On Synchronous Robotic Networks—Part II: Time Complexity of Rendezvous and Deployment Algorithms

Sonia Martínez, *Member, IEEE*, Francesco Bullo, *Senior Member, IEEE*, Jorge Cortés, *Senior Member, IEEE*, and Emilio Frazzoli, *Member, IEEE*

***Abstract*—This paper analyzes a number of basic coordination algorithms running on synchronous robotic networks. We provide upper and lower bounds on the time complexity of the move-toward-average and circumcenter laws, both achieving rendezvous, and of the centroid law, achieving deployment over a region of interest. The results are derived via novel analysis methods, including a set of results on the convergence rates of linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices.**

***Index Terms*—Circumcenter and centroid laws, coordination algorithms, deployment, rendezvous, robotic networks, time complexity.**

I. Introduction

A. Problem Motivation

RECENT YEARS have witnessed the emergence of numerous coordination algorithms for networked mobile systems. Despite remarkable progress, fundamental limits in terms of achievable performance, energy consumption, and operational time remain largely unknown. This is partially explained by the inherent difficulty in integrating the various sensing, computing, and communication aspects of problems involving groups of mobile agents. In this paper, we analyze the performance of several coordination algorithms achieving rendezvous and deployment. To achieve this goal, we rely on the general framework proposed in the companion paper [1] to formally model the behavior of robotic networks. Our research effort aims at developing tools and results to assess to what extent coordination algorithms are scalable and implementable in large networks of mobile agents. Ultimately, we aim to characterize the minimum amount of communication, sensing, and control that is necessary to reliably perform a desired task, and we aim to design algorithms that achieve those limits.

B. Literature Review

A description of the literature on cooperative mobile robotics and on control and communication issues is given in the companion paper [1]. Specific topics related to the present treatment include rendezvous [2]–[5], cyclic pursuit [6], [7], deployment [8], [9], swarm aggregation [10], gradient climbing [11], flocking [12], [13], vehicle routing [14], and consensus [15], [16].

C. Statement of Contributions

The companion paper [1] proposes a general framework to model robotic networks and formally analyze their behavior. In particular, [1] defines notions of time and communication complexity aimed at capturing the performance and cost of the execution of coordination algorithms. Here, we focus on establishing time complexity estimates for basic algorithms that achieve rendezvous and deployment.

The time complexity of an algorithm is the minimum number of communication rounds required by the agents to achieve the task. This is a classical notion in the study of distributed algorithms for networks with fixed communication topology, e.g., see [17]. From a controls perspective, the notion of time complexity is related to concepts such as settling time and speed of convergence. For a robotic network, it is natural to expect that these notions will depend on the number of agents. In this paper, we provide asymptotic characterizations of the time complexity of various coordination algorithms as the number of agents of the network grows. Arguably, this characterization serves as a measure of the scalability properties of the cooperative strategies under study.

We start by analyzing a simple averaging law for a network of locally connected agents moving on a line. This law is related to the widely known Vicsek's model; see [12] and [18]. We show that the averaging law achieves rendezvous (without preserving connectivity) and that its time complexity belongs to $\Omega(n)$ and $O(n^5)$. Second, for a network of locally connected agents moving on a line or on a segment, we show that the well-known circumcenter algorithm by [2] has time complexity of order $\Theta(n)$. (This algorithm achieves rendezvous while preserving connectivity with a communication graph with $O(n^2)$

Manuscript received May 4, 2005; revised July 1, 2006. Recommended by Associate Editor A. Garulli. This work was supported in part by the U.S. Office of Naval Research (ONR) Young Investigator Program (YIP) under Award N00014-03-1-0512, the National Science Foundation (NSF) under SENSORS Award IIS-0330008, the DARPA/AFOSR MURI under Award F49620-02-1-0325, and the NSF under CAREER Awards CCR-0133869, ECS-0546871, and CMS-0643679.

S. Martínez and J. Cortés are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093 USA (e-mail: soniamd@ucsd.edu; cortes@ucsd.edu).

F. Bullo is with the Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: bullo@engineering.ucsb.edu).

E. Frazzoli is with the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: frazzoli@mit.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2007.908304

links.) We then consider a network based on a different communication graph, called the limited Delaunay graph, which arises naturally in computational geometry and in the study of wireless communication topologies. For this less dense graph with $O(n)$ communication links, we show that the time complexity of the circumcenter algorithm grows to $\Theta(n^2 \log n)$. Intuitively, this tradeoff between the number of links in the communication graph and time complexity makes sense, as robotic networks where agents receive less information from their neighbors will need more communication rounds to achieve the desired task. For a network of agents moving on $\mathbb{R}^d$ (with a certain communication graph), we introduce a novel "parallel-circumcenter algorithm" and establish its time complexity of order $\Theta(n)$. Third, for a network of agents in a one-dimensional environment, we show that the time complexity of the deployment algorithm introduced in [8] is $O(n^3 \log n)$. To obtain these complexity estimates, we develop some novel analysis methods and build on the convergence results presented in [1]. An important observation is that the time complexity results presented here for the one-dimensional case induce lower bounds on the time complexity of the algorithms considered when executed in higher dimensions.

D. Organization

Section II briefly reviews the general approach to the modeling of robotic networks proposed in [1], presenting the notions of control and communication law, coordination tasks, and time complexity. Sections III and IV define the rendezvous and deployment coordination tasks, respectively, and present various coordination algorithms that achieve them. For both problems, we establish the asymptotic correctness of the proposed algorithms and characterize their time complexity. Finally, we present our conclusions in Section V. In the Appendix, we review some basic computational geometric structures employed along the discussion.

E. Notation

We let $\texttt{BooleSet} = \{\texttt{true, false}\}$. We let $\prod_{i\in\{1,\dots,n\}} S_i$ denote the Cartesian product of sets $S_1,\dots,S_n$. We let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the strictly positive and nonnegative real numbers, respectively. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the natural numbers and the nonnegative integers, respectively. For $x \in \mathbb{R}^d$, we let $\|x\|_2$ and $\|x\|_\infty$ denote the Euclidean and the ∞-norm of x, respectively (we also recall $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\|x\|_\infty$). For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$, we let $B(x,r)$ and $\bar{B}(x,r)$ denote the open and closed ball in $\mathbb{R}^d$ centered at x of radius r, respectively. We let $\mathrm{e}_1,\dots,\mathrm{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$. We define the vectors $\mathbf{0} = (0,\dots,0)$ and $\mathbf{1} = (1,\dots,1)$ in $\mathbb{R}^d$. For $f,g\colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$ (respectively, $|f(n)| \geq c|g(n)|$ for all $n \geq n_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$. We refer the reader to the Appendix for some useful geometric concepts. Finally, we will use the notation $\mathrm{Trid}_n(a,b,c), \mathrm{Circ}_n(a,b,c)$, and $\mathrm{ATrid}_n^{\pm}(a,b)$ to refer to various tridiagonal Toeplitz and circulant matrices as introduced in [1].

II. Synchronous Robotic Networks

The companion paper [1] proposes a formal model for robotic networks and defines notions of control and communication laws, coordination tasks, and time and communication complexity. To render this paper self-contained, we present here simplified versions of these notions.

