Finite Spectrum Assignment of Unstable Time-Delay Systems With a Safe Implementation

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***Abstract*—The instability mechanisms, related to the implementation of distributed delay controllers in the context of finite spectrum assignment, were studied in detail in the past few years. In this note we introduce a distributed delay control law that assigns a finite closed-loop spectrum and whose implementation with a sum of point-wise delays is safe. This property is obtained by implicitly including a low-pass filter in the control loop. This leads to a closed-loop characteristic quasipolynomial of retarded type, and not one of neutral type, which was shown to be a cause of instability in previous schemes.**

***Index Terms*—Delay equations, finite spectrum assignment.**

I. Introduction

Consider the linear finite-dimensional system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t-h) \tag{1}$$

where the matrices $A \in R^{n\times n}$, $B \in R^{n\times n}$ and h is the delay of the system. The matrix A is not Hurwitz and the pair (A, B) is controllable. An approach to stabilize the system (1), called finite spectrum assignment [7], [18], can be interpreted as follows: first a prediction of the state variable over one delay interval is generated and then a feedback of the predicted state is applied, thereby compensating the effect of the time-delay. This results in a closed-loop system with a finite number of eigenvalues, which can be assigned arbitrarily. Mathematically, with $x_p(t_1, t_2)$ the prediction of $x(t)$ at $t = t_2$, based on values of x and u for $t \leq t_1$, the control law

$$\begin{aligned} u(t) &= Kx_p(t, t+h) \\ &= K\left(e^{Ah}x(t) + \int_0^h e^{A\theta}Bu(t-\theta)d\theta\right) \end{aligned} \tag{2}$$

yields the closed-loop characteristic equation

$$\det(sI - A - BK) = 0. \tag{3}$$

The elimination of the delay is employed in the so-called process model control techniques [19], as for example, the celebrated Smith Predictor [16]. It can also be interpreted as a model transformation, the Artstein's model reduction technique [1].

The finite spectrum assignment feature of these control laws is a significant advantage from a design point of view because the stability and dynamic properties of polynomials can be readily analyzed, while those of quasipolynomials are usually a complex task. However, a difficulty in applying a control law of the form (2) consists of the practical implementation of the integral term, which needs to be calculated on-line. As explained in [7], obtaining this term as the solution to a differential equation must be discarded because it involves unstable pole-zero cancellations when A is unstable. As suggested in [7], a possibility is to approximate the distributed delay by a sum of point-wise delays by using a numerical quadrature rule. In this way, one ends up with a sequence of control-laws of the form

$$u(t) = K\left(e^{Ah}x(t) + \sum_{j=0}^{q} h_{j,q}e^{A\theta_{j,q}}Bu(t-\theta_{j,q})\right). \tag{4}$$

In the past few years the effect of such a semidiscretization on the stability of the closed-loop system has been examined thoroughly. In [17], it was demonstrated with a scalar example that for some parameter values, the control law (4) may *not* stabilize the system (1), for *arbitrarily large values* of q. In [2], [12], and [9], the underlying instability mechanism was investigated and various necessary and/or sufficient conditions for a safe implementation of the distributed delay as a sum of point-wise delays were provided. These conditions are all related to stability properties of the functional difference equation

$$u(t) = K\int_0^h e^{A\theta}Bu(t-\theta)d\theta \tag{5}$$

whose spectrum provides information on the position of the high-frequency modes of (1)–(4). They originate the fact that when the control law is approximated, the closed-loop equation is of *neutral type*, and an essential spectrum which is determined by the discretization of (5), is introduced.

In this note, we present a simple, yet effective way to overcome the previous instability problems. We will modify the control law in such a way that a *finite closed-loop spectrum* can still be assigned through standard design methods for linear systems, but, when the integral is approximated, the closed-loop characteristic quasipolynomial is of *retarded type* and, as we will see, a sensitivity of stability w.r.t. small implementation errors is not possible. Basically, our approach consists of including a low-pass element in the control loop.

The note is organized as follows: a motivating example is first given. Then we presents and analyze a class of control laws which assign a finite spectrum and allow a safe implementation. The note ends with a numerical example and concluding remarks.

