw}$ are also zeros, with at least the same multiplicity, of N_{zw}, and have thus been removed from $N_{zu}N_{yw}$ with the factor F_ζ. ■

We now complete the proof of Proposition IV.6. To do so requires us to perform two tasks. The first is to determine any CRHP zeros common to the polynomials N_K^C and D_K^C defined in (30). The second task is to compute the zeros of the resulting closed-loop characteristic polynomial, and show that none of these lies in the CRHP.

Recall that a possibly noncoprime factorization of K^C is given by (30). It follows from the definition of the zero polynomial that

$$D_K^C = N_{zw}N_{yu}D_{zu}D_{yw} - N_{zu}N_{yw}D_{zw}D_{yu} \tag{32}$$

$$= \frac{N_G D_{zw}D_{zu}D_{yw}D_{yu}}{\phi_G}. \tag{33}$$

By definition, the characteristic polynomial ϕ_G is a factor of $D_{zw}D_{zu}D_{yw}D_{yu}$, and the assumption of closed-loop stability implies that the CRHP zeros of ϕ_G are also zeros, with at least the same multiplicity, of D_{yu}. This fact and (33) imply that the CRHP zeros of D_K^C are equal to the union of the CRHP zeros of N_G, D_{zu}, D_{zw}, and D_{yw}. Let $m_K(\sigma)$ and $\gamma_K(\sigma)$ denote the multiplicities of σ as a zero of N_K^C and D_K^C, respectively. It follows that if p is a zero of N_G with multiplicity $m_G(p)$, then p is also a zero of D_K^C with multiplicity

$$\gamma_K(p) = m_G(p) + \gamma_{zw}(p) + \gamma_{zu}(p) + \gamma_{yw}(p). \tag{34}$$

Define

$$\beta(p) \triangleq \gamma_{zu}(p) + \gamma_{yw}(p) + m_{zw}(p). \tag{35}$$

We now show that

$$\beta(p) \geq \gamma_G(p) + m_{zu}(p) + m_{yw}(p) + \gamma_{zw}(p). \tag{36}$$

To show (36), we first note that (32) implies

$$\lim_{s\to p} \frac{D_K^C(s)}{(s-p)^{\beta(p)}} = \lim_{s\to p}\left(\frac{N_{zw}(s)N_{yu}(s)D_{zu}(s)D_{yw}(s)}{(s-p)^{\beta(p)}} - \frac{N_{zu}(s)N_{yw}(s)D_{zw}(s)D_{yu}(s)}{(s-p)^{\beta(p)}}\right). \tag{37}$$

If (36) fails to hold, then the second term on the right hand side of (37) is equal to zero. By definition of $\beta(p)$, the first term is finite and nonzero, and thus p is a zero of D_K^C with multiplicity $\gamma_K(p) = \beta(p)$. Equating (35) with (34), we see that $m_G(p) + \gamma_{zw}(p) = m_{zw}(p)$. Because $\gamma_G(p) \geq 1$, this contradicts (15), and thus (36) must hold.

It follows that if (15) holds, then this fact, together with the intermediate inequality (36), imply that

$$\gamma_K(p) \geq \gamma_G(p) + \gamma_{zu}(p) + \gamma_{yw}(p) + m_{zw}(p) \tag{38}$$

$$\geq 2\gamma_G(p) + m_{zu}(p) + m_{yw}(p) + \gamma_{zw}(p). \tag{39}$$

Note next that p is a zero of N_K^C with multiplicity

$$m_K(p) = \gamma_G(p) + \gamma_{zu}(p) + \gamma_{yw}(p) + m_{zw}(p). \tag{40}$$

It follows from (38) and (40) that $\gamma_K(p) \geq m_K(p)$. Hence if p is an unstable pole of G_{yu}, then p must be a zero of both N_K^C and D_K^C with multiplicity $m_K(p)$. As a result, N_K^C and D_K^C must have a common factor F_p that has zeros equal to the unstable poles of G_{yu}, and the multiplicity the zero at the unstable pole