Definition II.1 (Robotic Networks): A uniform network of robotic agents (*or* robotic network) S is a tuple $(I, \mathcal{A}, E_{\mathrm{cmm}})$ *consisting of* the following:

1) $I = \{1,\dots,n\}$; I is called the *set of unique identifiers (UIDs)*;
2) $\mathcal{A} = \{A^{[i]}\}_{i\in I}$, with $A^{[i]} = (X, U, X_0, f)$, is a set of identical control systems called *physical agents*;
3) E_{cmm} is a map from $\prod_{i\in I} X$ to the subsets of $I \times I$ called the *communication edge map*. •

Definition II.2 (Control and Communication Law): A control and communication law $\mathcal{CC}$ for $\mathcal{S}$ consists of the sets $\mathbb{T} = \{t_\ell\}_{\ell\in\mathbb{N}_0} \subset \mathbb{R}_{\geq 0}$ (*an increasing sequence of time instants, called communication schedule*) and $\mathcal{L}$ (*the* communication alphabet), and of the maps $\mathrm{msg} : X \times I \to \mathcal{L}$ (called *message-generation function*) and $\mathrm{ctl} : X \times \mathcal{L}^n \to \mathcal{U}$ (called *control function*). •

In the language of the companion paper [1], the control and communication law in Definition II.2 is a static, uniform, data-sampled, and time-independent law.

Definition II.3 (Evolution): The *evolution* of $(\mathcal{S}, \mathcal{CC})$ from initial conditions $x_0^{[i]} \in X_0^{[i]}, i \in I$, is the collection of curves $x^{[i]} : [t_0, +\infty) \to X, i \in I$, satisfying

$$\dot{x}^{[i]}(t) = f\left(x^{[i]}(t), \mathrm{ctl}^{[i]}\left(x^{[i]}(\lfloor t\rfloor_{\mathbb{T}}), y^{[i]}(\lfloor t\rfloor_{\mathbb{T}})\right)\right),$$

where $\lfloor t\rfloor_{\mathbb{T}} = \max\{t_\ell \in \mathbb{T} \mid t_\ell < t\}$ and $x^{[i]}(t_0) = x_0^{[i]}, i \in I$. Here, the curve $y^{[i]}\colon \mathbb{T} \to \mathcal{L}^n$ (describing the messages received by agent i) has jth component $y_j^{[i]}(t_\ell) = \mathrm{msg}^{[j]}(x^{[j]}(t_\ell), i)$, if $(j,i) \in E_{\mathrm{cmm}}(x^{[1]}(t_\ell),\dots,x^{[n]}(t_\ell))$, and $y_j^{[i]}(t_\ell) = \texttt{null}$, otherwise. •

When the messages interchanged among the network agents are just the agents' states, the corresponding alphabet is $\mathcal{L} = X \cup \{\texttt{null}\}$, and the message generation function $\mathrm{msg}_{\mathrm{std}}\colon X \times I \to X$ is $\mathrm{msg}_{\mathrm{std}}(x,j) = x$, referred to as the *standard message-generation function*. Next, let us introduce some useful examples of robotic networks.

Example II.4 (Locally Connected First-Order Agents in $\mathbb{R}^d$): Consider n agents $x^{[1]},\dots,x^{[n]}$ in $\mathbb{R}^d, d \geq 1$, obeying $\dot{x}^{[i]}(t) = u^{[i]}(t)$. These are identical agents of the form $A = (\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, (\mathbf{0}, \mathbf{e}_1,\dots,\mathbf{e}_d))$. Assume each agent can communicate to any other agent within distance r, that is, adopt $E_{r\text{-disk}}$ (defined in the Appendix) as the communication edge map. These data define the uniform robotic network $\mathcal{S}_{r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$. •

Example II.5 (LD-Connected First-Order Agents in $\mathbb{R}^d$): Consider the set of physical agents defined in the previous example. For $r \in \mathbb{R}_{>0}$, adopt the r-limited Delaunay map $E_{r\text{-LD}}$ defined by $(i,j) \in E_{r\text{-LD}}(x^{[1]},\dots,x^{[n]})$ if and only if

$$\left(V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)\right) \cap \left(V^{[j]} \cap \bar{B}\left(x^{[j]}, \frac{r}{2}\right)\right) \neq \emptyset, \qquad i \neq j,$$

where $\{V^{[1]}, \ldots, V^{[n]}\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x^{[1]}, \ldots, x^{[n]}\}$; see the Appendix. These data define the uniform robotic network $\mathcal{S}_{r\text{-LD}} = (I, \mathcal{A}, E_{r\text{-LD}})$. •

Example II.6 (Locally ∞-Connected First-Order Agents in $\mathbb{R}^d$): Consider the set of physical agents defined in the previous two examples. For $r \in \mathbb{R}_{>0}$, define the proximity edge map $E_{r\text{-square}}$ by $(i,j) \in E_{r\text{-square}}(x^{[1]}, \ldots, x^{[n]})$ if and only if

$$\|x^{[i]} - x^{[j]}\|_\infty \leq r, \qquad i \neq j.$$

These data define the uniform robotic network $\mathcal{S}_{r\text{-square}} = (I, \mathcal{A}, E_{r\text{-square}})$. •

Next, we define the notion of coordination task and of task achievement by a robotic network.

Definition II.7 (Coordination Task): Let $\mathcal{S}$ be a robotic network. A *coordination task* for $\mathcal{S}$ is a map $\mathcal{T} \colon \prod_{i \in I} X^{[i]} \to$ $\mathtt{BooleSet}$. The control and communication law $\mathcal{CC}$ achieves $\mathcal{T}$ if, for all initial conditions $x_0^{[i]} \in X_0^{[i]}, i \in I$, the corresponding evolution $t \mapsto x(t)$ has the property that there exists $T \in \mathbb{R}_{>0}$ with $\mathcal{T}(x(t)) = \mathtt{true}$ for all $t \geq T$. •

In the language of the companion paper [1], the coordination task in Definition II.7 is a static task. The notion of time complexity describes the performance of a law while achieving a coordination task.

Definition II.8 (Time Complexity): Let $\mathcal{S}$ be a robotic network, let $\mathcal{T}$ be a coordination task for $\mathcal{S}$, and let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$. The *time complexity to achieve* $\mathcal{T}$ with $\mathcal{CC}$ from $x_0 \in \prod_{i \in I} X_0^{[i]}$ is

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) = \inf\{\ell \mid \mathcal{T}(x(t_k)) = \mathtt{true}, \text{for all } k \geq \ell\},$$

where $t \mapsto x(t)$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from x_0. The *time complexity to achieve* $\mathcal{T}$ with $\mathcal{CC}$ is

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup\left\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) \mid x_0 \in \prod_{i \in I} X_0^{[i]}\right\}. \qquad \bullet$$

III. Rendezvous

In this section, we introduce rendezvous coordination tasks and analyze various coordination algorithms that achieve them, providing upper and lower bounds on their time complexity. Along the section, we will consider the networks $\mathcal{S}_{r\text{-disk}}$ and $\mathcal{S}_{r\text{-LD}}$ presented in Examples II.4 and II.5, respectively.

