On Output Tracking Using Dynamic Output Feedback Discrete-Time Sliding Mode Controllers

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Abstract—**In this paper an output feedback based discrete-time sliding mode control scheme is proposed. It incorporates a steady-state tracking requirement through the use of integral action. Previous work has shown that with an appropriate choice of sliding surface, discrete-time sliding mode control can be applied to non-minimum phase systems. The original scheme employed static output feedback and this imposed restrictions on the class of systems to which it was applicable – specifically a certain 'fictitious' sub-system was required to be output feedback stabilizable. The scheme proposed in this paper includes a compensator which broadens the class of systems for which the results are applicable. In the presence of bounded matched disturbances, ultimate boundedness results are obtained. It is also shown that in the presence of a class of sector bounded uncertainty, asymptotic stability can be achieved.**

Index Terms—**sliding modes, discrete-time, output feedback, LMIs**

I. Introduction

Many conventional (continuous-time) sliding mode control design schemes assume that all the states of the plant are directly accessible. In real systems this is not tenable and usually not all system states are fully available or measurable. One solution is to use an observer to reconstruct the system states [20], [23], [6]. Alternatively output feedback strategies can be employed in which the control law only requires knowledge of measured outputs [5], [8], [1]. There are, however, inherent system restrictions on using output feedback sliding mode control design methods: normally, the system must be relative degree one and minimum phase [7]. In some situations the relative degree condition can be relaxed by considering higher order sliding mode schemes. However the minimum phase restriction arises from the fact that the system zeros appear amongst the poles governing the sliding motion.

Compared with continuous time sliding mode strategies, the design problem in discrete-time has received much less coverage in the literature. With the exception of the early work in [19], most of the literature assumes all states are available [4], [9], [10], [11], [12], [22]. Schemes which have restricted themselves to output measurements alone have invariably utilized observers. Recent exceptions have been [14], [17] and the discrete-time versions of certain higher-order sliding mode control schemes in [2]. In particular, [14] considered an output tracking problem for an uncertain linear system using sliding mode ideas which requires output information only. It was shown in [14] that the relative degree and minimum phase requirements could be overcome by the use of a novel sliding surface. In order that a stable (ideal) discrete-time sliding motion exists, necessary and sufficient conditions were given in terms of the stabilizability, by static output feedback, of a fictitious system triple obtained from the real system. This fictitious system can easily be isolated once the real system is transformed into a special canonical form described in [14]. The stabilizability condition is a significant restriction on the class of systems to which the results are applicable, and of course for general multivariable systems this condition can, at best, only be tested numerically. In [14], a static output feedback structure was considered and so the fact that there is a limitation on the class of systems to which it is applicable is not surprising. This paper builds on this earlier work and proposes a specific compensator structure to circumvent this restriction. The resulting controller is applied to a High Incidence Research Model (HIRM) aircraft system as an example of a real engineering system.

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II. A Discrete-Time Sliding Mode Formulation

Consider the square discrete-time system with matched uncertainties

$$x(k+1) = Gx(k) + H(u(k) + \xi(k)) \quad (1)$$

$$y(k) = Cx(k) \quad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m = p < n$. Assume that the input and output distribution matrices H and C are full rank. In addition, assume the triple (G, H, C) is minimal. The matched uncertainties, represented by $\xi(k)$, are assumed to be unknown but bounded.

Consider the problem of determining an appropriate sliding surface $\mathcal{S}$ formed from a linear combination of the states, and a control law depending only on the measured outputs such that:

- for the nominal linear system when $\xi \equiv 0$ an *ideal sliding motion* is obtained in finite time i.e. $x(k) \in \mathcal{S}$ for all $k > k_s$;
- for uncertain systems the effect of the matched uncertainty ξ is minimized and an appropriate bounded motion about $\mathcal{S}$ is maintained.

The discrete-time sliding mode situation is quite different from its continuous time counterpart: in continuous time a discontinuous control strategy can be employed to maintain ideal sliding in the presence of bounded matched uncertainty and in theory its effect is completely rejected; in discrete-time, complete rejection of the uncertainty is not possible due to the sampled nature of the control signal. Furthermore, the inclusion of a switched term in the control law may be detrimental to the performance [21], [11], [12].

