# On the discrete-time integral sliding mode control

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# On the Discrete-Time Integral Sliding Mode Control

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*Abstract*—**A new discrete-time integral sliding mode control (DISMC) scheme is proposed for sampled-data systems. The new control scheme is characterized by a discrete-time integral switching surface which inherits the desired properties of the continuous-time integral switching surface, such as full order sliding manifold with eigenvalue assignment, and elimination of the reaching phase. In particular, comparing with existing discrete-time sliding mode control, the new scheme is able to achieve more precise tracking performance. It will be shown in this work that, the new control scheme achieves $O(T^2)$ steady-state error for state regulation with the widely adopted delay-based disturbance estimation. Another desirable feature is, the proposed DISMC prevents the generation of overlarge control actions, which are usually inevitable due to the deadbeat poles of a reduced order sliding manifold designed for sampled-data systems. Both the theoretical analysis and illustrative example demonstrate the validity of the proposed scheme.**

## I. INTRODUCTION

Research in discrete-time control has intensified in recent years. A primary reason is that most control strategies nowadays are implemented in discrete-time. This also necessitated a rework in the sliding mode control strategy for sampled-data systems. In such systems, the switching frequency in control variables is limited by $T^{-1}$; where $T$ is the sampling period. Although high-frequency switching is theoretically desirable from the robustness point of view, it is usually hard to achieve in practice because of physical constraints, such as processor computational speed, A/D and D/A conversion delays, actuator bandwidth, etc. The use of a discontinuous control law in a sampled-data system will bring about chattering phenomenon in the vicinity of the sliding manifold, hence lead to a boundary layer with thickness $O(T)$, [1].

In [2] a discrete-time equivalent control in the prescribed boundary layer, whose size is defined by the restriction applied to the control variables is proposed. This approach results in the motion in $O(T^2)$ vicinity of the sliding manifold. The main difficulty in implementation of this control law is that we need to know the disturbances for calculating the equivalent control. Lack of such information leads to an $O(T)$ error boundary.

Control proposed in [1] drives the sliding mode to $O(T^2)$ in one-step owing to the existence of deadbeat poles in the closed-loop system. State regulation was not considered in [1]. In fact, as far as the state regulation is concerned, the same SMC design will produce an accuracy in $O(T)$ instead of $O(T^2)$ boundary. Moreover, the SMC with deadbeat poles requires large control efforts which might be undesirable in practice. Introducing saturation in the control input endangers the global stability or accuracy of the closed-loop system.

In this work, aiming at improving control performance for sampled-data systems, a discrete-time integral switching surface (ISM) is proposed. With the full control of the system closed-loop poles and the elimination of the reaching phase, the closed-loop system can reach the desired control performance characteristics while avoiding the generation of large control inputs. It is worth highlighting that the discrete-time ISM control can not only drive the sliding mode into the $O(T^2)$ boundary, but also achieve the $O(T^2)$ boundary for state regulation.

In this work, eigenvalue assignment of the full-order sliding mode is discussed, as well as, the error dynamics during sliding motion.

The paper is organized as follows: The problem formulation and a revisit of the existing SMC properties in sampled-data systems are presented in Section II. Appropriate discrete-time integral switching surface and SMC design for state regulation will be presented in Section III. An illustrative examples demonstrating the validity of the proposed scheme is shown in Section IV. Section V gives the conclusions.

## II. PROBLEM FORMULATION

### A. Sampled-Data System

Consider the following continuous-time system with a nominal linear time invariant model and matched disturbance

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B(\mathbf{u}(t) + \mathbf{f}(t)) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases} \tag{1}$$

where the state $\mathbf{x} \in \Re^n$, the output $\mathbf{y} \in \Re^m$, the control $\mathbf{u} \in \Re^m$, and the disturbance $\mathbf{f} \in \Re^m$ is assumed smooth and bounded. The discretized counterpart of (1) can be given by

$$\begin{cases} \mathbf{x}_{k+1} = \Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k = C\mathbf{x}_k \end{cases} \tag{2}$$

where

$$\Phi = e^{AT}, \quad \Gamma = \int_0^T e^{A\tau} d\tau B$$

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau)d\tau,$$

and $T$ is the sampling period. Here the disturbance $\mathbf{d}_k$ represents the influence accumulated from $kT$ to $(k+1)T$, in the sequel it shall directly link to $\mathbf{x}_{k+1} = \mathbf{x}((k+1)T)$. From the definition of $\Gamma$ it can be shown that

$$\Gamma = BT + \frac{1}{2!}ABT^2 + \cdots = BT + MT^2 + O(T^3)$$

$$\Rightarrow BT = \Gamma - MT^2 + O(T^3) \tag{3}$$

where $M$ is a constant matrix because $T$ is fixed. To proceed further, the following definition is necessary:

**Definition:** *The magnitude of a variable $v$ is said to be $O(T^n)$ if and only if*

$$\lim_{T \to 0} \frac{v}{T^n} \neq 0 \quad and \quad \lim_{T \to 0} \frac{v}{T^{n-1}} = 0$$

where $n$ is an integer. Denote $O(T^0) = O(1)$.