II. Motivation

If the scalar system

$$\dot{x}(t) = x(t) + u(t-1) \tag{6}$$

is subject to the control law

$$u(t) = -2x_p(t, t+1) = -2(ex(t) + \int_0^1 e^{\theta}u(t-\theta)d\theta) \tag{7}$$

there is one closed-loop eigenvalue at $s = -1$. Approximating the integral term with a sum of point-wise delays using the *trapezoidal* rule

$$\begin{aligned} u(t) = -2\Bigg\{ex(t) + \frac{1}{q}\Bigg(&\frac{1}{2}u(t) \\ &+ \sum_{l=1}^{q-1} e^{l/q}u(t-\frac{l}{q}) + \frac{1}{2}eu(t-1)\Bigg)\Bigg\} \end{aligned} \tag{8}$$

results in the closed-loop eigenvalues, depicted in Fig. 1(a) for $q = 10$ and $q = 20$. Clearly, the eigenvalues, introduced by the approximation make the closed-loop system unstable. Moreover, as shown in [2], [9] the closed-loop system is unstable for *arbitrarily large* values of q.

A simple remedy to overcome the previous stability problem is based on the observation that the instability mechanism is a *high-frequency* mechanism, related to the occurrence of unstable eigenvalues with *arbitrarily large* imaginary parts. A closer look at the problem reveals

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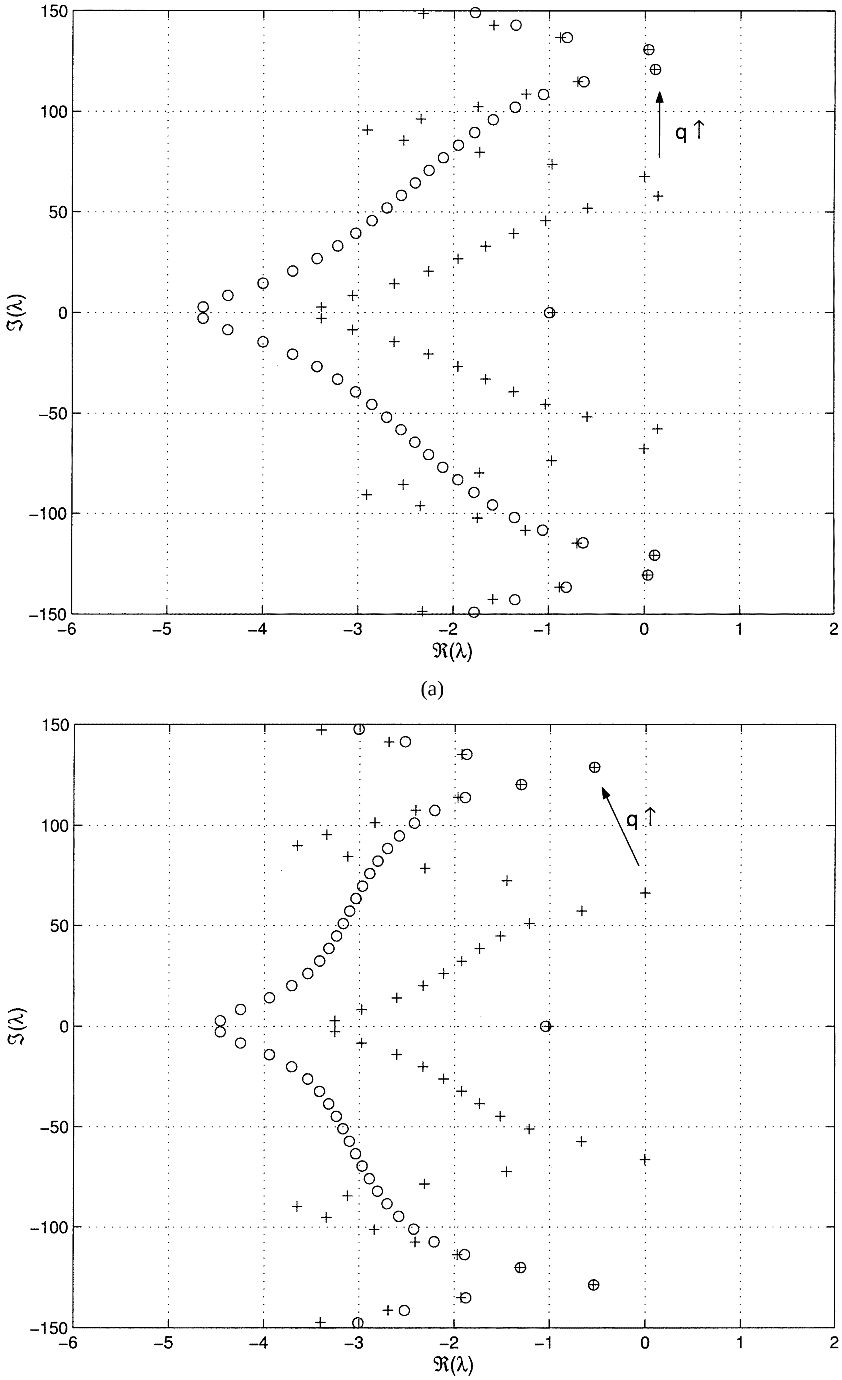