p is equal to $m_K(p)$. Moreover, as we saw in Lemma B.1, (14) implies that N_K^C and D_K^C have a common factor F_ζ whose zeros are equal to those zeros of G_{zu} and G_{yw} that are not also poles of G. We have thus shown that the polynomials $\bar{N}_K^C$ and $\bar{D}_K^C$ in (12) are given by $N_K^C = \bar{N}_K^C F_\zeta F_p$ and $D_K^C = \bar{D}_K^C F_\zeta F_p$. It remains to compute the closed loop characteristic polynomial (13)

$$\phi_T = N_{zu} N_{yw} D_{zw} D_{yu}^2 / F_\zeta F_p.$$

It follows from (38)–(40) that p is a zero of F_p with multiplicity at least equal to the right hand side of (39). Hence any CRHP zeros of $N_{zu} N_{yw} D_{zw} D_{yu}^2$ that are poles of G will be removed by dividing by F_p, and any CRHP zeros of $N_{zu} N_{yw}$ that are not poles of G will be removed by dividing by F_ζ. The controller (12) thus yields internal stability.

Appendix C
Proof of Proposition IV.7

We show that $S, T, K_p^C S$, and SG_{yu} are stable. Stability of the first three follows from the facts that K^C is nominally stabilizing and B is stable. Stability of SG_{yu} follows by rearranging

$$SG_{yu} = (1 - B) G_{yu} + S^C G_{yu} B \tag{41}$$

where S^C is given by (8). The first term on the right-hand side of (41) is stable because $(1 - B)$ has zeros at the CRHP poles of G_{yu}. Proposition IV.6 shows that K^C stabilizes G_{yu} and, hence, the resulting sensitivity function S^C must have zeros at the CRHP poles of G_{yu}. Hence, the second term on the right hand side of (41) is stable.

Appendix D
Proof of Proposition V.7

a) It may be verified that $\tilde{T}_{zw}$ defined in Definition V.1 satisfies the hypotheses of Corollary [1, Cor. A.6.3], and thus

$$\int_0^\infty \log |T_{zw}(j\omega)| W(\zeta, \omega)\, d\omega = \pi \log |\tilde{T}_{zw}(\zeta)|. \tag{42}$$

Proposition IV.1 a) implies that $G_{zw}(\zeta) = \tilde{T}_{zw}(\zeta) B_\xi(\zeta)$. The hypothesis that ζ is not a zero of G_{zw} implies that $\tilde{T}_{zw}(\zeta) = G_{zw}(\zeta) B_\xi^{-1}(\zeta)$, and the integral for T_{zw} follows. Next, it follows from the factorization $R_{zw} = \tilde{R}_{zw} B_\gamma B_\rho B_\alpha^{-1}$ and Proposition IV.1 c) that the integral for R_{zw} follows.

b) We must evaluate the term $\tilde{T}_{zw}(\zeta)$ on the right hand side of the Poisson integral (42). Inequality (9) implies that the Blaschke product B_β must contain precisely m_{zw} zeros at ζ and that $B_\gamma(\zeta) \neq 0$. Hence, the second term on the right-hand side of

$$\tilde{T}_{zw} = \tilde{G}_{zw} B_\alpha B_\rho^{-1} B_\gamma^{-1} + G_{zu} K S G_{yw} B_\beta^{-1} B_\gamma^{-1}$$

is equal to zero at ζ, which implies that $\tilde{T}_{zw}(\zeta) = \lim_{s \to \zeta} G_{zw}(s) B_\xi^{-1}(s)$, and the integral for T_{zw} follows. Similar arguments yield the integral for R_{zw}.

Appendix E
Proof of Proposition V.10

We first characterize the CRHP poles and zeros of the cancellation controller (7) without assuming that the controller is stabilizing.

Lemma E.1: Assume that the hypotheses of Lemma III.5 are satisfied, and that $G_{zw} \not\equiv 0$.

a) The cancellation controller K^C has a CRHP pole p if and only if G has a CRHP transmission zero p satisfying the inequality (20).

b) The cancellation controller K^C has a CRHP zero z if and only if i) G has a CRHP pole z, or ii) G_{zw} has a CRHP zero z, satisfying (21).

Proof: Using the zero polynomial of G and coprime factorizations for the $G_{\alpha\beta}$, we may rewrite (7) as $K^C = N_{zw} \phi_G / D_{zw} N_G$.

a) By definition of the characteristic polynomial, all zeros of D_{zw} must also be zeros of ϕ_G. It follows that any CRHP pole of K^C must be due to a CRHP zero of N_G that satisfies (20).

b) It is clear that any CRHP zero of K^C must either be a CRHP zero of N_{zw} or a CRHP pole of G that satisfies (21).