*Remark 1:* Note that $O(T^n)$ can be single valued function or a vector valued function.
Associated with the above definition the following assumption is made.

**Assumption:** *The sampling interval is sufficiently small such that $v_1 \in O(T^n)$ and $v_2 \in O(T^{n+1})$ gives $v_1 \gg v_2$. Therefore, the following relations exist about the effective approximation*

$$O(T^{n+1}) + O(T^n) \approx O(T^n) \quad \forall n \in Z$$

$$O(T^n) \cdot O(1) \approx O(T^n) \quad \forall n \in Z$$

$$O(T^n) \cdot O(T^{-m}) \approx O(T^{n-m}) \quad \forall n, m \in Z$$

where $\approx$ stands for the effective approximation and $Z$ is the set of integers.

Based on (3) and the Definition, the magnitude of $\Gamma$ is $O(T)$.

Note that, as a consequence of sampling, the disturbance originally matched in continuous-time will contain mismatched components in the sampled-data system. This is summarized in the following lemma.

*Lemma 1:* If the disturbance $\mathbf{f}(t)$ in (1) is bounded and smooth, then

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau)d = \Gamma \mathbf{f}_k + \frac{1}{2}\Gamma \mathbf{v}_k T + O(T^3) \tag{4}$$

where $\mathbf{v}_k = \mathbf{v}(kT)$, $\mathbf{v}(t) = \frac{d}{dt}\mathbf{f}(t)$, $\mathbf{d}_k - \mathbf{d}_{k-1} \in O(T^2)$, and $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} \in O(T^3)$.

*Proof:* See appendix.

Note that the magnitude of the unmatched part in the disturbance $\mathbf{d}_k$ is of the order $O(T^3)$

The control objective is, for the sampled-data system (2), design a discrete-time integral switching surface and a discrete-time SMC law to acheive as precisely as possible state regulation. Meanwhile the closed-loop dynamics of the sampled-data system has all its closed-loop poles assigned to desired locations. It will be shown that the proposed scheme achieves a steady-state error of order $O(T^2)$ for state regulation.

*B. Discrete-Time Sliding Mode Control Revisited*

Consider the well established discrete-time sliding-surface given below [1]-[2]

$$\boldsymbol{\sigma}_k = D\mathbf{x}_k \tag{5}$$

where $D$ is a constant matrix of rank $m$. The objective is to steer the states towards and stay on the switching surface $\boldsymbol{\sigma}_k = 0$ at every sampling instant. The property for this class of sampled-data SMC is given by the following lemma.

*Lemma 2:* For $\boldsymbol{\sigma}_k = D\mathbf{x}_k$ and control based on a disturbance estimate

$$\hat{\mathbf{d}}_k = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1},$$

the closed-loop system has the following properties

$$\lim_{k \to \infty} \|\mathbf{x}_k\| \leq O(T)$$

*Proof:* Discrete-time equivalent control is defined by solving $\boldsymbol{\sigma}_{k+1} = 0$, [1]. This leads to

$$\mathbf{u}_k^{eq} = -(D\Gamma)^{-1}D(\Phi\mathbf{x}_k + \mathbf{d}_k) \tag{6}$$

with $D$ selected such that the closed-loop system achieves desired performance and $D\Gamma$ is invertible, [3]. Under practical considerations, the control cannot be implemented in the same form as in (6) because of the lack of knowledge of $\mathbf{d}_k$ which requires *a priori* knowledge of the disturbance $\mathbf{f}(t)$. However, with some continuity assumptions on the disturbance, $\mathbf{d}_k$ can be estimated by its previous value $\mathbf{d}_{k-1}$, [1]. The substitution of $\mathbf{d}_k$ by $\mathbf{d}_{k-1}$ will at most result in an error of $O(T^2)$. Let