(a)

(b)

Fig. 1. (a) Eigenvalues of the closed-loop system (6)–(8) for $q = 10$ ("+") and $q = 20$ ("o"). (b) Eigenvalues of (6)–(9) for $f = 40$ and $q = 10, 20$.

that the latter are caused by the throughput at infinity of past inputs in (8) and, therefore, can be avoided by including a low-pass filter in the control loop. Adding the filter $f/s + f$ to (8) yields the control law shown in (9) at the bottom of the page. The closed-loop characteristic polynomial is now of retarded type and the eigenvalues are shown in Fig. 1(b) for $f = 40$ and $q = 10, 20$. The low-pass filter puts an upper bound on the values of the imaginary parts of the unstable eigenvalues, independent of q. Therefore, as $q \to \infty$ the real parts of the introduced eigenvalues move to the left half plane (actually their real parts move off to minus infinity) and stability is obtained.

$$\begin{cases} \dot{z}(t) = -fz(t) - 2f\left\{ex(t) + \frac{1}{q}\left(\frac{1}{2}u(t) + \sum_{l=1}^{q-1} e^{l/q}u(t - \frac{l}{q}) + \frac{1}{2}eu(t-1)\right)\right\} \\ u = z(t). \end{cases} \tag{9}$$

The previous ideas can be generalized. Since *any strictly proper linear system* $(A_f, B_f, C_f, 0)$ has a *low-pass* filtering property, one can construct the following *dynamic* controller for (1):

$$\begin{cases} \dot{z}(t) = A_f z(t) + B_f \underbrace{\left(e^{Ah}x(t) + \int_0^h e^{A\theta}Bu(t-\theta)d\theta\right)}_{x_p(t,t+h)} \\ u(t) = C_f z(t). \end{cases} \tag{10}$$

In this way the sensitivity problem of stability w.r.t. an implementation of the integral term is avoided. In addition, since (10) involves a prediction of the state, which *compensates* the input delay, the closed-loop system is the finite-dimensional system

$$\begin{cases} \dot{x}(t) = Ax(t) + BC_f z(t) \\ \dot{z}(t) = A_f z(t) + B_f x(t) \end{cases} \tag{11}$$

and standard design methods can be used.

In the rest of this note, we will perform a detailed stability analysis of this type of control laws in the special case where $C_f = I$ because this allows serious simplifications.

III. Dynamic Control Law

We study the multivariable linear time-invariant system with input delay h

$$\dot{x}(t) = Ax(t) + Bu(t-h),\ x \in R^n,\ u \in R^m,\ h \in R^+ \tag{12}$$

where the pair (A, B) is controllable with controllability indices $c_i,\ i = 1, \ldots, m$ in nonincreasing order, and the control law

$$\dot{u}(t) = A_f u(t) + B_f \left(e^{hA}x(t) + \int_0^h e^{A\theta}Bu(t-\theta)d\theta\right) \tag{13}$$

where $A_f \in R^{m\times m}$ and $B_f \in R^{m\times n}$. Without loosing generality, we assume that B has full-column rank. Note that (13) corresponds to (10) with $C_f = I$.

