ANALYSIS OF COORDINATION IN MULTI-AGENT SYSTEMS THROUGH PARTIAL DIFFERENCE EQUATIONS. PART II: NONLINEAR CONTROL

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Abstract: In this second part of the paper we exploit the framework of Partial difference Equations (PdEs) over graphs for analyzing the behavior of multi-agent systems equipped with potential field based control schemes. We consider agent dynamics affected by errors and model the collective dynamics through nonlinear PdEs. Hinging on the properties of the Laplacian operator on graph, discussed in Part I, we prove alignment and collision avoidance both in leaderless and leader-follower models.

Keywords: Networked autonomous vehicles, Partial difference equations, Formation control, Cooperative control. Decentralized control.

1. INTRODUCTION

Over the past decade, a large stream of research focused on the problem of controlling the motion of groups of autonomous vehicles communicating through a wireless network. In particular, rather than stabilizing each agent around a given trajectory, efforts have been devoted to devising decentralized control schemes where the actions of individual agents give rise to a global coordinated behavior. This control problem is even more challenging when each agent exchanges information just with few neighbors, a feature that is desirable for coping with the limited transmission power of small vehicles. In modern engineering, the interest in control of networked agents has been fostered by advances in electronics and mechanics that allow to construct small mobile entities (like robots and air or underwater vehicles) having on-board computing capabilities. Moreover, models of self-coordination play also a key role in other fields like biology (Flierl *et al.*, 1999), physics (Vicsek *et al.*, 1995) and computer graphics (Reynolds, 1987).

In the context of networked autonomous vehicles, the control action has to be designed in order to achieve various goal. In this paper we focus on the following three “flocking rules” inspired (but

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not identical) to the ones introduced by Reynolds (1987):

Alignment: all the vehicles asymptotically move with the same velocity.

Collision avoidance: at each time instant, the distance between two linked vehicles does not fall below a safety threshold.

Cohesion: the distance between pairs of linked vehicles asymptotically converges to a given setpoint.

In the Part I of this paper (Ferrari-Trecate *et al.*, 2005) we considered Laplacian control, where at each time instant the agent input depends linearly on the state of its neighbors. Linear control laws, although appealing for their simplicity do not guarantee, beside alignment, collision avoidance and cohesion. These goals called for the development of nonlinear feedback schemes, where collision avoidance is usually achieved through the use of potential functions. A typical example is provided by the control scheme considered in (Tanner *et al.*, 2003*a*) and inspired by the pioneering work of Reynolds (1987). Several other results on potential field based control schemes are now available in the literature, (Gazi and Passino, 2003), (Tanner *et al.*, 2003*b*), (Saber and Murray, 2003), (Saber, 2003).

In all the aforementioned works, the approach adopted is to *(i)* model the communication links through a graph, *(ii)* model the collective dynamics as a single nonlinear system, *(ii)* use graph theory and classic control tools for proving various properties of the resulting system. In particular, the equations describing the collective motion combine together the graph structure and the individual agents dynamics.

The main purpose of this paper is to adopt a new modeling framework for the analysis of multi-agent systems. Our approach exploits the formalism of Partial difference Equations (PdEs) over graphs proposed by Bensoussan and Menaldi (2004) and summarized in Section 2. In order to account for the temporal agent dynamics, we generalize the models of (Bensoussan and Menaldi, 2004) to *continuous-time PdEs.* We argue that the advantage of using PdE models is threefold: *(i)* many mathematical tools for analyzing PdEs are completely analogous to the ones available for PDEs, *(ii)* PdEs provide models where spatial interactions and temporal evolution are kept separated, *(iii)* the PdEs framework leads to equations that may be reminiscent of PDEs arising in physics and this can be of great help for conjecturing sensible properties of the collective dynamics.