As in [21], the class of sliding surfaces will be restricted to those which can be expressed as

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ H^{\mathrm{T}}Px = 0\} \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (s.p.d.) matrix. Associate with P a candidate Lyapunov function $V(k) = x(k)^{\mathrm{T}}Px(k)$ and define a Lyapunov difference function by

$$\Delta V(k) = V(k+1) - V(k) \tag{4}$$

Consider initially a regulation problem where no uncertainty is present (i.e. $\xi(k) \equiv 0$). In the absence of uncertainty an ideal sliding motion can be attained on $\mathcal{S}$ whereby

$$H^{\mathrm{T}}Px(k+1) = H^{\mathrm{T}}PGx(k) + H^{\mathrm{T}}PHu(k) = 0 \tag{5}$$

It follows from (5) that the equivalent control action necessary to maintain an ideal sliding motion on $\mathcal{S}$ from (3) is given by

$$u_{eq}(k) = -(H^{\mathrm{T}}PH)^{-1}H^{\mathrm{T}}PGx(k). \tag{6}$$

If P is such that the closed-loop system, obtained from using the control law (6) in (1), satisfies $\Delta V(k) < 0$ for all k, then from standard (discrete) Lyapunov theory the closed-loop system is asymptotically stable. It is clear that $\Delta V(k) \equiv -x^{\mathrm{T}}(k)Qx(k)$, where

$$Q := P - G_c^{\mathrm{T}}PG_c \tag{7}$$

and the closed-loop system matrix

$$G_c := G - H(H^{\mathrm{T}}PH)^{-1}H^{\mathrm{T}}PG \tag{8}$$

If $Q > 0$, the closed-loop system is stable.

Remark 1: For the uncertain discrete-time system in (1), the control law (6), with P chosen so that Q from (7) is s.p.d, has the property that it:

a) induces an ideal sliding motion on $\mathcal{S}$ from (3) in finite time when $\xi(k) \equiv 0$ (this follows immediately from (5) and (6));
b) minimizes the effect of $\xi(k)$ on the closed loop dynamics in a min-max sense i.e. the control law minimizes over all possible state feedback control laws the effect of the worst case uncertainty $\xi(k)$ on the Lyapunov difference $\Delta V(k)$ (see [21]);
c) minimizes in a min-max sense the deviation from the ideal sliding surface $\mathcal{S}$ (see [11]).

$\sharp$

If for a s.p.d. matrix P satisfying $P - G_c^{\mathrm{T}}PG_c > 0$ it is possible to solve

$$H^{\mathrm{T}}PG = FC \tag{9}$$

for some matrix $F \in \mathbb{R}^{m\times p}$ then provided $\det G \neq 0$, the controller from (6) can be realized through outputs alone as

$$u(k) = -(FCG^{-1}H)^{-1}Fy(k). \tag{10}$$

It is shown in [13] that two necessary conditions to solve the problem of synthesizing a s.p.d matrix P satisfying (9) which ensures Q from (7) is s.p.d, are

A1) the plant state transition matrix G is nonsingular;
A2) the matrix $CG^{-1}H$ has rank m.

Based on assumptions A1 and A2, a change of coordinates can be introduced which facilitates insight into the class of systems for which this problem is solvable. Define a new matrix

$$S := CG^{-1}. \tag{11}$$

This matrix will take the role of the output distribution matrix for a new, fictitious system (G, H, S), which will be useful for the theoretical developments which follow. In order to facilitate the analysis, a change of coordinates will be introduced for the fictitious system (G, H, S). From assumption A2 and the definition of S in (11), rank$(SH) = m$. As argued in [13], since rank$(SH) = m$, there exists a change of coordinates such that $x \mapsto \bar{x}$ and $(G, H, S) \mapsto (\bar{G}, \bar{H}, \bar{S})$ where

$$\bar{G} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}; \quad \bar{H} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix}; \quad \bar{S} = \begin{bmatrix} 0 & T \end{bmatrix} \tag{12}$$

where $G_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $H_2 \in \mathbb{R}^{m\times m}$ and is nonsingular and $T \in \mathbb{R}^{m\times m}$ is orthogonal. As argued in [13], *necessary and sufficient* conditions to solve the problem of synthesizing a s.p.d matrix P satisfying (9) which ensures Q from (7) is s.p.d, are that A1 and A2 hold, together with a third requirement:

A3) the matrix sub-block G_{11} from (12) is stable.

The coordinate system associated with (12) will be used as a basis for the results which follow.[1]

Remark 2: Assumption A1 appears in all the discrete-time output min-max literature: see for example [18]. However A1 means the approach in this paper is not applicable to discrete-time systems which contain pure time delays. Condition A2 is a necessary condition to find a s.p.d. matrix P and an $F \in \mathbb{R}^{m\times p}$ to solve (9). This can be easily verified: assuming $\det(G) \neq 0$, if (9) is satisfied then $H^{\mathrm{T}}PH = FCG^{-1}H$ and hence rank$(FCG^{-1}H) = m$. This implies $CG^{-1}H$ must be rank m. The second condition is equivalent to the triple (G, H, C) not having any invariant zeros at the origin since $CG^{-1}H = \mathcal{G}(0)$ where $\mathcal{G}(z) := C(zI - G)^{-1}H$ and so rank$(CG^{-1}H) = m$ implies $z = 0$ is not an invariant zero. Condition A3 is limiting and will be obviated in this paper by the introduction of a compensator.