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1} \tag{7}$$

where $\hat{\mathbf{d}}_k$ is the estimate of $\mathbf{d}_k$. Thus, the control law is modified to be

$$\mathbf{u}_k = -(D\Gamma)^{-1}D(\Phi\mathbf{x}_k + \mathbf{d}_{k-1}). \tag{8}$$

Applying this to the sampled-data system and using the conclusions in $Lemma\ 1$, yields

$$\boldsymbol{\sigma}_{k+1} = D(\Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k) = D(\mathbf{d}_k - \mathbf{d}_{k-1}) = O(T^2) \tag{9}$$

which is the result shown in [1]. The closed-loop dynamics is

$$\mathbf{x}_{k+1} = \left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)\mathbf{x}_k$$
$$+ \left(I - \Gamma(D\Gamma)^{-1}D\right)\mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1} \tag{10}$$

where the matrix $\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)$ has $m$ zero eigenvalues and $n - m$ eigenvalues to be assigned inside the unit disk in the complex *z-plane*. It is possible to simplify (10) further to

$$\mathbf{x}_{k+1} = \left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)\mathbf{x}_k + \boldsymbol{\delta}_k \tag{11}$$

where $\boldsymbol{\delta}_k = \left(I - \Gamma(D\Gamma)^{-1}D\right)\mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1}$. From Lemma 1,

$$\boldsymbol{\delta}_k = \mathbf{d}_k - \mathbf{d}_{k-1}$$
$$+ (I - \Gamma(D\Gamma)^{-1}D)\left(\Gamma\mathbf{f}_{k-1} + \frac{1}{2}\Gamma\mathbf{v}_{k-1}T + O(T^3)\right)$$
$$= O(T^2) + (I - \Gamma(D\Gamma)^{-1}D)O(T^3) \in O(T^2). \tag{12}$$

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In the above derivation, we use the relationship $O(1) \cdot O(T^3) \approx O(T^3)$, and the fact $\|I - \Gamma(D\Gamma)^{-1}D\| \le 1$. Note that since $m$ eigenvalues of $(\Phi - \Gamma(D\Gamma)^{-1}D\Phi)$ are deadbeat, it can be written as

$$\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right) = PJP^{-1} \tag{13}$$

where $P$ is a transformation matrix and $J$ is the Jordan matrix of the eigenvalues of $(\Phi - \Gamma(D\Gamma)^{-1}D\Phi)$. The matrix $J$ can be written as

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \tag{14}$$

where $J_1 \in \Re^{m \times m}$ and $J_2 \in \Re^{(n-m) \times (n-m)}$ and are given by

$$J_1 = \begin{bmatrix} \mathbf{0} & I_{m-1} \\ 0 & \mathbf{0} \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where $\lambda_j$ are the eigenvalues of $(\Phi - \Gamma(D\Gamma)^{-1}D\Phi)$. For simplicity it is assumed that the non-zero eigenvalues are designed to be distinct and that their continuous time counterparts are of order $O(1)$. Then the solution of (11) is

$$\mathbf{x}_k = PJ^kP^{-1}\mathbf{x}(0) + P\left(\sum_{i=0}^{k-1} J^i P^{-1}\boldsymbol{\delta}_{k-i-1}\right). \tag{15}$$

Rewritting (15)

$$\mathbf{x}_k = PJ^kP^{-1}\mathbf{x}(0) + P\left(\sum_{i=0}^{k-1} \begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1}\boldsymbol{\delta}_{k-i-1}\right)$$

$$+ P\left(\sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i \end{bmatrix} P^{-1}\boldsymbol{\delta}_{k-i-1}\right), \tag{16}$$

it is easy to verify that $J_1^i = \mathbf{0}$ for $i \ge m$. Thus, (16) becomes (for $k \ge m$)

$$\mathbf{x}_k = PJ^kP^{-1}\mathbf{x}(0) + P\left(\sum_{i=0}^{m} \begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1}\boldsymbol{\delta}_{k-i-1}\right)$$

$$+ \left(\sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i \end{bmatrix} P^{-1}\boldsymbol{\delta}_{k-i-1}\right). \tag{17}$$

Notice $\|J_1\| = 1$ and $\|J_2\| = \lambda_{max} = max\{\lambda_{m+1}, \cdots, \lambda_n\}$ ($\|\cdot\|$ indicates $\|\cdot\|_2$). Hence, from (17)

$$\lim_{k \to \infty} \|\mathbf{x}_k\| \le \|P\| \left(\sum_{i=0}^{m} \left\| \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|^i \|P^{-1}\|\|\boldsymbol{\delta}_{k-i-1}\| \right.$$

$$\left. + \sum_{i=0}^{k-1} \left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \right\|^i \|P^{-1}\|\|\boldsymbol{\delta}_{k-i-1}\| \right). \tag{18}$$