The paper is structured as follows. In Section 2 we introduce the class of potential field based control laws considered. They are similar to the ones proposed in (Tanner *et al.*, 2004), the only difference being that nonzero safety distances between pairs of communicating agents are allowed. Section 3 is devoted to a generalization of the LaSalle invariance principle for investigating convergence of solutions to PdEs on suitable subspaces. This constitutes the main technical tool used in Sections 4 and 5 for proving properties of the collective motion. Section 4 focuses on the analysis of leaderless models with agent dynamics perturbed by errors. As shown in Lemma 2, the collective dynamics can be modeled through nonlinear PdEs highlighting that the control action does not influence the average velocity of the group. Roughly speaking, this means that alignment and collision avoidance can be proved by considering only the zero-average components of velocities and errors (see Theorem 3). Finally, in Section 5, leader-follower models are considered. We first model the collective motion as a PdE with boundary conditions and then apply almost the same rationale adopted in the leaderless case for showing alignment to the leader velocity and collision avoidance.

2. THE COLLECTIVE DYNAMICS

We start modeling the communication network between agents in form of a graph. Let G be an undirected graph defined by a nonempty set $\mathcal{N}$ of N *nodes* and a set $\mathcal{E} \subset \mathcal{N}\times\mathcal{N}$ of *edges.* In our case, each node represents an agent and without loss of generality we assume that $\mathcal{N} = \{1, 2, \ldots, N\}$. Two nodes x and y are *neighbors* if $(x, y) \in \mathcal{E}$. This means that the agent x and y share the information about their position and velocity. We use the notation $x \sim y$ for neighboring nodes and assume that $x \sim x$ always holds. Two nodes x and y are connected by a path if there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $x_{i-1} \sim x_i$. The graph G is *connected* when each pair of nodes $(x, y) \in G \times G$ is connected by a path.

Next, we recap basic tools introduced by Bensoussan and Menaldi (2004) and recalled in Part I for functions on graphs $f : \mathcal{N} \to \mathbb{R}^q$. We define partial derivatives and integrals, respectively as

$$\partial_y f(x) \doteq f(y) - f(x)\ \forall\, x, y \in G\,, \quad \int_G f \doteq \sum_{x\in\mathcal{N}} f(x).$$

In the same manner, second order differential operator can be defined on graphs. We are interested in the Laplace operator

$$\triangle f(x) \doteq -\sum_{y\sim x} \partial_y^2 f(x) = +\sum_{y\sim x} \partial_y f(x).$$

We denote by $L^2(G|\mathbb{R}^q)$ the Hilbert space of functions $f : G \to \mathbb{R}^q$ endowed with the norm $\|f\|_{L^2}^2 \doteq \int_G \|f\|^2$. Now, we are in a position to introduce the agent dynamics. Let $r(x, t)$ be the position of the agent x at time t, where $r(\cdot,t) \in L^2(G|\mathbb{R}^q)$. Similarly, the agent velocity, input and errors are denoted with $v(x,t)$, $u(x,t)$, $e(x,t)$. By assuming that each agent obeys to a point-mass dynamics the collective model can be written as

$$\dot{r}(x,t) = v(x,t) \tag{1a}$$
$$\dot{v}(x,t) = u(x,t) + \beta e(x,t), \quad \beta \neq 0 \tag{1b}$$
$$\dot{e}(x,t) = -\alpha e(x,t), \quad \alpha > 0 \tag{1c}$$

equipped with the initial conditions $r(\cdot,0) = \tilde{r} \in L^2$, $v(\cdot,0) = \tilde{v} \in L^2$ and $e(\cdot,0) = \tilde{e} \in L^2$.

The control law $u(x,t)$ we consider has been inspired by the work of Reynolds (1987) and is similar to those studied in (Tanner *et al.*, 2003*a*), (Tanner *et al.*, 2003*b*), (Tanner *et al.*, 2004), (Saber, 2003) and (Saber and Murray, 2003). It order to introduce it, we need to set a few notation.

Definition 1. For $x \sim y$, the potential $V(x,y,\|\partial_y r(x,t)\|^2)$ is a differentiable, nonnegative, radially unbounded function of the squared distance $\|\partial_y r(x,t)\|^2$ such that:

(1) $V(x,y,\|\partial_y r(x,t)\|^2) = V(y,x,\|\partial_x r(y,t)\|^2)$
(2) $V(x,y,\|\partial_y r(x,t)\|^2) \to \infty$ as $\|\partial_y r(x,t)\|^2 \to \bar{r}_{xy}^2$ where $\bar{r}_{xy} \geq 0$ represent given *safety distances*;
(3) for all $x,y \in \mathcal{N}$, $V(x,y,\cdot)$ attains its unique minimum when $\|\partial_y r(x,t)\|^2 = \tilde{r}_{xy}^2$, where $\tilde{r}_{xy} > \bar{r}_{xy}$ are given *desired distances*.