Remark 3: It can be shown [13] that A3 is equivalent to the triple (G, H, S) being minimum phase since the eigenvalues of G_{11} represent the invariant zeros of (G, H, S). Note: this is quite different to the continuous-time min-max case where the system representation from the inputs to the true outputs must be minimum phase. As argued in [13], [14] it is quite possible for (G, H, C) to be non-minimum phase whilst satisfying A3.

This paper will consider the situation where a tracking requirement is required and will remove A3 by the use of a suitable dynamic compensator.

III. Main Results

Assume throughout the rest of the paper that A1 and A2 from §II hold. It follows from the canonical form (12) that the

[1] In fact [13] considers the more general situation where $p \geq m$ and a slightly more elaborate version of A3 is proved.

true output distribution matrix

$$\bar{C} = \bar{S}\bar{G} = \begin{bmatrix} TG_{21} & TG_{22} \end{bmatrix} \tag{13}$$

To incorporate a tracking element, integral action will also be included. The difference equation

$$x_r(k+1) = x_r(k) + \tau(r(k) - y(k)) \tag{14}$$

will be added where τ represents the sample interval[2]. The quantity $r(k)$ represents the signal to be tracked by the output. Furthermore assume $r(k) = r_s = \text{const}$ for $k > k_0$.

Partition the state vector $\bar{x}$ conformably as $\text{col}(x_1, x_2)$ where $x_1 \in \mathbb{R}^{(n-m)}$. Also introduce additional states $x_c \in \mathbb{R}^{(n-m)}$, which under certain circumstances represent an estimate of the states x_1.

The intention is to induce an ideal sliding motion on

$$\mathcal{S} = \{(x_1, x_c, x_r, x_2) : K_1 x_c + K_r x_r + x_2 + S_r r_s = 0\}, \tag{15}$$

where $K_1 \in \mathbb{R}^{m\times(n-m)}$ and $K_r \in \mathbb{R}^{m\times m}$ together with $S_r \in \mathbb{R}^{m\times m}$ represent design freedom. Let the compensator take the form

$$x_c(k+1) = G_{11}x_c(k) + G_{12}x_2(k) + L(y(k) - \hat{y}(k)) \tag{16}$$

where

$$\hat{y}(k) := TG_{21}x_c(k) + TG_{22}x_2(k) \tag{17}$$

and $L \in \mathbb{R}^{(n-m)\times m}$ is a design variable. During an ideal sliding motion, from (15),

$$x_2(k) = -K_1 x_c(k) - K_r x_r(k) - S_r r(k)$$

and so after some algebraic manipulation it can be shown that

$$x_c(k+1) = \Phi x_c(k) + \Gamma_1 y(k) + \Gamma_2 x_r(k) + \Gamma_3 r(k) \tag{18}$$

where

$$\Phi = G_{11} - LTG_{21} - G_{12}K_1 + LTG_{22}K_1 \tag{19}$$

and

$$\begin{aligned} \Gamma_1 &= L, & (20)\\ \Gamma_2 &= -G_{12}K_r + LTG_{22}K_r, & (21)\\ \Gamma_3 &= -G_{12}S_r + LTG_{22}S_r. & (22) \end{aligned}$$

It is assumed that as part of the design process, L is chosen to guarantee that $\det \Phi \neq 0$. Augment the system in (1), in the canonical form of (12), with the integral and compensator states from (14) and (18) to obtain:

$$x_a(k+1) = G_a x_a(k) + H_a(u(k) + \xi(k)) + H_r r(k), \tag{23}$$

where $x_a = \text{col}(x_1, x_c, x_r, x_2)$. (At first sight this represents a non-intuitive arrangement of the states but it leads to a simplification in the presentation.)

[2]If (1)-(2) is a genuinely discrete system and does not arise from sampling a continuous-time system, then (14) can be replaced by $x_r(k+1) = x_r(k) + r(k) - C_p x_p(k)$ and all the results which will subsequently be proved are still true when this equation is used in place of (14).