Since $\lambda_{max} < 1$ for a stable system,

$$\sum_{i=0}^{\infty} \|J_2\|^i = \frac{1}{1 - \lambda_{max}}, \quad \sum_{i=0}^{m} \|J_1\|^i = m.$$

Using Tustin's approximation

$$\lambda_{max} = \frac{2 + Tp}{2 - Tp} \Rightarrow \frac{1}{1 - \lambda_{max}}$$

$$= \frac{1}{1 - \frac{2+Tp}{2-Tp}} = \frac{2 - Tp}{-2Tp} \in O(T^{-1}) \tag{19}$$

where $p$ is the corresponding pole in continuous time and is assumed to be of order $O(1)$. Assuming $m \in O(1)$, and using the fact $\|P^{-1}\| = \|P\|^{-1}$, it can be derived from (18) that

$$\lim_{k \to \infty} \|\mathbf{x}_k\| \le O(1) \cdot O(T^2) + O(T^{-1}) \cdot O(T^2) = O(T). \tag{20}$$

$\blacksquare$

*Remark 2:* The SMC in [1] guarantees that the switching surface is of order $O(T^2)$, but cannot guarantee the same order of magnitude of steady-state errors for the ststem state variables. In the next section, we show that an integral sliding mode design can achieve a more precise state regulation.

## III. STATE REGULATION WITH ISM

Consider the sampled-data system defined by (2) and a new discrete-time integral sliding-surface defined below

$$\begin{aligned} \boldsymbol{\sigma}_k &= D\mathbf{x}_k - D\mathbf{x}(0) + \varepsilon_k \\ \varepsilon_k &= \varepsilon_{k-1} + E\mathbf{x}_{k-1} \end{aligned} \tag{21}$$

where $\varepsilon_k \in \Re^m$. This is the discrete-time counterpart of the following switching surface [5]

$$\boldsymbol{\sigma}(t) = D\mathbf{x}(t) - D\mathbf{x}(0) + \int_0^t E\mathbf{x}(\tau)d\tau = 0 \tag{22}$$

where $\boldsymbol{\sigma}(t) \in \Re^m$; $D$ and $E$ are constant matrices that provide two degrees of design freedom. The term $D\mathbf{x}(0)$ is used to eliminate the reaching phase.

*Theorem 1:* For the system (2) and the proposed switching surface (21), suppose that the pair $(\Phi, \Gamma)$ is controllable. Then there exist constant matrices $K$, $D\Gamma$ invertible, and

$$E = -D\left(\Phi - I + \Gamma K\right) \tag{23}$$

such that the closed-loop equation for system is

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K)\mathbf{x}_k + \boldsymbol{\zeta}_k \tag{24}$$

and $\boldsymbol{\zeta}_k \in \Re^n$ is $O(T^3)$ for the system with disturbance compensation (7). Furthermore, $\lim_{k \to \infty} \|\mathbf{x}_k\| \le O(T^2)$.

*Proof:* Consider a forward expression of (21)

$$\begin{aligned} \boldsymbol{\sigma}_{k+1} &= D\mathbf{x}_{k+1} - D\mathbf{x}(0) + \varepsilon_{k+1} \\ \varepsilon_{k+1} &= \varepsilon_k + E\mathbf{x}_k \end{aligned} \tag{25}$$

Substituting $\varepsilon_{k+1}$ and (2) into the expression of the switching surface in (25) leads to

$$\boldsymbol{\sigma}_{k+1} = (D\Phi + E)\mathbf{x}_k + D(\Gamma\mathbf{u}_k + \mathbf{d}_k) + \varepsilon_k - D\mathbf{x}(0). \tag{26}$$

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The equivalent control is found by solving for $\boldsymbol{\sigma}_{k+1} = 0$. Thus,

$$\mathbf{u}_k^{eq} = (D\Gamma)^{-1}D\mathbf{x}(0) - (D\Gamma)^{-1}\left((D\Phi + E)\mathbf{x}_k + D\mathbf{d}_k + \boldsymbol{\varepsilon}_k\right). \tag{27}$$

Similar to the classical case with control given by (6), implementation of (27) would require *a priori* knowledge of the disturbance $\mathbf{f}(t)$. Hence, $\mathbf{d}_k$ in (27) will be replaced by its estimate $\hat{\mathbf{d}}_k$ defined by (7). Therefore, the control is becomes

$$\mathbf{u}_k = (D\Gamma)^{-1}D\mathbf{x}(0) - (D\Gamma)^{-1}\left((D\Phi + E)\mathbf{x}_k + D\hat{\mathbf{d}}_k + \boldsymbol{\varepsilon}_k\right) \tag{28}$$