We set

$$\tilde{V}(x,\{\partial_y r\}_{y\in G}) \doteq \sum_{y\sim x} V(x,y,\|\partial_y r(x,t)\|^2) \tag{2}$$
$$U(r(\cdot,t))(x) \doteq \nabla_{r(x,t)}\tilde{V}(x,\{\partial_y r\}_{y\sim x}). \tag{3}$$

The control $u(x,t)$ is then chosen as

$$u(x,t) \doteq \triangle v - U(r(\cdot,t)) \tag{4}$$

and it will be termed "elastic control". Roughly speaking, elastic control mimic nonlinear elasticity phenomena among agents, where "large deformations" correspond to pairs of agents too close or too far away and produce repulsive and attractive forces, respectively. Intuitively, the divergence at infinity of V, as $\|\partial_y r(x,t)\| \to \bar{r}_{xy}$, prevents the distance between neighboring agents from falling below $\bar{r}_{xy}$.

We conclude this Section by justifying the error model (1c). Consider the following *uncertain* agent model

$$\dot{r}(x,t) = v(x,t) \tag{5a}$$
$$\dot{v}(x,t) = \epsilon v(x,t) + u(x,t) + \tilde{u}(x,t) \tag{5b}$$

where $\epsilon \in \mathbb{R}\backslash\{0\}$ represents an unknown perturbation coefficient and $\tilde{u}(x,t)$, is an internal feedback action, i.e. $\tilde{u}$ depends on $r(x,t)$ and $v(x,t)$ only. In Part I, we showed that if u is the Laplacian control and $\tilde{u} = 0$, alignment to a nonzero velocity cannot be achieved. Similarly, in Remark 1 we show that alignment is compromised even when elastic control is adopted and $\tilde{u} = 0$. Then, analogously to the case of Laplacian control, an internal feedback must be designed to compensate for perturbations. As recalled in Part I and detailed in (Ferrari-Trecate *et al.*, 2004), model (1) results from (5) for suitable choices of $\tilde{u}$.

3. LASALLE INVARIANCE PRINCIPLE

Let $z(x,t) : \mathcal{N} \times \mathbb{R}^+ \mapsto \mathbb{R}^q$ be a function of two variables and consider the initial value problem

$$\dot{z}(x,t) = F(z(\cdot,t)) \tag{6a}$$
$$z(x,0) = \tilde{z}(x) \tag{6b}$$

where $F : L^2(G|\mathbb{R}^q) \mapsto L^2(G|\mathbb{R}^q)$ is a continuous operator. We call the equality (6a) a continuous-time PdE with initial conditions (6b) and refer to $z(x,t)$ as the *state* of the PdE. For the existence and uniqueness of solutions to (6) we defer the reader to Part I.

Next, we consider the problem of characterizing the convergence of solutions to (6). In particular, we are interested in the effect of perturbations on the *projection* of $z(x,t)$ on suitable subspaces. Consider a subspace $\mathcal{V} \subset L^2(G|\mathbb{R}^q)$ and denote with $f_{\mathcal{V}} = P_{\mathcal{V}} f$ the projection of $f \in L^2(G|\mathbb{R}^q)$ on $\mathcal{V}$.

Definition 2. A set $\Omega \subset \mathcal{V}$ is positively $\mathcal{V}$-invariant with respect to (6a) if $\tilde{z}_{\mathcal{V}} \in \Omega \Rightarrow z_{\mathcal{V}}(\cdot,t) \in \Omega$, $\forall t \geq 0$

Now, we are in a position to introduce the LaSalle invariance principle on subspaces.