The avialable outputs associated with this system are given by $y_a = \text{col}(x_c, x_r, y)$. It is easily verified that

$$G_a = \begin{bmatrix} G_{11} & 0 & 0 & G_{12} \\ \Gamma_1 TG_{21} & \Phi & \Gamma_2 & \Gamma_1 TG_{22} \\ -\tau TG_{21} & 0 & I_m & -\tau TG_{22} \\ G_{21} & 0 & 0 & G_{22} \end{bmatrix}; \; H_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix} \tag{24}$$

and the output distribution matrix

$$C_a = \begin{bmatrix} 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ TG_{21} & 0 & 0 & TG_{22} \end{bmatrix}, \tag{25}$$

where $y_a := C_a x_a$. Modify the control law in (10) to include the reference signal so that

$$u(k) = -(F_a C_a G_a^{-1} H_a)^{-1} F_a C_a x_a(k) + F_r r(k), \tag{26}$$

where now both F_a and $F_r \in \mathbb{R}^{m\times m}$ are to be determined (in terms of L, K_1, K_r and S_r). The objective is to select F_a and a matrix $F_2 \in \mathbb{R}^{m\times m}$ so that the surface

$$\mathcal{S}_a = \{x_a \; : \; F_a C_a G_a^{-1} x_a + F_2 S_r r_s = 0\} \tag{27}$$

is identical to the surface $\mathcal{S}$ in (15), and then to select K_1, K_r and L to ensure a stable ideal sliding motion when confined to $\mathcal{S}$.

Providing the design matrix F_a is chosen to ensure the eigenvalues of

$$G_c = G_a - H_a(F_a C_a G_a^{-1} H_a) F_a C_a \tag{28}$$

are inside the unit disk, $(I - G_c)$ is invertible. Define $x_s = (I - G_c)^{-1}(H_r + H_a F_r) r_s$ then using (26) and defining

$$e(k) = x_a(k) - x_s \tag{29}$$

it follows from simple algebraic manipulation that

$$e(k+1) = G_c e(k) + H_a \xi(k) \tag{30}$$

In the absence of uncertainty $e(k) \to 0$ as $k \to \infty$, and since steady state is achieved, it follows from (14) that $y(k) = r_s$ and so tracking is achieved. Furthermore it can be shown that

$$\begin{aligned} F_a C_a G_a^{-1} x_s &= F_a C_a G_a^{-1}(I - G_c)^{-1}(H_r + H_a F_r) r_s \\ &\equiv F_a C_a G_a^{-1}(H_r + H_a F_r) r_s \end{aligned}$$

and consequently if F_r is chosen as

$$F_r := -(F_a C_a G_a^{-1} H_a)^{-1} \left(F_a C_a G_a^{-1} H_r + F_2 S_r\right) \tag{31}$$

then $F_a C_a G_a^{-1} x_s + F_2 S_r r_s = 0$ and so $x_s \in \mathcal{S}$. From (30), and following similar arguments to those presented in §II concerning (7) and (9), the problem is therefore to find an F_a and a s.p.d. matrix $P_a \in \mathbb{R}^{2n\times 2n}$ such that

$$F_a C_a = H_a^{\mathrm{T}} P_a G_a \tag{32}$$

and

$$G_c^{\mathrm{T}} P_a G_c - P_a < 0 \tag{33}$$

Proposition 1: Assuming conditions A1 and A2 are satisfied, then there exist matrices F_a and P_a such that (32)-(33) hold.

Proof If A1 and A2 are satisfied then all the development in §III is valid. Define for the augmented system (30) and the output distribution matrix in (25) a fictitious output distribution matrix $S_a := C_a G_a^{-1}$. After some algebra it can be shown that

$$\begin{aligned} S_a &= \left[\begin{array}{c|ccc} 0 & \Phi^{-1} & -\Phi^{-1}\Gamma_2 & -\Phi^{-1}\Gamma_1 T - \tau\Phi^{-1}\Gamma_2 T \\ 0 & 0 & I_m & \tau T \\ 0 & 0 & 0 & T \end{array} \right] \\ &=: \left[\begin{array}{cc} 0 & T_a \end{array} \right], \end{aligned} \tag{34}$$

where $T_a \in \mathbb{R}^{(n+m)\times(n+m)}$ and $\det T_a \neq 0$. Define a matrix

$$F_a := F_2 \left[\begin{array}{ccc} K_1\Phi & K_1\Gamma_2 + K_r & K_1\Gamma_1 - K_r\tau + T^{\mathrm{T}} \end{array} \right], \tag{35}$$

where $F_2 \in \mathbb{R}^{m\times m}$ and is nonsingular. The matrix F_2 has no effect on the dynamics of the ideal sliding motion but is required to solve the constraint (32). After a little algebra it can be shown that

$$F_a S_a = F_a C_a G_a^{-1} = F_2 \left[\begin{array}{cccc} 0 & K_1 & K_r & I_m \end{array} \right]. \tag{36}$$