Substitution of $\mathbf{u}_k$ defined by (28) into (2) leads to the closed-loop equation during sliding mode,

$$\mathbf{x}_{k+1} = [\Phi - \Gamma(\Gamma D)^{-1}(D\Phi + E)]\mathbf{x}_k - \Gamma(D\Gamma)^{-1}\boldsymbol{\varepsilon}_k$$

$$+ \Gamma(D\Gamma)^{-1}D\mathbf{x}(0) + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\hat{\mathbf{d}}_k. \tag{29}$$

Solving $\boldsymbol{\varepsilon}_k$ in (21) in terms of $\mathbf{x}_k$ and $\boldsymbol{\sigma}_k$

$$\boldsymbol{\varepsilon}_k = \boldsymbol{\sigma}_k - D\mathbf{x}_k + D\mathbf{x}(0). \tag{30}$$

and substituting into (29) the following closed-loop dynamics is obtained

$$\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D(\Phi - I) + E)\right]\mathbf{x}_k$$

$$- \Gamma(D\Gamma)^{-1}\boldsymbol{\sigma}_k + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\hat{\mathbf{d}}_k. \tag{31}$$

Next let us derive the sliding dynamics. Rewriting (25)

$$\boldsymbol{\sigma}_{k+1} = D\mathbf{x}_{k+1} + E\mathbf{x}_k - D\mathbf{x}(0) + \boldsymbol{\varepsilon}_k. \tag{32}$$

Substitution of (29) into (32) leads to

$$\boldsymbol{\sigma}_{k+1} = D\mathbf{d}_k - D\hat{\mathbf{d}}_k = D\mathbf{d}_k - D\mathbf{d}_{k-1} = O(T^2), \tag{33}$$

that is, the introduction of ISMC leads to the same sliding dynamics as in [1]. In (31), $\boldsymbol{\sigma}_k$ can be substituted by $\boldsymbol{\sigma}_k = D\mathbf{d}_{k-1} - D\mathbf{d}_{k-2}$ as can be inferred from (33). Also, under (23), $D(\Phi - I) + E = D\Gamma K$. Therefore, $\Phi - \Gamma(D\Gamma)^{-1}(D(\Phi - I) + E) = \Phi - \Gamma K$. Since the pair $(\Phi, \Gamma)$ is controllable, there exists a matrix $K$ such that eigenvalues of $\Phi - \Gamma K$ can be placed anywhere inside the unit disk. Note that, the selection of matrix $D$ is arbitrary as long as it guarantees the invertibility of $D\Gamma$ while matrix $E$, computed using (23), guarantees the desired closed-loop performance. Thus, we have

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K)\mathbf{x}_k + \mathbf{d}_k$$

$$- \Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-1} - \Gamma(D\Gamma)^{-1}D(\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \tag{34}$$

Note that in (34), the disturbance estimate $\hat{\mathbf{d}}_k$ has been replaced by $\mathbf{d}_{k-1}$. Further simplification of (34) leads to

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K)\mathbf{x}_k + \boldsymbol{\zeta}_k \tag{35}$$

where

$$\boldsymbol{\zeta}_k = \mathbf{d}_k - 2\Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-1} + \Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-2} \tag{36}$$

if $2\mathbf{d}_{k-1}$ and $\mathbf{d}_{k-2}$ are added and subtracted from the r.h.s of (36) and the result rearranged

$$\boldsymbol{\zeta}_k = (\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2})$$

$$+ (I - \Gamma(D\Gamma)^{-1}D)(2\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \tag{37}$$

In *Lemma* 1, it is shown that $(\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}) \in O(T^3)$ and

$$(I - \Gamma(D\Gamma)^{-1}D)(2\mathbf{d}_{k-1} - \mathbf{d}_{k-2})$$

$$= (I - \Gamma(D\Gamma)^{-1}D)(\Gamma(2\mathbf{f}_{k-1} + \mathbf{f}_{k-2})$$

$$+ \frac{T}{2}\Gamma(2\mathbf{v}_{k-1} + \mathbf{v}_{k-2}) + O(T^3)) = O(T^3).$$