The closed-loop system analysis and design procedure is based on the concept of coprime factorizations or normal external descriptions (NED) for linear systems; see, e.g., [20].

System (12) and the control (13) are, respectively, described in the frequency domain by

$$(sI - A)x(s) = Be^{-sh}u(s) \tag{14}$$

and

$$B_f e^{hA}x(s) = \left\{sI - A_f - B_f\int_0^h e^{-(sI-A)\theta}Bd\theta\right\}u(s). \tag{15}$$

Let $(N(s), D(s))$ be a NED for the pair $(sI - A, B)$ such that

$$(sI - A)^{-1}B = N(s)D(s)^{-1}. \tag{16}$$

where $N(s)$ and $D(s)$ are right coprime, $D(s)$ is column reduced with nonincreasingly ordered column degrees $c_i,\ i = 1, \ldots, m$. The invariant factors of $D(s)$ and of $sI - A$ are the same up to unitary invariant factors and the column degrees of $N(s)$ are smaller than those of $D(s)$), see e.g., [20].

Using (16) the characteristic matrix of the closed-loop system (12)–(13) can be written in the absence of uncertainty as

$$D^{\text{ideal}}_{A_f,B_f}(s) \triangleq \left\{sI - A_f - B_f\int_0^h e^{-(sI-A)\theta}Bd\theta\right\}D(s) - B_f e^{hA}N(s)e^{-hs}. \tag{17}$$

Substituting

$$\int_0^h e^{-(sI-A)\theta}d\theta = (I - e^{-(sI-A)h})(sI - A)^{-1} \tag{18}$$

yields

$$D^{\text{ideal}}_{A_f,B_f}(s) = (sI - A_f)D(s) - B_f N(s) \tag{19}$$

hence, the number of closed-loop eigenvalues is finite. Next, we prove that it is always possible to find constant matrices A_f and B_f so that a prescribed spectrum consisting of $n + m$ eigenvalues is assigned.

Proposition 1: Let a polynomial $p(s)$ of degree $\sum_{i=1}^m (c_i + 1) = n + m$ be given with leading coefficient equal to that of $\det(D(s))$. Then, there exist constant matrices A_f and B_f such that the characteristic polynomial of system (12) in closed-loop with the control law (13) is $p(s)$.

Proof: There always exists a polynomial matrix $D^{\text{ideal}}_{A_f,B_f}(s)$ with column degrees $c_i + 1,\ i = 1, \ldots, m$, same highest degree coefficient matrix as $D(s)$ and determinant $p(s)$ (see, for, instance [20]). This implies that the matrix $D_{A_f,B_f}(s)-$ diag $\{s\}_m D(s)$ has column degrees smaller are equal than those of $D(s)$. The matrix polynomial (19) can be written in the form

$$XD(s) + YN(s) = D^{\text{ideal}}_{A_f,B_f}(s) - \text{diag}\{s\}_m D(s) \tag{20}$$

where $X = -A_f$ and $Y = -B_f$. For the previous choice of $D^{\text{ideal}}_{A_f,B_f}$ it follows from [6] that the matrix polynomial (20) has a constant solution. ∎

Remark 2: A particular case of the previous design procedure is the clever choice for A_f and B_f made in [7], which clearly illustrates the filtering property of the dynamic control law, and which decouples the design in that of a pole placement controller and that of a filter. The control law (13) with the choice

$$A_f = F + KB \quad B_f = K(A - G) \text{ such that } FK = KG$$

where $F \in R^{m\times m}$, $G \in R^{n\times n}$ and $K \in R^{m\times n}$ yields the closed-loop characteristic matrix

$$D_{F,K}(s) = (sI - F)(D(s) - KN(s))$$

the closed-loop polynomial

$$\det D_{F,K}(s) = \det(sI - F)\det(sI - A - BK).$$

IV. Effect of the Approximation of the Integral Term

When the integral in (13) is approximated with a sum of point-wise delays, the closed-loop characteristic matrix is

$$D_{A_f,B_f}(s) = \left\{sI - A_f - B_f\sum_{j=1}^q h_{j,q}e^{-\theta_{j,q}(sI-A)}B\right\}D(s) - B_f e^{-h(sI-A)}N(s). \tag{21}$$