To facilitate choosing the parameters L, K_1 and K_r, change coordinates according to the transformation $x_a \mapsto \tilde{T} x_a =: \tilde{x}$ where

$$\tilde{T} := \left[\begin{array}{cccc} I_{n-m} & -I_{n-m} & 0 & 0 \\ 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & K_1 & K_r & I_m \end{array} \right]. \tag{37}$$

This effectively forces the last m states of the new coordinates to represent what, in continuous-time sliding mode control, is called the 'switching function' $\sigma = K_r x_r + K_1 x_c + x_2$ associated with the surface $\mathcal{S}$ in (15). It follows that the matrices $\tilde{G} = \tilde{T} G_a \tilde{T}^{-1}$, $\tilde{H} = \tilde{T} H_a$, $\tilde{H}_r = \tilde{T} H_r$, $\tilde{C} = C_a \tilde{T}^{-1}$ and $\tilde{S} = S_a \tilde{T}^{-1}$. After some straightforward algebra

$$\tilde{H} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ H_2 \end{array} \right] \quad \text{and} \quad \tilde{H}_r = \left[\begin{array}{c} -\Gamma_3 \\ \Gamma_3 \\ \tau I_m \\ K_1\Gamma_3 + \tau K_r \end{array} \right]. \tag{38}$$

From equation (36) it follows that

$$F_a \tilde{S} = \left[\begin{array}{cccc} 0 & 0 & 0 & F_2 \end{array} \right]. \tag{39}$$

Some algebra reveals the closed-loop system matrix

$$\tilde{G}_c = \tilde{G} - \tilde{H}(F_a \tilde{S} \tilde{H})^{-1} F_a \tilde{C} =: \left[\begin{array}{cc} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & 0 \end{array} \right], \tag{40}$$

where

$$\tilde{G}_{11} = \left[\begin{array}{c|c} G_{11} - LTG_{21} & 0 \\ \hline \begin{array}{c} LTG_{21} \\ -\tau TG_{21} \end{array} & \tilde{G}_m \end{array} \right] \tag{41}$$

and

$$\tilde{G}_m := \left[\begin{array}{cc} G_{11} - G_{12}K_1 & -G_{12}K_r \\ -\tau TG_{21} + \tau TG_{22}K_1 & I_m + \tau TG_{22}K_r \end{array} \right] \tag{42}$$

This is most easily seen from the definition of $\tilde{S} = \tilde{C}\tilde{G}^{-1}$ and the fact that $\tilde{G}_c = (I - \tilde{H}(F_a\tilde{S}\tilde{H})^{-1}F_a\tilde{S})\tilde{G}$. From (39) and (38) it can be easily shown that

$$(I - \tilde{H}(F_a\tilde{S}\tilde{H})^{-1}F_a\tilde{S}) = \mathrm{diag}(I_{n-m}, I_{n-m}, I_m, 0_{m\times m})$$

and hence the structure in (40) follows immediately. It is clear from (40) and (41) that

$$\sigma(\tilde{G}_c) = \{0\}^m \cup \sigma(G_{11} - LTG_{21}) \cup \sigma(\tilde{G}_m),$$

where

$$\tilde{G}_m = \underbrace{\left[\begin{array}{cc} G_{11} & 0 \\ -\tau TG_{21} & I_m \end{array} \right]}_{G_{11}^a} - \underbrace{\left[\begin{array}{c} G_{12} \\ -\tau TG_{22} \end{array} \right]}_{G_{12}^a} \left[\begin{array}{cc} K_1 & K_r \end{array} \right] \tag{43}$$

and $\sigma(\cdot)$ denotes the spectrum of a matrix. Since the matrix pair (G_{11}, G_{21}) is observable (see for example [13]) and T is nonsingular, the pair (G_{11}, TG_{21}) is observable. Consequently L can be chosen to make $(G_{11} - LTG_{21})$ stable. Likewise it can be shown that provided (G, H, C) does not have invariant zeroes at unity, the pair (G_{11}^a, G_{12}^a) is controllable and hence the choice of K_1 and K_r constitutes a state-feedback problem. Consequently K_1, K_r and L can be chosen to make $\tilde{G}_{11}$ from (41) stable. In the new set of coordinates $\tilde{x}$, let the Lyapunov matrix be represented by $\tilde{P}$. Using the definition of $\tilde{S}$, equation (32) becomes

$$\tilde{H}^{\mathrm{T}}\tilde{P} = F_a\tilde{C}\tilde{G}^{-1} = F_a\tilde{S} \tag{44}$$