Thus, this leads to the conclusion that

$$\boldsymbol{\zeta}_k \in O(T^3).$$

Comparing (35) with (11), we note the difference lies in that $\boldsymbol{\delta}_k \in O(T^2)$ but $\boldsymbol{\zeta}_k \in O(T^3)$. Further, by doing a similarity decomposition for dynamics of (35), it only has $J_2$ matrix of dimension $n$. Thus the derivation procedure shown in (11)-(20) holds for (35). Its solution is given as

$$\mathbf{x}_k = (\Phi - \Gamma K)^k\mathbf{x}(0) + \sum_{i=0}^{k-1}(\Phi - \Gamma K)^i\boldsymbol{\zeta}_{k-i-1}. \tag{38}$$

Assuming distinct eigenvalues of $\Phi - \Gamma K$ and following the procedure that resulted in (20), it can be shown that

$$\lim_{k\to\infty}\left\|\sum_{i=0}^{k-1}(\Phi - \Gamma K)^i\boldsymbol{\zeta}_{k-i-1}\right\| \in O(T^2). \tag{39}$$

Thus, it is concluded that

$$\lim_{k\to\infty}\|\mathbf{x}_k\| \leq O(T^2). \tag{40}$$

■

*Remark 3:* It is possible to enhance the robustness properties of the closed-loop system under ISM if a nonlinear term such as $\beta sat(\boldsymbol{\sigma}_k/\rho)$ is included in the control law (28) with $\beta$ selected such that the closed-loop system is stable and $\rho$ being the error bound on $\boldsymbol{\sigma}_k$. This would ensure that $\boldsymbol{\sigma} \in O(T^2)$. On the other hand, we cannot expect such a switching term to play the key role of feedback as its continuous-time counterpart, due to the fundamental limitation of bandwidth in sampled-data systems, and the low gain limitation in discrete-time systems.

IV. ILLUSTRATIVE EXAMPLE

Consider system (1) with the following parameters

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\mathbf{f}(t) = \begin{bmatrix} 0.3\sin(4\pi t) \\ 0.3\cos(4\pi t) \end{bmatrix}.$$

The initial states are $\mathbf{x}(0) = [1 \quad 1 \quad -1]$. The system will be simulated for a sample interval of 1ms. For the classical

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SMC, the $D$ matrix is chosen such that the non-zero pole of the sliding dynamics is $p = -5$ in continuous-time, or $z = 0.9950$ in discrete-time. Hence, the poles of the system with SM are $[0 \quad 0.9950 \quad 0]^T$ respectively. Accordingly the $D$ matrix is

$$D = \begin{bmatrix} 0.2621 & -0.3108 & -0.0385 \\ 3.4268 & 2.4432 & 1.1787 \end{bmatrix}.$$

Using the same $D$ matrix given above, the system with ISM is designed such that the dominant (non-zero) pole remains the same, but, the remaining poles are not deadbeat. The poles are selected as $z = [0.9048 \quad 0.9950 \quad 0.8958]^T$ respectively.

Using the pole placement command of Matlab, the gain matrices can be obtained

$$K = \begin{bmatrix} 66.6705 & 9.4041 & 15.8872 \\ 18.2422 & 21.3569 & 8.5793 \end{bmatrix}.$$

According to (23)

$$E = \begin{bmatrix} 0.0297 & -0.0313 & -0.0034 \\ 0.3147 & 0.2366 & 0.1115 \end{bmatrix}.$$

The delayed disturbance compensation is used. Fig.1a shows that the system state $x_1$ is asymptotically stable for both discrete-time SMC and ISMC, which show almost the same behavior globally. On the other hand, the difference in the steady-state response between discrete-time SMC and ISMC can be seen from Fig.1b. The control inputs are shown in Fig.2. Note that the control inputs for the system with SMC has much larger values at the initial phase in comparison to the system with ISMC, due to the deadbeat poles. Another reason for the lower value of the control input in the ISMC is the elimination of the reaching phase by compensating for the non-zero intial condition in (28).

To demonstrate the quality of both designs, the open-loop transfer matrices for the systems with SM and ISM are computed and the Bode plot for several elements are shown on Fig.3. In the plot, $G_{ij}^{OL}$ stands for the $i,j$-th element of the open-loop matrix-valued transfer function. It shows that ISMC, while stable, presents higher gains, hence achieves better control performance.

The frequency responses of two sensitivity functions from disturbance components $f_1$ and $f_2$ to the state $x_1$ are plotted in Fig.4. It can be seen from Fig.3 that ISMC is able to reduce the effect of the disturbances as compared to SMC, consequently achieves the desirable robustness.