For such approximations based on fixed-step method [8], the sum in (21) is as follows:

$$\int_0^h e^{-(sI-A)\theta}d\theta \approx \sum_{j=1}^q h_{j,q}e^{-\theta_{j,q}(sI-A)} = \sum_{p=0}^q \eta_p e^{-p/qh(sI-A)} \tag{22}$$

where q is the number of intervals of equal length h/q and η_p are scalars that depend on the selected integration rule. The characteristic matrix (21) can then be rewritten as

$$\begin{aligned} D_{A_f,B_f}(s) = &\{sI - A_f - \eta_0 B_f B\} D(s) \\ &- \sum_{p=1}^{q-1} \eta_p B_f e^{Ap/qh} BD(s) e^{-p/qhs} \\ &- B_f e^{hA}(\eta_q BD(s) + N(s)) e^{-hs} \end{aligned}$$

which corresponds to an equation of *retarded* type. Delay equations of retarded type have an *infinite* number of roots of arbitrarily large magnitude, located on logarithmic sectors in the left half-plane. However, there are only a *finite* number of roots in any right half plane, $\Re(s) \geq \alpha$, $\alpha \in \mathbb{R}$, unlike equations of neutral type [5].

We now prove that when the ideal closed-loop system is asymptotically stable, the approximation of the integral term (22) preserves stability for large q. We need the following technical result.

Lemma 3: Assume that the parameters A_f and B_f in control law (13) are such that the ideal closed-loop (19) is Hurwitz. Then, the closed-loop characteristic matrix (21) is asymptotically stable if

$$\bar{\sigma}\left(B_f\, H(s,q,h,A,B)\, BD(s) D^{\text{ideal}}_{A_f,B_f}(s)^{-1}\right) < 1 \qquad (23)$$

where

$$H(s,q,h,A,B) \triangleq \int_0^h e^{-(sI-A)\theta} d\theta - \sum_{p=0}^{q} \eta_p e^{-p/qh(sI-A)}$$

for all $s \in \mathbb{C}$ with $\Re(s) \geq 0$.

Proof: The characteristic equation of the closed-loop system can be written as

$$\begin{aligned} \det\left\{D_{A_f,B_f}(s)\right\} = &\det\left\{D^{\text{ideal}}_{A_f,B_f}(s)\right\} \\ &\cdot \det\left\{I + B_f\, H(s,q,h,A,B)\, BD(s) D^{\text{ideal}}_{A_f,B_f}(s)^{-1}\right\}. \end{aligned}$$

Under the assumption of (23) the second determinant has no roots in the closed right half plane. $D^{\text{ideal}}_{A_f,B_f}(s)$ is Hurwitz by hypothesis, hence, the lemma is proven. ∎

The main result is as follows.

Proposition 4: Assume that the parameters A_f and B_f in the control law (13) are such the ideal closed-loop (19) is Hurwitz. When the integral term in (13) is approximated by a fixed step method as (22), then there exists a precision of the approximation such that the closed-loop characteristic matrix (21) is asymptotically stable.

Proof: It is sufficient to prove that the condition of Lemma 3 is satisfied for large q.

For all s with $\Re(s) \geq 0$, we have

$$\begin{aligned} &\left\| B_f \left\{ \int_0^h e^{-(sI-A)\theta} d\theta - \sum_{p=0}^{q} \eta_p e^{-p/qh(sI-A)} \right\} B \right\| \\ &\quad \leq \int_0^h \left\| B_f e^{A\theta} B \right\| d\theta + \sum_{p=0}^{q} \eta_p \left\| B_f e^{p/qhA} B \right\| \leq M \qquad (24) \end{aligned}$$

where M can be chosen *independent* of q.