A. Rendezvous Tasks

First, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\mathrm{rndzvs}} : X^n \to \mathtt{BooleSet}$ for $\mathcal{S}$ is the static task defined by $\mathcal{T}_{\mathrm{rndzvs}}(x^{[1]}, \ldots, x^{[n]}) = \mathtt{true}$ if and only if

$$x^{[i]} = x^{[j]}, \qquad \text{for all } (i,j) \in E_{\mathrm{cmm}}\left(x^{[1]}, \ldots, x^{[n]}\right).$$

Second, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network with agents' state space $X \subset \mathbb{R}^d$. Examples of networks of this form are $\mathcal{S}_{r\text{-disk}}$ (see Examples II.4 and III.B) and $\mathcal{S}_{r\text{-LD}}$ (see Example II.5). For $\varepsilon > 0$, the *ε-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}} :$ $X^n \to \mathtt{BooleSet}$ for $\mathcal{S}$ is defined by $\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) = \mathtt{true}$ if and only if

$$\left\|x^{[i]} - \mathrm{avrg}\left(\left\{x^{[i]}\right\} \cup \left\{x^{[j]} \mid (i,j) \in E_{\mathrm{cmm}}(x)\right\}\right)\right\|_2 < \varepsilon,$$

for all $i \in I$, where avrg computes the average of a finite point set in $\mathbb{R}^d$, that is, $\mathrm{avrg}(\{x_1, \ldots, x_h\}) = (x_1 + \cdots + x_h)/h$, and where we let $x = (x^{[1]}, \ldots, x^{[n]}) \in X^n \subset (\mathbb{R}^d)^n$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is $\mathtt{true}$ at $x \in (\mathbb{R}^d)^n$ if, for all $i \in I, x^{[i]}$ is at distance less than ε from the average of its own position with the position of its E_{cmm}-neighbors.

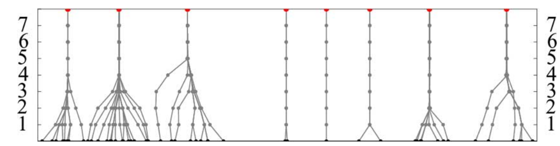


Fig. 1. Evolution of a robotic network under the move-toward-average control and communication law in Section III-B implemented over the r-disk graph, with $r = 1.5$. The vertical axis corresponds to the elapsed time and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

B. Rendezvous Without Connectivity Constraint Via the Move-Toward-Average Control and Communication Law

From Example II.4, consider the uniform network $\mathcal{S}_{r\text{-disk}}$ of locally connected first-order agents in $\mathbb{R}^d$. We now define a control and communication law that we refer to as the move-toward-average law and that we denote by $\mathcal{CC}_{\mathrm{avrg}}$. We loosely describe it as follows.

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent transmits its position. Between communication rounds, each agent moves towards and reaches the point that is the average of its neighbors' positions; the average point is computed including the agent's own position.

Note that this law is related to the Vicsek's model discussed in [12] and [18], where, however, different communication topologies are adopted and where the coordination task is that of heading alignment rather than rendezvous. Next, we formally define the law as follows. First, we take $\mathbb{T} = \mathbb{N}_0$ and we assume that each agent operates with the standard message-generation function, i.e., we set $\mathcal{L} = \mathbb{R}^d \cup \{\mathtt{null}\}$ and $\mathrm{msg}(x,j) = \mathrm{msg}_{\mathrm{std}}(x,j) = x$. Second, we define the control function $\mathrm{ctl} : \mathbb{R}^d \times \mathcal{L}^n \to \mathbb{R}^d$ by

$$\mathrm{ctl}(x_{\mathrm{smpld}}, y) = \mathrm{avrg}(\{x_{\mathrm{smpld}}\} \cup \{x_{\mathrm{rcvd}} \mid x_{\mathrm{rcvd}} \text{ is a nonnull message in } y\}) - x_{\mathrm{smpld}}.$$

In summary, we set $\mathcal{CC}_{\mathrm{avrg}} = (\mathbb{N}_0, \mathbb{R}^d, \mathrm{msg}_{\mathrm{std}}, \mathrm{ctl})$. An implementation of this control and communication law is shown in Fig. 1 for $d = 1$. Note that, along the evolution, the following are true: 1) several agents *rendezvous*, i.e., agree upon a common location, and 2) some agents are connected at the simulation's beginning and not connected at the simulation's end.

Our main objective here is to characterize the complexity of this law.

Theorem III.1 (Time Complexity of Move-Towards-Average Law): For $d = 1$, the network $\mathcal{S}_{r\text{-disk}}$, the law $\mathcal{CC}_{\text{avrg}}$, and the task $\mathcal{T}_{\text{rndzvs}}$ satisfy $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in O(n^5)$ and $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in \Omega(n)$. •

Proof: One can easily prove that, along the evolution of the network, the ordering of the agents is preserved, i.e., if $x^{[i]}(\ell) \leq x^{[j]}(\ell)$, then $x^{[i]}(\ell+1) \leq x^{[j]}(\ell+1)$. However, links between agents are not necessarily preserved (see, e.g., Fig. 1). Indeed, connected components may split along the evolution. However, merging events are not possible. Consider two contiguous connected components C_1 and C_2, with C_1 to the left of C_2. By definition, the rightmost agent of C_1 and the leftmost agent of C_2 are at a distance strictly bigger than r. Now, by executing the algorithm, they can only but increase that distance, since the rightmost agent of C_1 will move to the left and the leftmost agent of C_2 will move to the right. Therefore, connected components do not merge.

Consider first the case of an initial configuration of the network for which the communication graph remains connected throughout the evolution. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[n]}(0) = (x_0)_n$. Let $\alpha \in \{3, \dots, n\}$ have the property that agents $\{2, \dots, \alpha-1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in one time instant, and the statement in theorem is easily seen to be true.) Note that we can assume that agents $\{2, \dots, \alpha-1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$x^{[1]}(1) = \frac{1}{\alpha-1} \sum_{k=1}^{\alpha-1} x^{[k]}(0),$$
$$x^{[\gamma]}(1) \in \left[\frac{1}{\alpha} \sum_{k=1}^{\alpha} x^{[k]}(0), *\right], \qquad \gamma \in \{2, \dots, \alpha-1\},$$

where "$*$" denotes a certain unimportant point. Now, we show

$$x^{[1]}(\alpha-1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}. \tag{1}$$

Let us first show the inequality for $\alpha = 3$. Note that the fact that the communication graph remains connected implies that agent 2 is still a neighbor of agent 1 at the time instant $\ell = 1$. Therefore, $x^{[1]}(2) \geq \frac{1}{2}(x^{[1]}(1) + x^{[2]}(1))$, and from here, we deduce

$$\begin{aligned} &x^{[1]}(2) - x^{[1]}(0) \\ &\quad \geq \frac{1}{2}\left(x^{[2]}(1) - x^{[1]}(0)\right) \\ &\quad \geq \frac{1}{2}\left(\frac{1}{3}\left(x^{[1]}(0) + x^{[2]}(0) + x^{[3]}(0)\right) - x^{[1]}(0)\right) \\ &\quad \geq \frac{1}{6}\left(x^{[3]}(0) - x^{[1]}(0)\right) \geq \frac{r}{6}. \end{aligned}$$