Theorem 1. Assume that there exists a unique solution to (6), $\forall t \geq 0$ and that $P_{\mathcal{V}}F = FP_{\mathcal{V}}$. Let $\Omega \subset \mathcal{V}$ be a positively $\mathcal{V}$-invariant compact set in L^2 and let $W : \mathcal{V} \mapsto \mathbb{R}$ be a continuously differentiable functional verifying $\dot{W}(z_{\mathcal{V}}(x,t)) \leq 0$. Consider the set $E = \{v \in \mathcal{V} : \dot{W}(v) = 0\}$. Then, for every initial condition verifying $\tilde{z}_{\mathcal{V}} \in \Omega$, the projected solution $z_{\mathcal{V}}(x,t)$ approaches E, i.e.

$$\lim_{t\to\infty} \inf_{p\in E} \|z_{\mathcal{V}}(\cdot,t) - p\|_{L^2} = 0.$$

The proof of Theorem 1 can be found in (Ferrari-Trecate *et al.*, 2004). The main difference between Theorem 1 and the standard LaSalle principle is the additional assumption that the projection on $\mathcal{V}$ and the operator F *commute*. As detailed in (Ferrari-Trecate *et al.*, 2004), this allows to give a meaningful definition of invariance for the limit set associated to a solution to the PdE. Note also that the condition $P_{\mathcal{V}}F = FP_{\mathcal{V}}$, implies that $\mathcal{V}$ is positively invariant with respect to (6).

4. COORDINATION IN LEADERLESS MODELS

As in Part I, we introduce now the "Sobolev" space $H^1(G|\mathbb{R}^q)$ composed by all functions in $L^2(G|\mathbb{R}^q)$ with zero average. We denote by $H^1_\perp$ the orthogonal supplement (with respect to the L^2 scalar product) of H^1, i.e. the space of constant functions over the graph G. We recall Theorem 1 of Part I.

Theorem 2. The operator $\triangle : H^1 \mapsto H^1$ has $(N-1)q$ strictly negative eigenfunctions and the corresponding eigenvectors form a basis for H^1.

The next Lemma, proved in (Ferrari-Trecate *et al.*, 2004), shows that both $\triangle$ and U do not affect the average velocity of the formation.

Lemma 1. For all $r(\cdot,t)$, $v(\cdot,t) \in L^2$, the functions $U(r(\cdot,t))$ and $\triangle v(\cdot,t)$ belong to H^1.

Note that the previous statement means exactly that:

$$\int_G (\triangle v)^T c = \int_G U(r(x,t))^T c \equiv 0 \quad \forall\, c \in H^1_\perp.$$

Remark 1. Lemma 1 can be used for showing that if the velocity dynamics is affected by a persistent perturbation, as in (5b) for $\tilde{u}(x,t)=0$, alignment with nonzero velocity cannot be achieved. In fact, equation (5b) can be recast into the PdE

$$\dot{v} = (\epsilon + \triangle)v - U(r(\cdot,t)). \tag{7}$$

Assume, by contradiction, that the alignment condition holds, i.e. there exists $\bar{v} \in H^1_\perp$, $\bar{v} \neq 0$, verifying

$$0 = (\epsilon + \triangle)\bar{v} - U(r(\cdot,t)). \tag{8}$$

Since $\triangle \bar{v} = 0$, equation (8) reduces to $\epsilon \bar{v} = U(r(\cdot,t))$. However, in view of Lemma 1, we have that $U(r(\cdot,t)) \in H^1$ and the last identity can not hold.

The next aim is to show that $u(x,t)$ guarantees alignment and collision avoidance. We introduce the safety set

$$\mathcal{R} \doteq \left\{ r \in L^2 : \|\partial_y r(x)\| > \bar{r}_{xy}, \forall (x,y) \in \mathcal{E} \right\}. \tag{9}$$

Collision avoidance is ensured as soon as $\mathcal{R}$ is an invariant set for the formation dynamics.

As in Part I, we consider the decompositions $v = v_1 + \bar{v}$, $e = e_1 + \bar{e}$, with $v_1,\ e_1 \in H^1$, and $\bar{v}, \bar{e} \in H^1_\perp$. The collective dynamics (1-4) can be split according to the following Lemma.