The column degrees of $D^{\text{ideal}}_{F,K}(s)$ are those of $D(s)$ plus one. As a consequence

$$\left\| D(s) D^{\text{ideal}}_{A_f,B_f}(s)^{-1} \right\| \to 0 \text{ as } |s| \to \infty. \qquad (25)$$

Expressions (24) and (25) imply the existence of a radius R_0 such that

$$\sup_{\Re(s)\geq 0,\ |s|\geq R_0} \bar{\sigma}\left(B_f\, H(s,q,h,A,B)\, BD(s) D^{\text{ideal}}_{A_f,B_f} \times (s)^{-1}\right) < 1 \qquad (26)$$

for *all* values of q.

Given a value of R_0 satisfying (26) it only remains to prove that there always exists a precise enough approximation q, so that the condition (23) is also satisfied for $\Re(s) \geq 0$, $|s| \leq R_0$. Recall that for a smooth real function f the error between the integral value $I(f) = \int_0^h f(\zeta) d\zeta$ and its approximation $I_q(f)$, obtained via a fixed-step method, depends on the value of q and is given by an expression of the form

$$I(f) - I_q(f) = -\frac{h^{\beta+1}}{\alpha q^{\beta}} f^{(\gamma)}(\zeta_0)$$

where the point ζ_0 belongs to the interval $[-h,\ 0]$ and where α, β and γ are positive integers that depend on the chosen method (for instance, in the trapezoidal method, $\beta = 2$, $\gamma = 2$ and $\alpha = 12$) [8]. It is easy to extend this result to a smooth function $g: \mathbb{R} \to \mathbb{C}$ and obtain a bound of the form

$$|I(g) - I_q(g)| \leq \frac{\kappa}{q^{\beta}} \max_{\theta \in [-h,0]} |g^{(\gamma)}(\theta)|. \qquad (27)$$

Let λ_i $(i = 1, \ldots, k)$ denote the eigenvalues of matrix A and η_i the order of multiplicity of λ_i with respect to the characteristic polynomial of A, and let Z_{ij} $(i = 1, \ldots, k,\ j = 1, \ldots, \eta_i)$ be the components of matrix A (see [4, Ch. 5]). We then have

$$\begin{aligned} H(s,q,h,A,B) &= \int_{-h}^{0} e^{\theta(sI-A)} d\theta - \sum_{p=0}^{q} \eta_p e^{-p/qh(sI-A)} \\ &= \sum_{i=1}^{k} \sum_{j=1}^{\eta_i} Z_{ij} \left(I\left(\theta^{j-1} e^{\theta(s-\lambda_i)}\right) \right. \\ &\qquad \left. - I_q\left(\theta^{j-1} e^{\theta(s-\lambda_i)}\right)\right). \end{aligned}$$

Using inequality (27) with $g(\theta) = \theta^{j-1} e^{\theta(s-\lambda_i)}$, it is straightforward to prove that there is a constant m such that

$$\sup_{\Re(s)\geq 0,\ |s|\leq R_0} \bar{\sigma}(H(s,q,h,A,B)) \leq \frac{m}{q^{\beta}}. \qquad (28)$$

Since $D^{\text{ideal}}_{A_f,B_f}(s)$ has no roots in the compact set $\Re(s) \geq 0 \cap |s| \leq R_0$, this implies

$$\sup_{\Re(s)\geq 0,\ |s|\leq R_0} \bar{\sigma}\left(B_f\, H(s,q,h,A,B)\, BD \times (s) D^{\text{ideal}}_{A_f,B_f}(s)^{-1}\right) < 1$$

for large q and the proof is complete. ∎

Remark 5: The proof of the proposition is *constructive*. Using (24), (25), and (28), a minimal precision q_0 (and the radius R_0) can be computed explicitly.

Because the closed-loop characteristic matrix (21) is of *retarded* type, the achieved stability will not be sensitive to arbitrarily small perturbations of the parameters $h_{j,q}, \theta_{j,q}$ of the integration rule; see [5]. Proposition 4 extends to other types of quadrature rules under mild conditions, since basically only a generally satisfied convergence result of $I_q(f)$ to $I(f)$ as $q \to \infty$ is required in its proof. Notice that these two properties do not hold in general for the discretization of the classical FSA controller (4) as a sum of point-wise delays, as shown in [9].