Let us now proceed by induction. Assume that inequality (1) is valid for $\alpha - 1$ and let us prove it for α. Consider first the possibility when at the time instant $\ell = 1$, the agent $\alpha-1$ is still a neighbor of agent 1. In this case, $x^{[1]}(2) \geq \frac{1}{\alpha-1} \sum_{k=1}^{\alpha-1} x^{[k]}(1)$, and from here, we deduce

$$\begin{aligned} &x^{[1]}(2) - x^{[1]}(0) \\ &\quad \geq \frac{1}{\alpha-1}\left(x^{[\alpha-1]}(1) - x^{[1]}(0)\right) \\ &\quad \geq \frac{1}{\alpha-1}\left(\frac{1}{\alpha}\sum_{k=1}^{\alpha} x^{[k]}(0) - x^{[1]}(0)\right) \\ &\quad \geq \frac{1}{\alpha(\alpha-1)}\left(x^{[\alpha]}(0) - x^{[1]}(0)\right) \geq \frac{r}{\alpha(\alpha-1)}, \end{aligned}$$

which, in particular, implies (1). Consider then the case when agent $\alpha - 1$ is not a neighbor of agent 1 at the time instant $\ell = 1$. Let $\beta < \alpha$ such that agent $\beta - 1$ is a neighbor of agent 1 at $\ell = 1$, but agent β is not. Since $\beta < \alpha$, we have by induction $x^{[1]}(\beta) - x^{[1]}(1) \geq \frac{r}{\beta(\beta-1)}$. From here, we deduce that $x^{[1]}(\alpha-1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}$.

It is clear that after $\ell_1 = \alpha - 1$, we could again consider two complementary cases (either agent 1 has all others as neighbors or not) and repeat the same argument once again. In that way, we would find ℓ_2 such that the distance traveled by agent 1 after ℓ_2 rounds would be lower bounded by $\frac{2r}{n(n-1)}$. Repeating this argument iteratively, the worst possible case is one in which agent 1 keeps moving to the right and there is always another agent which is not a neighbor. Since $\operatorname{diam}(x_0, I) \leq (n-1)r$, in the worst possible situation, there exists some time ℓ_k such that $\frac{kr}{(n-1)n} = O(r(n-1))$. This implies that $k = O((n-1)^2 n)$. Now, we can upper bound the total convergence time ℓ_k by $\ell_k = \sum_{i=1}^{k} \alpha_i - k \leq k(n-1)$, where we have used that $\alpha_i \leq n$ for all $i \in \{1, \dots, n\}$. From here, we see that $\ell_k = O((n-1)^3 n)$ and hence, we deduce that in $O(n(n-1)^3)$ time instants there cannot be any agent which is not a neighbor of the agent 1. Hence, all agents rendezvous at the next time instant. Consequently

$$\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}, x_0) = O(n(n-1)^3).$$

Finally, for a general initial configuration x_0, because there is a finite number of agents, only a finite number of splittings (at most $n-1$) of the connected components of the communication graph can take place along the evolution. Therefore, we conclude $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) = O(n^5)$.

Let us now prove the lower bound. Consider an initial configuration $x_0 \in \mathbb{R}^n$ where all agents are positioned in increasing order according to their identity, and exactly at a distance r apart, say $(x_0)_{i+1} - (x_0)_i = r, i \in \{1, \dots, n-1\}$. Assume, for simplicity, that n is odd—when n is even, one can reason in an analogous way. Because of the symmetry of the initial condition, in the first time step, only agents 1 and n move. All the remaining agents remain in their position because it coincides with the average of its neighbors' position and its own. At the second time step, only agents $1, 2, n-1$, and n move, and the others remain still because of the symmetry. Applying this idea iteratively, one deduces the time step when agents $\frac{n-1}{2}$ and $\frac{n+3}{2}$ move for the first time is lower bounded by $\frac{n-1}{2}$. Since both agents have still at least a neighbor (agent $\frac{n+1}{2}$), the task $\mathcal{T}_{\text{rndzvs}}$ has not been achieved yet at this time step. Therefore, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}, x_0) \geq \frac{n-1}{2}$ and the result follows. ■

C. Rendezvous With Connectivity Constraint via Circumcenter Control and Communication Laws

Here, we define the *circumcenter control and communication law* $\mathcal{CC}_{\text{crcmcntr}}$ for both networks $\mathcal{S}_{r\text{-disk}}$ and $\mathcal{S}_{r\text{-LD}}$. This is a static, uniform, data-sampled, time-independent law originally introduced by [2] and later studied in [4] and [5]. The circumcenter of a point set is the center of the smallest radius sphere that encloses the set. Loosely speaking, the evolution of the network under the $\mathcal{CC}_{\text{crcmcntr}}$ law can be described as follows.

[Informal description] Communication rounds take place at each natural instant of time. At each communication round, each agent performs the following tasks: 1) it transmits its position and receives its neighbors' positions; 2) it computes the circumcenter of the point set comprised of its neighbors and of itself; and 3) it moves toward this circumcenter while maintaining connectivity with its neighbors.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0, \mathcal{L} = \mathbb{R}^d \cup \{\texttt{null}\}$ and $\text{msg}^{[i]} = \text{msg}_{\text{std}}, i \in I$. We define the control function in three steps. First, given an agent state x and an array of messages y, define the point

$$x_{\text{goal}}(x,y) = \text{Circum}\,(\{x\} \cup \{x_{\text{rcvd}} \mid \text{for all nonnull } x_{\text{rcvd}} \in y\}),$$

where $\text{Circum}\,(q_1,\dots,q_l)$ is the circumcenter of the set of points $q_1,\dots,q_l$; see definition in the Appendix. This definition is well posed because the nonnull messages $y^{[i]}(\ell)$ received by the agent $i \in I$ at any time $\ell \in \mathbb{N}_0$ are the positions of its neighbors. Second, connectivity is maintained by restricting the allowable motion of each agent in the following appropriate manner. If agents i and j are neighbors at time $\ell \in \mathbb{N}_0$, then we require their subsequent positions to belong to

$$\bar{B}\left(\frac{x^{[i]}(\ell) + x^{[j]}(\ell)}{2}, \frac{r}{2}\right).$$

If an agent i has its neighbors at locations $\{q_1,\dots,q_l\}$ at time ℓ, then its *constraint set* $\mathcal{D}_r(x^{[i]}(\ell), \{q_1,\dots,q_l\})$ is

$$\mathcal{D}_r(x^{[i]}(\ell), \{q_1,\dots,q_l\}) = \bigcap_{q \in \{q_1,\dots,q_l\}} \bar{B}\left(\frac{x^{[i]}(\ell) + q}{2}, \frac{r}{2}\right).$$