In order to show that $\tilde{P}$ is a Lyapunov matrix for $\tilde{G}_c$ it must be established that

$$\tilde{Q} := \tilde{P} - \tilde{G}_c^{\mathrm{T}}\tilde{P}\tilde{G}_c > 0 \tag{45}$$

It can be seen from the structures of $\tilde{H}$ and $F_a\tilde{S}$ in (38) and (39) and from the fact that $\det H_2 \neq 0$ that in order to satisfy (44), $\tilde{P}$ must have a block diagonal structure:

$$\tilde{P} = \left[\begin{array}{cc} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{array} \right], \tag{46}$$

where $\tilde{P}_1 \in \mathbb{R}^{(2n-m)\times(2n-m)}$, $\tilde{P}_2 \in \mathbb{R}^{m\times m}$ and

$$F_2 = H_2^{\mathrm{T}}\tilde{P}_2. \tag{47}$$

In terms of the partition in (40), (45) can be written as

$$\tilde{Q} = \left[\begin{array}{cc} \tilde{P}_1 - \tilde{G}_{11}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{11} & -\tilde{G}_{11}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{12} \\ -\tilde{G}_{12}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{11} & \tilde{P}_2 - \tilde{G}_{12}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{12} \end{array} \right]. \tag{48}$$

Let $\tilde{P}_1 > 0$ be a solution to

$$\tilde{P}_1 - \tilde{G}_{11}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{11} > 0. \tag{49}$$

Such a solution $\tilde{P}_1$ is guaranteed to exist since $\tilde{G}_{11}$ is stable. Then from the Schur complement, inequality (48) is satisfied if and only if

$$\begin{aligned} \tilde{P}_2 \;>\; & \tilde{G}_{12}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{11}(\tilde{P}_1 - \tilde{G}_{11}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{11})^{-1}(\tilde{G}_{11}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{12}) \\ & + \tilde{G}_{12}^{\mathrm{T}}\tilde{P}_1\tilde{G}_{12} \end{aligned} \tag{50}$$

Any pair $(\tilde{P}_1, \tilde{P}_2)$ satisfying (49) and (50) ensures $\tilde{P}$ from (46) satisfies (44) and (45). Therefore F_a as defined in (35) and $P_a = \tilde{T}^{\mathrm{T}}\tilde{P}\tilde{T}$ where $\tilde{T}$ is given in (37) constitutes a solution to (32)-(33) and the proposition is proved. ■

Corollary 1: The sliding surface $\mathcal{S}_a$ given in (27) is identical to $\mathcal{S}$ given in (15).

Proof From the choice of F_r in (31) and $F_a C_a G_a^{-1}$ in (36), the equivalence of $\mathcal{S}_a$ with $\mathcal{S}$ from (15) is clear since F_2 is nonsingular. ■

Remark 4: It is easy to see from that the control law is independent of the choice of matrix F_2.

IV. Robustness

This subsection considers the robustness properties of the controller developed in §III. Suppose the matched uncertainty $\xi(k)$ in (1), in the coordinates associated with (12), satisfies

$$\|\xi(k)\| < \rho_1\|\bar{x}(k)\| + \rho_0, \tag{51}$$

where ρ_1 and ρ_0 are positive constants. Let

$$N := \begin{bmatrix} I_{n-m} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}. \tag{52}$$

In this section, assume initially that $r(k) \equiv 0\ \forall k$. Then $\bar{x} = Nx_a = Ne$ where e is defined in (29) since $x_s = 0$ because $r_s = 0$ by hypothesis. Consequently inequality (51) can be written as

$$\|\xi(k)\| < \rho_1\|Ne(k)\| + \rho_0. \tag{53}$$

The design freedom associated with the Lyapunov matrix has been shown to be represented by the pair of s.p.d. matrices $\tilde{P}_1$ and $\tilde{P}_2$. Although the pair $(\tilde{P}_1, \tilde{P}_2)$ must satisfy the matrix inequalities (49) and (50), there is some inherent design freedom. The selection of these matrices has no effect on the compensator dynamics (18) or indeed the control law. Assume the relevant design parameters have been selected to ensure $\tilde{G}_c$ (and in particular $\tilde{G}_{11}$ from (40)) is stable. Define

$$\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \mu) := -e^{\mathrm{T}}\tilde{T}^{\mathrm{T}}\tilde{Q}\tilde{T}e + \xi^{\mathrm{T}}\tilde{H}^{\mathrm{T}}\tilde{P}\tilde{H}\xi + (\mu e^{\mathrm{T}}N^{\mathrm{T}}Ne - \xi^{\mathrm{T}}\xi) \tag{54}$$

where $\tilde{Q}$ is defined in (48), $\tilde{T}$ is given in (37), $\tilde{P}$ is defined in (46) and the scalar $\mu > 0$.