■

In Lemma E.1, we did not require that K^C be stabilizing, and thus did not rule out unstable pole/zero cancellations between K^C and G_{yu}. We now characterize the CRHP poles and zeros of a stabilizing K^C. Before stating the result, we require a technical lemma.

Lemma E.2: Let σ be a complex scalar that is not a pole of G.

a) Define

$$\mu(\sigma) \stackrel{\triangle}{=} \min\{m_{zw}(\sigma) + m_{yu}(\sigma), m_{zu}(\sigma) + m_{yw}(\sigma)\}. \tag{43}$$

Then, $m_G(\sigma) \geq \mu(\sigma)$.

b) Assume that

$$m_{zw}(\sigma) + m_{yu}(\sigma) \neq m_{zu}(\sigma) + m_{yw}(\sigma). \tag{44}$$

Then, $m_G(\sigma) = \mu(\sigma)$.

c) A *necessary* condition for the inequality $m_G(\sigma) > \mu(\sigma)$ to hold is that $m_{zw}(\sigma) + m_{yu}(\sigma) = m_{zu}(\sigma) + m_{yw}(\sigma)$.

Proof: We may factor $\det G(s) = f(s)(s - \sigma)^{\mu(\sigma)}$, where $f(\sigma)$ is finite. Then, a) follows immediately. If (44) is satisfied, then $f(\sigma) \neq 0$ and b) and c) follow. ■

We now use Lemmas E.1–E.2 to complete the proof of Proposition V.10. The CRHP poles and zeros of a *stabilizing* K^C are equal to the subset of the poles and zeros described in Lemma E.1 that also satisfy the hypotheses of Proposition IV.6.

a) Suppose p is a pole of K^C as described in Lemma E.1. If p is also a pole of G, then inequality (15) is automatically satisfied. Hence we must consider only the case for which

p is a CRHP zero of G_{zu} or G_{yw} and is not also a pole of G. In this case, (20) reduces to

$$m_G(p) > m_{zw}(p). \tag{45}$$

Assume first that $G_{yu}(p) \neq 0$. Then, the necessary condition in Lemma E.2 c) is equivalent to $m_{zw}(p) = m_{zu}(p) + m_{yw}(p)$, and (14) follows. Assume next that $G_{yu} \equiv 0$. Then $m_G(p) = m_{zu}(p) + m_{yw}(p)$, and inequalities (45) and (14) are mutually exclusive. Finally, assume that $G_{yu}(p) = 0$, but is not identically zero. Then, (45) holds *only if* the necessary condition in Lemma E.2 c) is satisfied, and since $m_{yu}(p) > 0$, it follows that (14) is violated.

b) Suppose that z is a zero of K^C that is a pole of G. Then (15) and (21) are mutually incompatible. Suppose next that z is a zero of G_{zw} that is not a pole of G, and that satisfies (21), which simplifies to

$$m_{zw}(z) > m_G(z). \tag{46}$$

It follows from Lemma E.2 (a) that either (i) $m_G(z) \geq m_{zw} + m_{yu}(z)$ or (ii) $m_G(z) \geq m_{zu} + m_{yw}(z)$. In case (i), it follows immediately that $m_G(z) \geq m_{zw}(z)$, which contradicts (46) and hence cannot occur. Now consider case (ii), and suppose that (14) is false, so that $m_{zu}(z) + m_{yw}(z) > m_{zw}(z)$. Because we are considering case (ii), it follows that $m_G(z) > m_{zw}(z)$, which contradicts (46). Hence if z is a zero of G_{zw} that is not a pole of G, and that satisfies (21), then the condition (14) must be satisfied, and hence all such zeros will be present in a stabilizing K^C.

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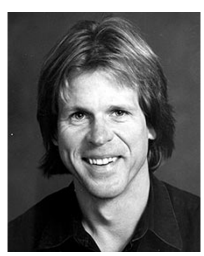

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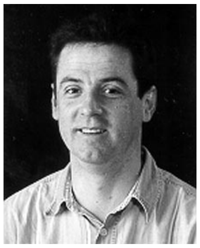

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