Third, we define a function that encodes the desire to move from a first point to a second point while remaining inside a convex set. For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, define the "from to inside" function by

$$\text{fti}(q_0,q_1,Q) = \begin{cases} q_1, & \text{if } q_1 \in Q \\ [q_0,q_1] \cap \partial Q, & \text{if } q_1 \notin Q, \end{cases}$$

where $[q_0,q_1]$ denotes the closed segment with endpoints q_0 and q_1. With these three ingredients, we are now ready to define the last ingredient of $\mathcal{CC}_{\text{crcmcntr}}$. We define the control function $\text{ctl} : \mathbb{R}^d \times \mathcal{L}^n \to \mathbb{R}^d$ by

$$\text{ctl}(x_{\text{smpld}}, y) = \text{fti}\Bigg(x_{\text{smpld}}, x_{\text{goal}}(x_{\text{smpld}}, y), \mathcal{D}_r(x_{\text{smpld}}, \{x_{\text{rcvd}} \mid \text{for all nonnull } x_{\text{rcvd}} \in y)\Bigg). \tag{2}$$

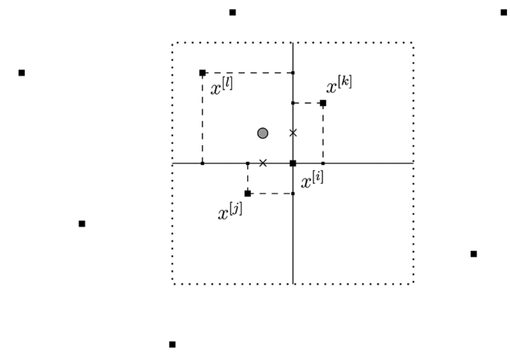


Fig. 2. Parallel circumcenter control and communication law in $\mathbb{R}^2$. The target point for the agent i is plotted in light gray and has coordinates $(\text{Circum}\,(\tau_1(\mathcal{M}^{[i]})), \text{Circum}\,(\tau_2(\mathcal{M}^{[i]})))$.

Evolving under this control law, each agent i moves during the interval $[\lfloor t \rfloor, \lfloor t \rfloor + 1]$ from the point $x^{[i]}(\lfloor t \rfloor)$ towards the point $x_{\text{goal}}(x^{[i]}(\lfloor t \rfloor), y(\lfloor t \rfloor))$ as much as possible while remaining inside an appropriate connectivity set.

Next, we consider the network $\mathcal{S}_{r\text{-square}}$ of locally ∞-connected first-order agents in $\mathbb{R}^d$; see Example II.6. For this network, we define the *parallel circumcenter law* $\mathcal{CC}_{\text{pll-crcmcntr}}$ by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is static, uniform, data-sampled, and time-independent. We set $\mathbb{T} = \mathbb{N}_0, \mathcal{L} = \mathbb{R}^d \cup \{\texttt{null}\}$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}, i \in I$. We define the control function $\text{ctl} : \mathbb{R}^d \times \mathcal{L}^n \to \mathbb{R}^d$ by

$$\text{ctl}(x_{\text{smpld}}, y) = (\text{Circum}\,(\tau_1(\mathcal{M})) - (x_{\text{smpld}})_1, \dots, \text{Circum}(\tau_d(\mathcal{M})) - (x_{\text{smpld}})_d), \tag{3}$$

where $\mathcal{M} = \{x_{\text{smpld}}\} \cup \{x_{\text{rcvd}} \mid \text{ for all nonnull } x_{\text{rcvd}} \in y\}$ and where $\tau_1,\dots,\tau_d : \mathbb{R}^d \to \mathbb{R}$ denote the canonical projections of $\mathbb{R}^d$ onto $\mathbb{R}$. See Fig. 2 for an illustration of this law in $\mathbb{R}^2$.

Asymptotic Behavior and Complexity Analysis: The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter law.

Theorem III.2 (Correctness of the Circumcenter Laws): For $d \in \mathbb{N}, r \in \mathbb{R}_{>0}$, and $\varepsilon \in \mathbb{R}_{>0}$, the following statements hold:
1) on the network $\mathcal{S}_{r\text{-disk}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;
2) on the network $\mathcal{S}_{r\text{-LD}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;
3) on the network $\mathcal{S}_{r\text{-square}}$, the law $\mathcal{CC}_{\text{pll-crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;
4) the evolutions of $(\mathcal{S}_{r\text{-disk}}, \mathcal{CC}_{\text{crcmcntr}})$, of $(\mathcal{S}_{r\text{-LD}}, \mathcal{CC}_{\text{crcmcntr}})$, and of $(\mathcal{S}_{r\text{-square}}, \mathcal{CC}_{\text{pll-crcmcntr}})$ have the property that, if two agents belong to the same connected component of the communication graph at $\ell \in \mathbb{N}_0$, then they continue to belong to the same connected component for all subsequent times $k \geq \ell$. •

Proof: The results on $\mathcal{S}_{r\text{-disk}}$ appeared originally in [2]. The proof for the results on $\mathcal{S}_{r\text{-LD}}$ is provided in [5]. We postpone the proof for $\mathcal{S}_{r\text{-square}}$ to the proof of Theorem III.3. ■

Next, we analyze the time complexity of $\mathcal{CC}_{\text{crcmcntr}}$. We provide complete results for the case $d = 1$. As we see next, the

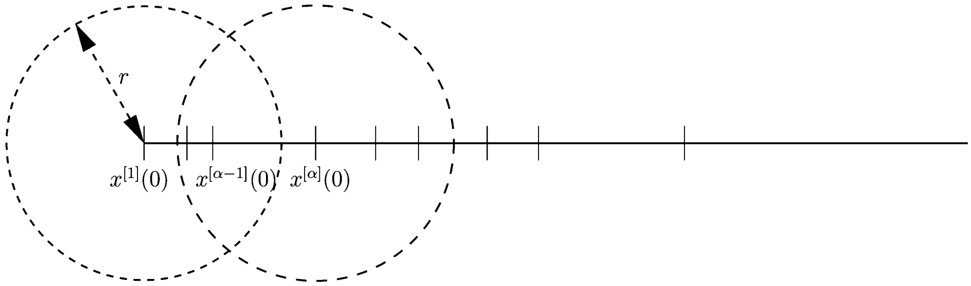


Fig. 3. Definition of $\alpha \in \{3, \ldots, n\}$ for an initial network configuration.

complexity of $\mathcal{CC}_{\text{crcmcntr}}$ differs dramatically when applied to the two robotic networks with different communication graphs.

Theorem III. 3 (Time Complexity of Circumcenter Laws): For $r \in \mathbb{R}_{>0}$ and $\varepsilon \in]0,1[$, the following statements hold:

1) for $d = 1$, on the network $\mathcal{S}_{r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(n)$;
2) for $d = 1$, on the network $\mathcal{S}_{r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(n^2 \log(n\varepsilon^{-1}))$;
3) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{r\text{-square}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(n)$. •

Proof: Let $x_0 \in \mathbb{R}^n$. Throughout the proof, we let $\pi_{\mathbb{R}}(y)$ denote the subset of nonnull messages in y.