## V. CONCLUSION

This work presents a form of the discrete-time integral sliding control design for sampled-date systems with state regulation. Using the new discrete-time integral type switching surface, the SMC design retains the deadbeat structure of the discrete-time sliding mode, and at the same time allocates the closed-loop eigenvalues for the full-order multi-input system. The discrete-time ISM based SMC achieves accurate control performance for both the sliding mode and state regulation, meanwhile eliminates the reaching phase and avoids overlarge control efforts. The theoretical results were confirmed through both theoretical analysis and a numerical example.

## APPENDIX

*Proof of Lemma 1*:
Consider the Taylor's series expansion of $\mathbf{f}((k+1)T - \tau)$

$$\mathbf{f}(kT + T - \tau) = \mathbf{f}_k + \mathbf{v}_k(T - \tau) + \frac{1}{2!}\mathbf{w}_k(T - \tau)^2 + \cdots$$

$$= \mathbf{f}_k + \mathbf{v}_k(T - \tau) + \boldsymbol{\xi}(T - \tau)^2 \quad (41)$$

where $\mathbf{v}(t) = \frac{d}{dt}\mathbf{f}(t)$, $\mathbf{w}(t) = \frac{d^2}{dt^2}\mathbf{f}(t)$ and $\boldsymbol{\xi} = \frac{1}{2!}\mathbf{w}(\mu)$ and $\mu$ is a time value between $kT$ and $(k+1)T$. Substituting (41) into the expression of $\mathbf{d}_k$

$$\mathbf{d}_k = \int_0^T e^{A\tau}\mathbf{B}\mathbf{f}_k d\tau$$

$$+ \int_0^T e^{A\tau}B\mathbf{v}_k(T - \tau)d\tau + \int_0^T e^{A\tau}B\boldsymbol{\xi}(T - \tau)^2 d\tau. \quad (42)$$

For clarity, each integral will be analyzed separately. Since $\mathbf{f}_k$ is independent of $\tau$ it can be taken out of the first integral

$$\int_0^T e^{A\tau}B\mathbf{f}_k d\tau = \int_0^T e^{A\tau}Bd\tau \mathbf{f}_k = \Gamma\mathbf{f}_k. \quad (43)$$

In order to solve the second integral term, it is necessary to expand $e^{A\tau}$ into series form. Thus,

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T - \tau)d\tau$$

$$= \int_0^T \left[e^{A\tau}B - \left(B + AB\tau + \frac{1}{2!}A^2B\tau^2 + \cdots\right)\tau\right]d\tau\mathbf{v}_k. \quad (44)$$

Solving the integral leads to

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T - \tau)d\tau$$

$$= \left[\Gamma - \left(\frac{1}{2!}BT + \frac{1}{3!}ABT^2 + \cdots\right)\right]T\mathbf{v}_k. \quad (45)$$

Simplifying the result with the aid of (3)

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T - \tau)d\tau = \left[\Gamma - \frac{1}{2}\Gamma + \frac{1}{2}MT^2\right.$$

$$\left. - \left(\frac{1}{3!}ABT^2 + \frac{1}{4!}A^2BT^2 + \cdots\right)\right]T\mathbf{v}_k. \quad (46)$$

Simplifying the above expression further

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T - \tau)d\tau = \frac{1}{2}\Gamma\mathbf{v}_k T + \hat{M}\mathbf{v}_k T^3 \quad (47)$$

where $\hat{M}$ is a constant matrix. Finally, note that in (42) the third integral is $O(T^3)$, since, the term inside the integral is already $O(T^2)$, therefore

$$\int_0^T e^{A\tau}B\boldsymbol{\xi}(T - \tau)^2 d\tau = O(T^3). \quad (48)$$

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Thus, combining (43), (46) and (48) leads to

$$\mathbf{d}_k = \Gamma \mathbf{f}_k + \frac{1}{2}\Gamma \mathbf{v}_k T + \hat{M}T^3 \mathbf{v}_k + O(T^3)$$

$$= \Gamma \mathbf{f}_k + \frac{1}{2}\Gamma \mathbf{v}_k T + O(T^3). \qquad (49)$$

Note that

$$\mathbf{d}_k - \mathbf{d}_{k-1} \in O(T^2) \qquad (50)$$

since, $\mathbf{f}_k - \mathbf{f}_{k-1} \in O(T)$ and (3), $\Gamma \in O(T)$, if the assumptions on the boundedness and smoothness of $\mathbf{f}(t)$ hold. Similarly,

$$\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = \Gamma(\mathbf{f}_k - 2\mathbf{f}_{k-1}$$

$$+ \mathbf{f}_{k-2}) + \frac{1}{2}\Gamma(\mathbf{v}_k - 2\mathbf{v}_{k-1} + \mathbf{v}_{k-2}) + O(T^3)$$

Accordingly the magnitude of $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}$ is $O(T^3)$.