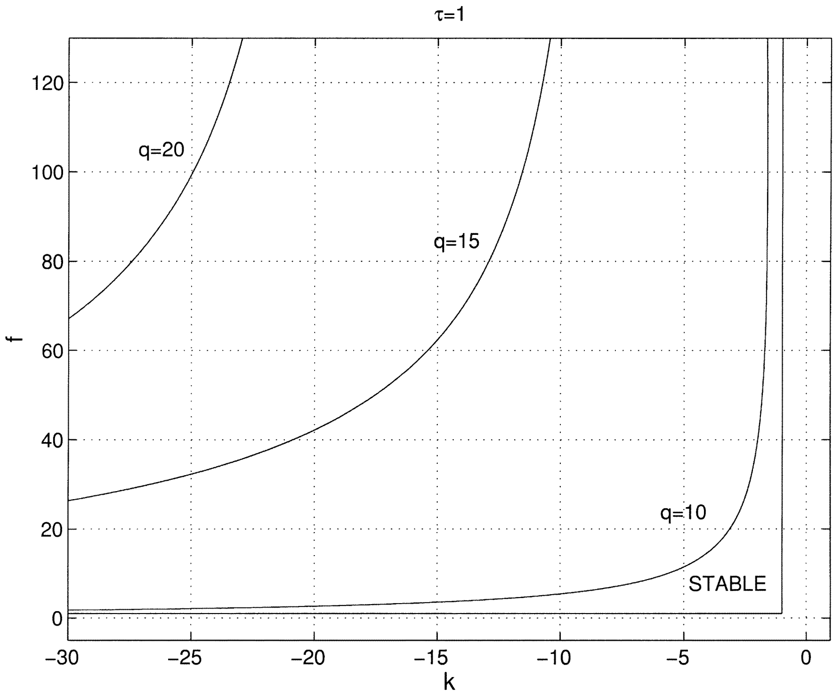

Fig. 2. Stability region of the closed-loop system (29)–(30).

If the precision of the approximation q is such that the approximation of the integral is not an issue, the stability of the closed-loop system when the system's delay differs from the nominal one can be studied by using numerical continuation [3]or the analytical approach spelled out in [11], [13]. It is also possible to perform a robustness analysis with respect to parameter uncertainty based on the concept of *stability radii* [10], [14].

V. Illustrative Example

We consider the monovariable system

$$\dot{x}(t) = x(t) + u(t-h) \tag{29}$$

and the control law

$$\begin{cases} \dot{z}(t) = -fz(t) + fk\left\{e^h x(t) + \int_0^h e^\theta u(t-\theta)d\theta\right\} \\ u(t) = z(t) \end{cases} \tag{30}$$

which corresponds to (13) with $A_f = f$ and $B_f = f.k$. This controller parametrization in (f, k) is used because, for large f, it can be seen as a cascade of the classical FSA controller $u(t) = k\,x_p(t, t+h)$ and the prefilter $f/(s+f)$.

When the computation of the integral is exact, the closed-loop quasipolynomial $p(s) = s^2 + (f-1)s - f(k+1)$ is stable for pairs $f,\ k$ such that $f > 1$ and $k < -1$.

When the integral is approximated, as in (7)–(8), the effect of the low-pass filtering property of the control law (30) on the introduced eigenvalues and the effect of increasing the precision q of the approximation have already been illustrated in Section II. For $h = 1$ and different values of q, a $\mathcal{D}$-subdivision analysis leads to the stability-instability boundaries of the closed-loop system in the (k, f)-plane, depicted in Fig. 2.

VI. Conclusion

We proposed a distributed delay control law for multivariable input delay systems that assigns a finite closed-loop spectrum and allows a safe implementation of the integral term with point-wise delays. This robustness property is obtained by *dynamic* feedback, which makes the control law behave as a low-pass filter. The idea of adding low pass elements to overcome the problem of sensitivity with respect to approximation method and to infinitesimal delay variations also applies to other predictor-based control laws such as the Smith Predictor.