Fact 1) Let us show that, for $d = 1$, the connectivity constraints on each agent $i \in I$ imposed by the constraint set $\mathcal{D}_r(x^{[i]}, \pi_{\mathbb{R}}(y))$ are superfluous, i.e., the control function in (2) equals $x_{\text{goal}}(x_{\text{smpld}}, y)$. To see this, assume that agents i and j are neighbors in the r-disk graph at time instant ℓ, define $\mathcal{M}^{[i]}$ as $\pi_{\mathbb{R}}(y^{[i]}(\ell)) \cup \{x^{[i]}(\ell)\}$, and let us show that $\mathrm{Circum}\,(\mathcal{M}^{[i]})$ belongs to $\bar{B}\left(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2}\right)$. Without loss of generality, let $x^{[i]}(\ell) \le x^{[j]}(\ell)$. Let $x_-^{[i]}(\ell), x_+^{[i]}(\ell)$ denote the positions of the leftmost and rightmost agents among the neighbors of agent i. Note that $x^{[i]}(\ell) \le x^{[j]}(\ell) \le x_+^{[i]}(\ell)$ and $\mathrm{Circum}(\mathcal{M}^{[i]}) = \frac{1}{2}(x_-^{[i]}(\ell) + x_+^{[i]}(\ell))$. Then,

$$\begin{aligned}
&\left|\mathrm{Circum}(\mathcal{M}^{[i]}) - \frac{1}{2}\left(x^{[i]}(\ell) + x^{[j]}(\ell)\right)\right| \\
&\quad = \frac{1}{2}\left|x_-^{[i]}(\ell) - x^{[i]}(\ell) + x_+^{[i]}(\ell) - x^{[j]}(\ell)\right| \\
&\quad \le \frac{1}{2}\max\left\{\left|x_-^{[i]}(\ell) - x^{[i]}(\ell)\right|, \left|x_+^{[i]}(\ell) - x^{[j]}(\ell)\right|\right\} \le \frac{r}{2},
\end{aligned}$$

as claimed. Therefore, we have that $x^{[i]}(\ell+1) = \mathrm{Circum}(\mathcal{M}^{[i]})$. Likewise, one can deduce $\mathrm{Circum}(\mathcal{M}^{[i]}) \le \mathrm{Circum}(\mathcal{M}^{[j]})$, and therefore, the order of the agents is preserved.

Consider the case when $E_{r\text{-disk}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \le \cdots \le x^{[n]}(0) = (x_0)_n$. Let $\alpha \in \{3, \ldots, n\}$ have the property that agents $\{2, \ldots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) See Fig. 3 for an illustration of these definitions. Note that we can assume that agents $\{2, \ldots, \alpha - 1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α, rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$\begin{aligned}
x^{[1]}(1) &= \frac{x^{[1]}(0) + x^{[\alpha-1]}(0)}{2}, \\
x^{[\gamma]}(1) &\in \left[\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2}, \frac{x^{[1]}(0) + x^{[\gamma]}(0) + r}{2}\right],
\end{aligned}$$

for $\gamma \in \{2, \ldots, \alpha - 1\}$. These equalities imply that $x^{[1]}(1) - x^{[1]}(0) = \frac{1}{2}\big(x^{[\alpha-1]}(0) - x^{[1]}(0)\big) \le \frac{1}{2}r$. Analogously, we deduce $x^{[1]}(2) - x^{[1]}(1) \le \frac{1}{2}r$, and therefore

$$x^{[1]}(2) - x^{[1]}(0) \le r. \tag{4}$$

On the other hand, from $x^{[1]}(2) \in \left[\frac{1}{2}(x^{[1]}(1) + x^{[\alpha-1]}(1)), *\right]$ (where the symbol "$*$" represents a certain unimportant point in $\mathbb{R}$), we deduce

$$\begin{aligned}
&x^{[1]}(2) - x^{[1]}(0) \\
&\quad \ge \frac{1}{2}\left(x^{[1]}(1) + x^{[\alpha-1]}(1)\right) - x^{[1]}(0) \\
&\quad \ge \frac{1}{2}\left(x^{[\alpha-1]}(1) - x^{[1]}(0)\right) \\
&\quad \ge \frac{1}{2}\left(\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2} - x^{[1]}(0)\right) \\
&\quad = \frac{1}{4}\left(x^{[\alpha]}(0) - x^{[1]}(0)\right) \ge \frac{1}{4}r.
\end{aligned} \tag{5}$$

Inequalities (4) and (5) mean that, after at most two time instants, agent 1 has traveled an amount larger than $r/4$. In turn, this implies that

$$\begin{aligned}
\frac{\mathrm{diam}(x_0, I)}{r} &\le \mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}, x_0) \\
&\le \frac{4\mathrm{diam}(x_0, I)}{r}.
\end{aligned}$$

If $E_{r\text{-disk}}(x_0)$ is not connected, note that along the network evolution, the connected components of the r-disk graph do not change. Therefore, using the previous characterization on the amount traveled by the leftmost agent of each connected component in at most two time instants, we deduce

$$\begin{aligned}
&\frac{1}{r}\max_{C \in \mathcal{C}_{E_{r\text{-disk}}(x_0)}} \mathrm{diam}(x_0, C) \\
&\quad \le \mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}, x_0) \\
&\quad \le \frac{4}{r}\max_{C \in \mathcal{C}_{E_{r\text{-disk}}(x_0)}} \mathrm{diam}(x_0, C).
\end{aligned}$$

Note that the connectedness of each $C \in \mathcal{C}_{E_{r\text{-disk}}}(x_0)$ implies that $\operatorname{diam}(x_0, C) \leq (n-1)r$, and therefore, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in O(n)$. Moreover, for $x_0 \in \mathbb{R}^n$ such that $(x_0)_{i+1} - (x_0)_i = r, i \in \{1,\ldots,n-1\}$, we have $\operatorname{diam}(x_0, I) = (n-1)r$, and therefore, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) \geq n-1$. We conclude that

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(n).$$

Fact 2) In the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Also, note that the r-limited Delaunay graph and the r-disk graph have the same connected components (cf. [9]). Using an argument similar to the one previously mentioned, one can show that the connectivity constraints imposed by the constraint sets $\mathcal{D}_r(x^{[i]}(\lfloor t \rfloor), \pi_{\mathbb{R}}(y))$ are again superfluous.

Consider first the case when $E_{r\text{-LD}}(x_0)$ is connected. Note that this is equivalent to $E_{r\text{-disk}}(x_0)$ being connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[n]}(0) = (x_0)_n$. The evolution of the network under $\mathcal{CC}_{\mathrm{crcmcntr}}$ can then be described as the discrete-time dynamical system

$$\begin{aligned} x^{[1]}(\ell+1) &= \frac{1}{2}\left(x^{[1]}(\ell) + x^{[2]}(\ell)\right), \\ x^{[2]}(\ell+1) &= \frac{1}{2}\left(x^{[1]}(\ell) + x^{[3]}(\ell)\right), \\ &\vdots \\ x^{[n-1]}(\ell+1) &= \frac{1}{2}\left(x^{[n-2]}(\ell) + x^{[n]}(\ell)\right), \\ x^{[n]}(\ell+1) &= \frac{1}{2}\left(x^{[n-1]}(\ell) + x^{[n]}(\ell)\right). \end{aligned}$$

Note that this evolution respects the ordering of the agents. Equivalently, we can write $x(\ell+1) = A\,x(\ell)$, where A is the $n \times n$ matrix given by

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Note that $A = \mathrm{ATrid}_n^+\left(\frac{1}{2}, 0\right)$ as defined in [1]. Reference [1, Theorem A.4, Case 1)] implies that, for $x_{\mathrm{ave}} = \frac{1}{n}\mathbf{1}^T x(0)$, we have that $\lim_{\ell \to +\infty} x(\ell) = x_{\mathrm{ave}}\mathbf{1}$, and that the maximum time required for $\|x(\ell) - x_{\mathrm{ave}}\mathbf{1}\|_2 \leq \eta \|x(0) - x_{\mathrm{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^n$) is $\Theta(n^2 \log \eta^{-1})$. (Note that this also implies that agents rendezvous at the location given by the average of their initial positions. In other words, the asymptotic rendezvous position for this case can be expressed in closed form, as opposed to the case with the r-disk communication graph.)