Lemma 2. The dynamics (1-4) are equivalent to

$$\partial_y \dot{r} = \partial_y v = \partial_y v_1 \ \forall y \in G \tag{10a}$$
$$\dot{v} = \triangle v - U(r(\cdot,t)) + \beta e \tag{10b}$$
$$\dot{e} = -\alpha e. \tag{10c}$$

and also to

$$\Sigma_{1,R} : \begin{cases} \partial_y \dot{r} = \partial_y v_1 \ \forall y \in G \\ \dot{v}_1 = \triangle v_1 - U(r(\cdot,t)) + \beta e_1 \\ \dot{e}_1 = -\alpha e_1 \end{cases} \quad \bar{\Sigma}_R : \begin{cases} \dot{\bar{v}} = \beta \bar{e} \\ \dot{\bar{e}} = -\alpha \bar{e} \end{cases} \tag{11}$$

equipped with the initial conditions $\partial_y r = \partial_y \tilde{r}$, $v_1(\cdot,0) = P_{H^1}\tilde{v}$, $e_1(\cdot,0) = P_{H^1}\tilde{e}$, $\bar{v}(0) = P_{H^1_\perp}\tilde{v}$ and $\bar{e}(0) = P_{H^1_\perp}\tilde{e}$.

Proof: The dynamics of $\partial_y r(\cdot,t)$ is obtained directly from (1a), realizing that $\partial_y \bar{v} \equiv 0\ \forall y \in G$. The splitting (11) is a consequence of Lemma 1 (see (Ferrari-Trecate *et al.*, 2004) for a detailed proof). ■

The PdE $\bar{\Sigma}_R$ shows that elastic control does not influence the average velocity of the group. Moreover, $\bar{v}$ converges to a value that depends only on the errors and initial conditions. Similarly to the case of Laplacian control (discussed in Part I), it follows that the problem of checking alignment is reduced to the problem of checking the convergence of v_1 to zero, as $t \to +\infty$.

For proving alignment and collision avoidance we consider the subspace

$$\mathcal{V}_R \doteq \Big\{ \left[\, \partial_1 r^T \ldots \partial_N r^T \ v_1^T \ e_1^T \,\right]^T \in L^2(G|\mathbb{R}^{q(N+2)}) \text{ with } v_1 \in H^1, e_1 \in H^1 \Big\}. \tag{12}$$

along with the energy $W_R : \mathcal{V}_R \mapsto \mathbb{R}$,

$$W_R(\{\partial_i r\}_{i=1}^N, v_1, e_1) = \frac{1}{2}\int_G \tilde{V}(x, \{\partial_y r\}_{y\in G}) + \frac{1}{2}\|v_1\|^2_{L^2} + \frac{\gamma}{2}\|e_1\|^2_{L^2} \tag{13}$$

where $\gamma > 0$ is a parameter and $\tilde{V}$ has been defined in (2). The next Theorem, inspired by the results of (Tanner *et al.*, 2003*a*) and (Tanner *et al.*, 2004), exploits the LaSalle invariance principle, stated in Theorem 1, on the subspace $\mathcal{V}_R$.

Theorem 3. Assume that the initial positions satisfy the collision avoidance condition $r(\cdot,0) \in \mathcal{R}$. Then,

(1) $r(\cdot,t) \in \mathcal{R}$, $\forall t \geq 0$ (collision avoidance at all times).
(2) $v_1 \to 0$ as $t \to \infty$ (alignment).

Proof: Let $\delta = W_R\left(\left\{\tilde{d}_i\right\}_{i=1}^N, P_{H^1}\tilde{v}, P_{H^1}\tilde{e}\right)$ and consider the set

$$\Omega_\delta = \{\xi \in \mathcal{V}_R : W_R(\xi) \leq \delta\}. \tag{14}$$

Applying exactly the same argument used in the proof of (Tanner *et al.*, 2004, Theorem 1), one can show that Ω_δ is a compact set. Now, we need to compute $\dot{W}_R$ and prove that $\dot{W}_R \leq 0$. We exploit the identity:

$$\frac{1}{2}\frac{d}{dt}\int_G \tilde{V}(x, \{\partial_y r\}_{y\in G}) = \int_G v^T U(r(\cdot,t)). \quad (15)$$

and Lemma 1 to obtain:

$$\dot{W}_R = \int_G v_1^T(\triangle v_1 + \beta e_1) - \alpha\gamma \int_G \|e_1\|^2.$$

Simple algebraic manipulations show that $\dot{W}_R \leq 0$ if the parameter γ is chosen big enough (see (Ferrari-Trecate *et al.*, 2004) for further details). As a consequence, the set Ω_δ is positively $\mathcal{V}$-invariant. The implication

$$\begin{aligned} &\left[\, \partial_1 r^T \ \cdots \ \partial_N r^T \ v_1^T \ e_1^T \,\right]^T \in \Omega_\delta, \ \forall t \geq 0 \\ &\quad \Rightarrow r(\cdot,t) \in \mathcal{R}, \ \forall t \geq 0, \end{aligned} \quad (16)$$

can be easily checked by contradiction.