Proposition 2: Suppose $\tilde{P}_1$, $\tilde{P}_2$ and μ are chosen so that $\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \mu) < 0$ and $\rho_1 < \sqrt{\mu}$, then in the absence of external disturbances (i.e. when $\rho_0 = 0$), asymptotic stability of the closed-loop system (30) is guaranteed.

Proof Let

$$V(k) := e^{\mathrm{T}}(k)P_a e(k), \tag{55}$$

where $P_a := \tilde{T}^{\mathrm{T}}\tilde{P}\tilde{T}$. Then from (30) it follows that

$$\Delta V(k) = -e^{\mathrm{T}}(k)\tilde{T}^{\mathrm{T}}\tilde{Q}\tilde{T}e(k) + \xi^{\mathrm{T}}(k)\ \underbrace{\tilde{H}^{\mathrm{T}}\tilde{P}\tilde{H}}_{=H_a^{\mathrm{T}}P_aH_a}\ \xi(k), \tag{56}$$

Since by hypothesis $\rho_1 < \sqrt{\mu}$ and $\rho_0 = 0$, it follows from (53) that $\|\xi\| < \sqrt{\mu}\|Ne\|$. Consequently $\mu e^{\mathrm{T}}N^{\mathrm{T}}Ne - \xi^{\mathrm{T}}\xi > 0$, which in conjunction with (54), implies $\Delta V(k) < 0$. It then follows from standard Lyapunov arguments that $e(k) \to 0$ as $k \to \infty$ and asymptotic stability is proved. ∎

Corollary 2: The states $e(k)$ are forced to evolve in such a way that the deviation from $\mathcal{S}_a$ tends to zero with respect to time.

Proof It follows from (30) that

$$F_aC_aG_a^{-1}e(k+1) = H_a^TP_aH_a\xi(k) \tag{57}$$

for $k = 1, 2, \ldots$ and thus $\|H_a^TP_aH_a\xi(k)\|$ represents the deviation from the ideal sliding surface

$$\begin{aligned} \mathcal{S}_a &= \{x_a \mid FC_aG_a^{-1}x_a + F_2S_rr_s = 0\} \\ &\equiv \{e \mid FC_aG_a^{-1}e = 0\} \end{aligned}$$

As argued above $\|\xi(k)\| \leq \rho_1\|Ne(k)\|$ for all k and thus $\|H_a^TP_aH_a\xi(k)\| \leq \rho_1\|H_a^TP_aH_a\|\|e(k)\| \to 0$ as $k \to \infty$ as claimed, since $\|e(k)\| \to 0$ from Proposition 2. ∎

The condition $\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \mu) < 0$ is guaranteed if

$$\mu N^{\mathrm{T}}N < \tilde{T}^{\mathrm{T}}\tilde{Q}\tilde{T}, \tag{58}$$

$$\tilde{H}^{\mathrm{T}}\tilde{P}\tilde{H} < I_m \tag{59}$$

are satisfied subject to $\tilde{P} > 0$. A logical way to proceed is to choose $\tilde{P}_1, \tilde{P}_2$ satisfying (58)-(59) to maximize μ. This represents a convex optimization problem with decision variables $\tilde{P}_1, \tilde{P}_2$ and μ. Linear Matrix Inequality (LMI) methods [3] can be used to obtain the optimal values of the decision matrices as a generalized eigenvalue problem.

Finally if $\rho_0 \neq 0$ and/or $r_s \neq 0$ then (51) becomes

$$\|\xi(k)\| < \rho_1\|Ne(k)\| + \underbrace{(\rho_0 + \rho_1\|Nx_s\|)}_{\bar{\rho}_0}$$

rather than (53). Now quadratic stability is lost, but if $\rho_1 < \sqrt{\mu}$, ultimate boundedness can still be guaranteed using arguments similar to those in Proposition 2.