Next, let us convert the contraction inequality on 2-norms into an appropriate inequality on ∞-norms. Note that $\operatorname{diam}(x_0, I) \leq (n-1)r$ because $E_{r\text{-LD}}(x_0)$ is connected. Therefore

$$\begin{aligned} &\|x(0) - x_{\mathrm{ave}}\mathbf{1}\|_\infty \\ &\qquad = \max_{i \in I}\left|x^{[i]}(0) - x_{\mathrm{ave}}\right| \leq \left|x_0^{[1]} - x_0^{[n]}\right| \leq (n-1)r. \end{aligned}$$

For ℓ of order $n^2 \log \eta^{-1}$, we use this bound on $\|x(0) - x_{\mathrm{ave}}\mathbf{1}\|_\infty$ and the basic inequalities $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n}\|v\|_\infty$ for all $v \in \mathbb{R}^n$, to obtain

$$\begin{aligned} \|x(\ell) - x_{\mathrm{ave}}\mathbf{1}\|_\infty &\leq \|x(\ell) - x_{\mathrm{ave}}\mathbf{1}\|_2 \leq \eta\|x(0) - x_{\mathrm{ave}}\mathbf{1}\|_2 \\ &\leq \eta\sqrt{n}\|x(0) - x_{\mathrm{ave}}\mathbf{1}\|_\infty \leq \eta\sqrt{n}(n-1)r. \end{aligned}$$

This means that $(r\varepsilon)$-rendezvous is achieved for $\eta\sqrt{n}(n-1)r = r\varepsilon$, that is, in time $O(n^2 \log \eta^{-1}) = O(n^2 \log(n\varepsilon^{-1}))$. Next, we show the lower bound. Consider the unit-length eigenvector $\mathbf{v}_n = \sqrt{\frac{2}{n+1}}\left(\sin\frac{\pi}{n+1}, \ldots, \sin\frac{n\pi}{n+1}\right)^T \in \mathbb{R}^n$ of $\mathrm{Trid}_{n-1}\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ corresponding to the largest singular value $\cos\left(\frac{\pi}{n}\right)$. For $\mu = \frac{-1}{10\sqrt{2}} r n^{5/2}$, we then define the initial condition $x_0 = \mu P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{n-1} \end{bmatrix} \in \mathbb{R}^n$. One can show that $(x_0)_i < (x_0)_{i+1}$ for $i \in \{1, \ldots, n-1\}$, that $(x_0)_{\mathrm{ave}} = 0$, and that $\max\{(x_0)_{i+1} - (x_0)_i | i \in \{1, \ldots, n-1\}\} \leq r$. Using [1, Lemma A.5] and because $\|w\|_\infty \leq \|w\|_2 \leq \sqrt{n}\|w\|_\infty$ for all $w \in \mathbb{R}^n$, we compute

$$\begin{aligned} \|x_0\|_\infty &= \frac{rn^{5/2}}{10\sqrt{2}}\left\|P_+\begin{bmatrix} 0 \\ \mathbf{v}_{n-1}\end{bmatrix}\right\|_\infty \\ &\geq \frac{rn^2}{10\sqrt{2}}\left\|P_+\begin{bmatrix} 0 \\ \mathbf{v}_{n-1}\end{bmatrix}\right\|_2 \\ &\geq \frac{rn}{10\sqrt{2}}\|\mathbf{v}_{n-1}\|_2 = \frac{rn}{10\sqrt{2}}. \end{aligned}$$

The trajectory $x(\ell) = \left(\cos\left(\frac{\pi}{n}\right)\right)^\ell x_0$, therefore, satisfies

$$\|x(\ell)\|_\infty = \left(\cos\left(\frac{\pi}{n}\right)\right)^\ell \|x_0\|_\infty \geq \frac{rn}{10\sqrt{2}}\left(\cos\left(\frac{\pi}{n}\right)\right)^\ell.$$

Therefore, $\|x(\ell)\|_\infty$ is larger than $\frac{1}{2}r\varepsilon$ so long as $\frac{1}{10\sqrt{2}}n\left(\cos\left(\frac{\pi}{n}\right)\right)^\ell > \frac{1}{2}\varepsilon$, that is, so long as

$$\ell < \frac{\log(\varepsilon^{-1}n) - \log(5\sqrt{2})}{-\log\left(\cos\left(\frac{\pi}{n}\right)\right)}.$$

The rest of the proof is analogous to the case 1) of Theorem A.3 in [1] for the lower bound result.

If $E_{r\text{-LD}}(x_0)$ is not connected, along the network evolution, the connected components do not change. Therefore, the previous reasoning can be applied to each connected component. Since the number of agents in each connected component is strictly less that n, the time complexity can only but improve. Therefore, we conclude that

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta\left(n^2 \log(n\varepsilon^{-1})\right).$$

Fact 3) Finally, we prove the statements regarding $\mathcal{S}_{r\text{-square}}$ and $\mathcal{CC}_{\text{pll-crcmcntr}}$ in Fact 3) and in the previous Theorem III.2. By definition, agents i and j are neighbors at time $\ell \in \mathbb{N}_0$ if and only if $\|x^{[i]}(\ell) - x^{[j]}(\ell)\|_\infty \leq r$, which is equivalent to

$$\left|\tau_k\left(x^{[i]}(\ell)\right) - \tau_k\left(x^{[j]}(\ell)\right)\right| \leq r, \qquad k \in \{1, \ldots, d\}.$$

Recall from the proof of Fact 1) that the connectivity constraints of $\mathcal{CC}_{\mathrm{crcmcntr}}$ on each agent are trivially satisfied in the one-dimensional case. This fact has the following important consequence: from the expression for the control function in $\mathcal{CC}_{\text{pll-crcmcntr}}$, we deduce that the evolution under $\mathcal{CC}_{\text{pll-crcmcntr}}$ of the robotic network $\mathcal{S}_{r\text{-square}}$ (in d-dimensions) can be alternatively described as the evolution under $\mathcal{CC}_{\mathrm{crcmcntr}}$ of d robotic networks $\mathcal{S}_{r\text{-disk}}$ in $\mathbb{R}$. The correctness and the time complexity results now follow from the analysis of $\mathcal{CC}_{\mathrm{crcmcntr}}$ at $d = 1$. ■