The $\mathcal{V}_R$-invariance of Ω_δ implies also that v_1 is bounded, $\forall t \geq 0$. Then, the solution to the PdE (10) is uniquely defined, $\forall t \geq 0$. Indeed, looking at (11), it is apparent that only v_1 could have a finite escape time but this cannot happen because v_1 stays bounded.

Now, we apply Theorem 1 to conclude. To this purpose, we verify that $P_{\mathcal{V}_R}F = FP_{\mathcal{V}_R}$. The operator F corresponding to the PdE (11) is given by

$$F(z) \doteq \left[\, \partial_1 v^T \cdots \partial_N v^T \, (\triangle v - U + \beta e)^T \, {-\alpha e^T} \,\right]^T \quad (17)$$

and the equality $P_{\mathcal{V}_R}F = FP_{\mathcal{V}_R}$ is an easy consequence of Lemma 2. The set E considered in Theorem 1 is given by

$$E = \Big\{ \left[\, \partial_1 r^T \cdots \partial_N r^T v_1^T e_1^T \,\right]^T \in L^2(G|\mathbb{R}^{q(N+2)}) : v_1 = 0 \text{ and } e_1 = 0 \Big\}. \quad (18)$$

and the fact that that $v_1 \to 0$ and $e_1 \to 0$, as $t \to \infty$, follows. ∎

Theorem 3 agrees with the results of (Tanner *et al.*, 2004) and (Tanner *et al.*, 2003*a*) where alignment and collision avoidance for elastic control with *zero* safety distances have been proved in absence of errors. Note that elastic control does not guarantee cohesion, i.e. that

$$\lim_{t\to\infty} \|\partial_y r(x,t)\|^2 = \tilde{r}_{xy}, \quad \forall (x,y) \in \mathcal{E}. \quad (19)$$

In fact, since the desired distances $\tilde{r}_{xy}$ are arbitrary, a necessary condition for cohesion is that there exists a function $r \in L^2$ fulfilling the conditions

$$\|\partial_y r(x)\|^2 = \tilde{r}_{xy}, \quad \forall (x,y) \in \mathcal{E}. \quad (20)$$

In (Tanner *et al.*, 2004) it has been proved that, in the errorless case, if the graph has a tree structure, then equation (20) can be always solved and it implies cohesion. The study of the solvability of (20) for general connected graphs, and the conjecture that (20) is sufficient for achieving cohesion, are still open issues.

5. COORDINATION IN LEADER-FOLLOWER MODELS

In this Section we use PdEs for analyzing the collective motion of the agents in presence of a leader. By leader, we mean a vehicle that moves with a prescribed constant velocity, independently of the motion of all other vehicles. However, followers connected to the leader use information on the leader state in order to compute their control inputs.

Let S be a subgraph of the connected graph G and let the boundary of S be defined by: $\partial S \doteq \{y \in G \setminus S \ : \ \exists x \in S \ : \ x \sim y\}$. The leader and the follower are indexed by the nodes of ∂S and S respectively. Since we assume that the leader is unique, we have $\partial S = \{x_L\}$. The closure of S is given by $\bar{S} \doteq S \cup \partial S = G$.

As in (Bensoussan and Menaldi, 2004), define the space $H_0^1(S) \doteq \{u \in L^2(\bar{S}) \ : \ u_{|\partial S} = 0\}$, equipped with the norm

$$\|f\|^2_{H_0^1} = \sum_{x\in\mathcal{N}} \sum_{y\sim x} \|\partial_y f(x)\|^2. \quad (21)$$

The next Theorem, proved in (Bensoussan and Menaldi, 2004), summarizes the key property of the Laplacian on $H_0^1(S)$.