## REFERENCES

[1] W. C. Su, S. Drakunov, U. Ozguner, "An $O(T^2)$ boundary layer in sliding mode for sampled-data systems", IEEE Transactions on Automatic Control, Vol.45, no.3, pp.482-485, 2000.
[2] V. Utkin, "Sliding mode control in discrete-time and difference systems", In Variable Structure and Lyapunov Control, Chapter 5, Editted by A.S.I. Zinober, Springer-Verlag, 193, pp.87-107, 1994.
[3] J. Ackermann, V. Utkin, "Sliding mode control design based on Ackermann's formula", IEEE Transactions on Automatic Control, Vol.43, no.2, pp.234-237, 1998.
[4] V. Utkin, J. Shi, "Integral sliding mode in systems operating under uncertainty conditions", Proc. Conference on Decision and Control, Kobe, Japan, pp.4591-4596, 1996.
[5] W. J. Cao and J. X. Xu, "Eigenvalue Assignment in Full-Order Sliding Mode Using Integral Type Sliding Surface", IEEE Transactions on Automatic Control, Vol.49, no.8, pp.1355-1360, 2004.
[6] L. Fridman, F. Castanos, N. M'Sirdi N. and Kharfa, "Decomposition and Robustness Properties of Integral Sliding Mode Controllers", Proc. IEEE VSS workshop, Spain, 2004.
[7] W. Gao, Y. Wang, and A. Homaifa, "Discrete-time Variable Structure Control Systems," IEEE Transactions on Industrial Electronics, Vol.42, no.2, pp.117-122, 1995.
[8] G. Golo and C. Milosavljevic, "Robust Discrete-time Chattering Free Sliding Mode Control," Systems and Control Letters, Vol.41, no.1, pp.19-28, 2000.
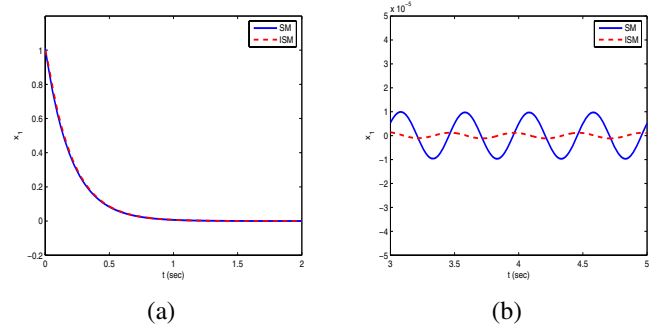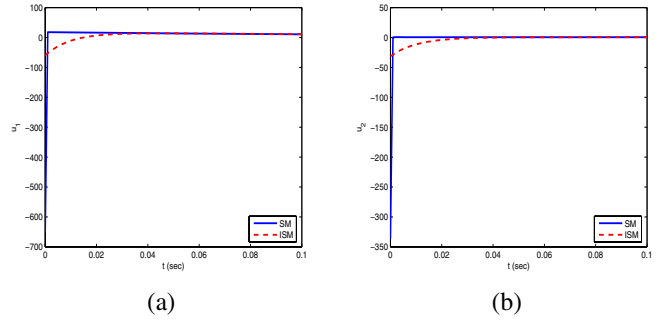
Fig. 1. System state $x_1$



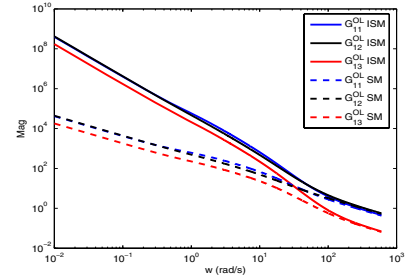Fig. 2. System control inputs $u_1$ and $u_2$



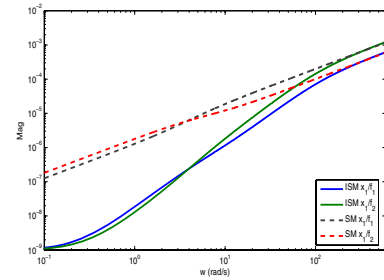Fig. 3. Bode plot of some of the elements of the open-loop transfer matrix



Fig. 4. Frequency response of $x_1$ with respect to $f_1$ and $f_2$

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