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On Global Tracking Control of a VTOL Aircraft Without Velocity Measurements

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***Abstract*—This note develops a nonlinear output-feedback controller to force a nonminimum phase, underactuated vertical take-off and landing aircraft to globally asymptotically track a reference trajectory generated by a reference model. The control development is based on a global exponential observer, some global coordinate transformations, Lyapunov's direct method and an extension of the backstepping technique. Interestingly, the proposed methodology also yields new results for the previously studied problems of stabilization and output tracking or regulation. Numerical simulations illustrate the effectiveness of the proposed controller.**

***Index Terms*—Backstepping, Lyapunov's direct method, output-feedback, vertical take-off and landing (VTOL).**

I. Introduction

Over the last few years, controlling a vertical take-off and landing (VTOL) aircraft has received a lot of attention from the control community. The main difficulty with controlling VTOL aircraft is that it is underactuated and nonminimum phase. An approximate input-output linearization approach was used in [1], [4], [8], and [9] to develop a controller for stabilization and output tracking/regulation of a VTOL aircraft. In these papers, the controller was initially designed by ignoring the coupling between rolling moment and thrust. The controller parameters were then selected to take into account the effects of the coupling. In [2], by noting that the output at a fixed point with respect to the aircraft body (Huygens center of oscillation) can be used, an interesting approach was introduced to design an output tracking controller. However, the proposed controller is not defined in the whole space. A simple approach was developed in [3] to provide a global controller for the stabilization of a VTOL aircraft. An optimal controller was provided in [6] for robust hovering control of a VTOL aircraft. In [5], dynamic inversion and robust control techniques were used to deal with the nonminimum phase dynamics. However, this approach imposed restrictions on the desired reference trajectories. Recently, a dynamic high-gain approach was used in [7] to design a controller to force the VTOL aircraft to globally practically track a reference trajectory generated by a reference model. In all of the aforementioned papers, all of the VTOL aircraft states are required for feedback.

From the above discussion, it is clear that the design of an output-feedback tracking controller for a VTOL aircraft without velocity measurements is an open problem. Under this controller, the VTOL aircraft should globally asymptotically track a reference trajectory generated by a reference model. Indeed, this tracking control problem should also include stabilization and output tracking/regulation problems studied in the above-mentioned papers.

This note provides a simple positive answer to the above challenging problem. The new result is facilitated by a global exponential observer, some nonlinear global coordinate transformations, Lyapunov's direct method and an extension of applying the backstepping technique. Numerical simulations illustrate the soundness of the proposed methodology.

II. Problem Formulation

A scaled mathematical model of a VTOL aircraft can be described as [1]

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -u_1\sin(\theta)+\varepsilon u_2\cos(\theta)\\ \dot{y}_1 &= y_2\\ \dot{y}_2 &= u_1\cos(\theta)+\varepsilon u_2\sin(\theta)-g\\ \dot{\theta} &= \omega\\ \dot{\omega} &= u_2\end{aligned}\tag{1}$$

where x_1, y_1, θ denote position of the aircraft center of mass and roll angle, x_2, y_2, ω denote linear and roll angular velocities of the aircraft, respectively, u_1 and u_2 are the vertical control force and rotational moment, $g > 0$ is the gravitational acceleration and ε is the constant coupling between the roll moment and the lateral force. It is seen that the aircraft model (1) is underactuated and that its zero dynamics are nonminimum phase for $\varepsilon \neq 0$ at the steady state when considering (x_1, y_1) as the output and θ as an internal state. This phenomenon can be seen from (1) by setting $x_1 = y_1 = x_2 = y_2 = 0$. We assume that the reference trajectory to be tracked is generated by

$$\begin{aligned}\dot{x}_{1r} &= x_{2r}\\ \dot{x}_{2r} &= -u_{1r}\sin(\theta_r)+\varepsilon u_{2r}\cos(\theta_r)\\ \dot{y}_{1r} &= y_{2r}\\ \dot{y}_{2r} &= u_{1r}\cos(\theta_r)+\varepsilon u_{2r}\sin(\theta_r)-g\\ \dot{\theta}_r &= \omega_r\\ \dot{\omega}_r &= u_{2r}\end{aligned}\tag{2}$$

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