Theorem 4. Let G be a connected graph. Then, the operator $\triangle : H_0^1(S|\mathbb{R}^q) \mapsto L^2(\bar{S}|\mathbb{R}^q)$ has $|S|q$ strictly negative eigenvalues where $|S|$ denotes the number of nodes of S. Moreover, the corresponding eigenfunctions form a basis for $H_0^1(S|\mathbb{R}^q)$.

Suppose that the leader x_L has a constant velocity v_L. Let, by abuse of notation $v_L(x) := v_L$ for all $x \in \bar{S}$. Note that $\Delta v_L = 0$, because $v_L \in H_\perp^1(\bar{S})$. It turns out that the agents velocity $v \in L^2(\bar{S})$ can be written as

$$v = v_0 + v_L, \quad v_0 \in H_0^1(S) \quad (22)$$

and alignment (to the leader velocity) corresponds to the condition $v_0 \to 0$ as $t \to \infty$.

When the followers obeys to the elastic control, the collective dynamics (1) can be directly recast into the *Dirichlet boundary value problem*

$$\begin{aligned} \partial_y \dot{r} &= \partial_y v_0 & x \in S, \ y \in G \quad &\text{(23a)} \\ \dot{v}_0 &= \triangle v_0 - U(r(\cdot,t)) + \beta e & x \in S \quad &\text{(23b)} \\ \dot{e} &= -\alpha e & x \in S \quad &\text{(23c)} \\ v_0 &= 0 & x \in \partial S \quad &\text{(23d)} \end{aligned}$$

with the initial conditions $\partial_y r(\cdot,0) = \partial_y \tilde{r} \in L^2$, $\forall y \in G$, $v_0(\cdot,0) = \tilde{v}_0 \in L^2$, $e(\cdot,0) = \tilde{e} \in L^2$. For a given v_L, equations (23) define a PdE with state $z \in \mathcal{V}_{LR}$, where

$$\mathcal{V}_{LR} \doteq \{ \left[\{\partial_i r^T\}_{1\leq i\leq N} \; v_0^T \; e^T \right]^T \in L^2(G|\mathbb{R}^{q(N+2)}) : v_0 \in H_0^1(S)\}. \tag{24}$$

The fact that $v_0 \to 0$ as $t \to \infty$ can be proved by exploiting LaSalle invariance principle. To this purpose we consider the energy $W_{LR} : \mathcal{V}_{LR} \mapsto \mathbb{R}$ defined as

$$W_{LR}\left(\{\partial_i r\}_{i=1}^N, v_0, e_0\right) = \frac{1}{2}\int_{\bar{S}} \tilde{V}(x, \{\partial_y r\}_{y\in G}) + \frac{1}{2}\|v_0\|^2_{L^2(S)} + \frac{\gamma}{2}\|e\|^2_{L^2(S)}. \tag{25}$$

Theorem 5. Assume that the initial positions verify the collision avoidance condition $r(\cdot,0) \in \mathcal{R}$. Then,

(1) $r(\cdot,t) \in \mathcal{R}$, $\forall t \geq 0$ (collision avoidance at all times);
(2) $v_0 \to 0$ as $t \to \infty$ (alignment to the leader velocity).

Proof: The proof is detailed in (Ferrari-Trecate *et al.*, 2004) and is similar to the one of Theorem 3, by replacing v_1 and e_1 with v_0 and e_0, respectively. ■

6. DISCUSSION AND CONCLUSIONS

In this second part of the paper we exploited the framework of continuous-time PdEs for analyzing coordination phenomena in multi-agent systems using potential field based control laws. We showed that elastic control guarantees collision avoidance and alignment both in leaderless and leader-follower models even when the agent dynamics is perturbed by exponentially decreasing errors.

Generally speaking, we believe that PdEs provide a useful mathematical framework even when dealing with *(i)* more complex agent models accounting for the effects of various perturbations (e.g. stochastic effect of wind on the motion of aerial vehicles or communication delays) *(ii)* more complex control laws guaranteeing also obstacle avoidance (Saber and Murray, 2003) *(iii)* time-varying communication links (Tanner *et al.*, 2003*b*), (Jadbabaie *et al.*, 2003).

Moreover, the profound similarity between PdEs and PDEs describing physical phenomena can be inspiring for devising new decentralized control schemes. As an example, linear and nonlinear elasticity models might be inspiring for designing distributed control laws regulating geometric features of the formation.

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