V. Example

In this section the longitudinal dynamics of the High Incidence Research Model (HIRM) aircraft will be considered [16]. A linearization of the nonlinear simulation [16] about Mach number 0.3 and an altitude of 5000ft has been used as the basis of the design. A discretized representation based on a sample interval of 0.025 secs is

$$G = \begin{bmatrix} 0.9862 & 0.0243 & 0 \\ -0.0264 & 0.9894 & 0 \\ -0.0003 & 0.0249 & 1.0000 \end{bmatrix};\ H = \begin{bmatrix} -0.0038 \\ -0.0810 \\ -0.0010 \end{bmatrix};$$

where $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ the states of the model are angle of attack (rad), pitch rate (rad/s) and pitch angle (rad). The inputs and outputs are symmetrical tail plane deflection and pitch respectively. In the coordinates of (12) it can be shown that

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 0.9874 & -0.1166 & 9.1989 \\ -0.0003 & 2.9806 & -157.0127 \\ \hline -0.0003 & 0.0251 & -0.9925 \end{array}\right]$$

and $T = 1$. Choosing

$$L = \begin{bmatrix} -4.5025 \\ 80.2795 \end{bmatrix}$$

and

$$\begin{bmatrix} K_1 & K_r \end{bmatrix} = \left[\begin{array}{cc|c} 0.4004 & 0.0090 & 0.6418 \end{array}\right]$$

means $\sigma(G_{11} - LTG_{21}) = \{0.97, 0.98\}$ and $\sigma(\tilde{G}_m) = \{0.80, 0.90, 0.98\}$. In the following design $S_r = 0.005$ has been selected. From equations (19)-(22) it follows that

$$\Phi = \begin{bmatrix} -0.9082 & -0.0460 \\ 30.9949 & 1.6593 \end{bmatrix}$$

and

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} = \left[\begin{array}{c|c|c} -4.5025 & -3.0357 & -0.0237 \\ 80.2795 & 49.6316 & 0.3867 \end{array}\right],$$

It can be verified that $\det(\Phi) = -0.0798$ and so Φ is invertible as required by the theory. From (35) it can be shown

$$F_a = F_2 \left[\begin{array}{cccc} -0.1277 & -0.0851 & -0.0035 & -0.0974 \end{array} \right]$$

where F_2 is a non-zero scalar. From (26)

$$\begin{aligned} u(k) &= \left[\begin{array}{cccc} -124.864 & -83.191 & -3.445 & -95.211 \end{array} \right] y_a(k) \\ &\quad +9.7553 r(k). \end{aligned}$$

The LMI optimization gives an optimal value of $\mu = 2.7792$ for which $F_2 = 976.7225$. In the following simulations, to test the robustness of the controller, uncertainty of the form

$$\xi(k) = \left[\begin{array}{ccc} 0.3 & 0 & 0.3 \end{array} \right] x(k)$$

has been included. Clearly $\|\xi\| \leq 0.3\|x\|$ and $0.3 \leq \sqrt{2.7792}$, from the theory developed earlier, asymptotic stability will be retained for a zero reference signal, and ultimate boundedness results will be achieved if $r_s \neq 0$.

Figures 1-2 show the response of the closed loop system obtained from implementing the above controller on the nominal and uncertain systems. Good output tracking is achieved in both situations, although of course, total invariance to the matched uncertainty is not obtained (Figure 1). In the nominal system there is no deviation from the sliding surface; however in the presence of uncertainty some deviation appears (Figure 2). Further results pertaining to the application of the theory developed in this paper to the HIRM aircraft benchmark are given in [15].

VI. Conclusions

This paper has proposed a new output feedback based discrete-time sliding mode control scheme. It incorporates a tracking requirement and is dynamic in nature. Previous work has shown that with an appropriate choice of surface, discrete-time sliding mode control can be applied to non-minimum phase systems. The original scheme was static output feedback in nature and so inherently this imposed restrictions on the class of systems to which it was applicable. The scheme which has been proposed here includes a compensator and so the output feedback restrictions have been removed. The key aspect of the new scheme proposed here is that it is still applicable to non-minimum phase systems.

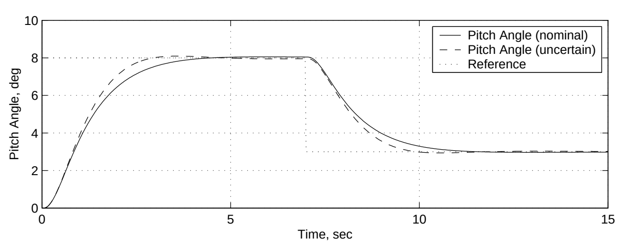

Fig. 1. Closed loop response to step changes in pitch angle

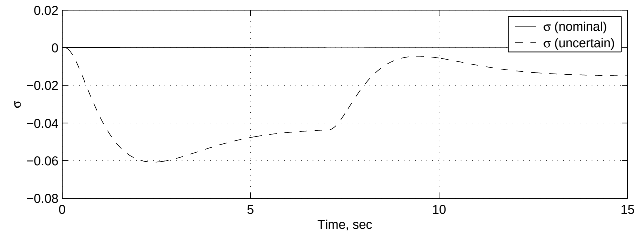

Fig. 2. Deviation from the sliding surface $\mathcal{S}$

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