Remark III.4 (Analysis in Higher Dimensions): The results in cases 1) and 2) of Theorem III.3 induce lower bounds on the time complexity of the circumcenter law in higher dimensions. Indeed, we have the following:

1) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(n)$;
2) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(n^2 \log(n\varepsilon^{-1}))$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{r\text{-disk}}$. We run the algorithm starting from generic initial configurations (where, in particular, agents' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$. We have consistently obtained that the time complexity to achieve $\mathcal{T}_{\mathrm{rndzvs}}$ with $\mathcal{CC}_{\mathrm{crcmcntr}}$ starting from these initial configurations is independent of the number of agents. This leads us to conjecture that initial configurations where all agents are aligned (equivalently, the one-dimensional case) give rise to the worst possible performance of the algorithm. In other words, we conjecture that, for $d \geq 2$, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) = \Theta(n)$. •

Remark III.5 (Congestion Effects): As discussed in [1, Remark II.9], one way of incorporating congestion effects into the network operation is to assume that the parameters of the physical components of the network depend upon the number of robots. For instance, it is common to assume that the communication range decreases with the number of robots. Theorem III.3 presents an alternative, equivalent way of looking at congestion: the results hold under the assumption that the communication range is constant, but allow for the diameter of the initial network configuration (the maximum interagent distance) to grow unbounded with the number of robots. •

IV. Deployment

In this section, we introduce the deployment coordination task and analyze a coordination algorithm that achieves it, providing upper and lower bounds on its time complexity. Along the section, we consider the uniform robotic network $\mathcal{S}_{r\text{-LD}}$ presented in Example II.5 with parameter $r \in \mathbb{R}_{>0}$. Given a convex polytope $Q \subset \mathbb{R}^d$, with an integrable density function $\phi : Q \to \mathbb{R}_{>0}$, we assume that the initial positions of the agents belong to Q and we intend to design a control law that keeps them in Q for subsequent times.

A. Deployment Task

By optimal deployment on the convex polytope $Q \subset \mathbb{R}^d$ with density function $\phi : Q \to \mathbb{R}_{>0}$, we mean the following objective: place the agents on Q so that the expected square Euclidean distance from a point in Q to one of the agents is minimized. To define this task formally, let us review some known preliminary notions; we will require some computational geometric notions from the Appendix. We consider the following network objective function $\mathcal{H}_{\mathrm{deplmnt}} : Q^n \to \mathbb{R}$:

$$\mathcal{H}_{\mathrm{deplmnt}}\left(x^{[1]}, \ldots, x^{[n]}\right) = \int_Q \min_{i \in I} \|q - x^{[i]}\|_2^2 \phi(q)\, dq.$$

This function and variations of it are studied in the facility location and resource allocation research literature; see [19] and [8]. It is convenient [9] to study a generalization of this function. For $r \in \mathbb{R}_{>0}$, define the saturation function $\mathrm{sat}_r : \mathbb{R} \to \mathbb{R}$ by $\mathrm{sat}_r(x) = x$ if $x \leq r$ and $\mathrm{sat}_r(x) = r$, otherwise. For $r \in \mathbb{R}_{>0}$, define the objective function $\mathcal{H}_{r\text{-deplmnt}} : Q^n \to \mathbb{R}$ by

$$\mathcal{H}_{r\text{-deplmnt}}\left(x^{[1]}, \ldots, x^{[n]}\right) = \int_Q \min_{i \in I} \mathrm{sat}_{\frac{r}{2}}\left(\|q - x^{[i]}\|_2^2\right) \phi(q)\, dq.$$

Note that if $r \geq 2\mathrm{diam}(Q)$, then $\mathcal{H}_{\mathrm{deplmnt}} = \mathcal{H}_{r\text{-deplmnt}}$. Let $\{V^{[1]}, \ldots, V^{[n]}\}$ be the Voronoi partition of Q associated with $\{x^{[1]}, \ldots, x^{[n]}\}$. The partial derivative of the cost function takes the following meaningful form (see [9]):

$$\begin{aligned}\frac{\partial \mathcal{H}_{r\text{-deplmnt}}}{\partial x^{[i]}} & \left(x^{[1]}, \ldots, x^{[n]}\right) \\ &= 2\,\mathrm{Mass}\left(V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)\right) \\ &\quad \cdot \left(\mathrm{Centroid}\left(V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)\right) - x^{[i]}\right), \qquad i \in I.\end{aligned}$$

(Here, as in the Appendix, $\mathrm{Mass}(S)$ and $\mathrm{Centroid}(S)$ are, respectively, the mass and the centroid of $S \subset \mathbb{R}^d$.) Clearly, the critical points of $\mathcal{H}_{r\text{-deplmnt}}$ are network states where $x^{[i]} = \mathrm{Centroid}\left(V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)\right)$. We call such configurations $\frac{r}{2}$-centroidal Voronoi configurations. For $r \geq 2\mathrm{diam}(Q)$, they coincide with the standard centroidal Voronoi configurations on Q. Fig. 4 illustrates these notions.

Motivated by these observations, we define the following deployment task. For $r, \varepsilon \in \mathbb{R}_{>0}$, define the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}} : Q^n \to$ `BooleSet` by $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) =$ `true` if and only if

$$\left\|x^{[i]} - \mathrm{Centroid}\left(V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)\right)\right\|_2 \leq \varepsilon, \quad \text{for all } i \in I.$$

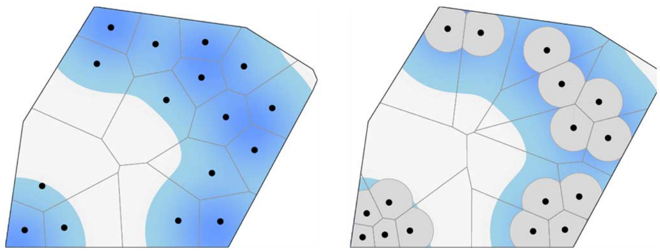


Fig. 4. Centroidal and $(r/2)$-centroidal Voronoi configurations. The density function ϕ is depicted by a contour plot. For each agent i, the set $V^{[i]} \cap \bar{B}(x^{[i]}, r/2)$ is plotted in light gray.

Roughly speaking, $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$ is $\texttt{true}$ for those network configurations where each agent i is sufficiently close to the centroid of its dominance region $V^{[i]} \cap \bar{B}\left(x^{[i]}, \frac{r}{2}\right)$.

B. Centroid Law

To achieve the ε-r-deployment task discussed in Section IV-A, we define the *centroid* control and communication law $\mathcal{CC}_{\text{centrd}}$. This is a static, uniform, data-sampled, time-independent law studied in [8] and [9]. Loosely speaking, the evolution of the network under the centroid control and communication law can be described as follows.

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round, each agent performs the following tasks: 1) it transmits its position and receives its neighbors' positions; 2) it computes the centroid of its dominance region (the intersection between the agent's Voronoi cell and a closed ball centered at its position of radius $\frac{r}{2}$), and 3) it moves toward this centroid.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0, \mathcal{L} = \mathbb{R}^d \cup \{\texttt{null}\}$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}, i \in I$. We define the control function $\text{ctl} : \mathbb{R}^d \times \mathcal{L}^n \to \mathbb{R}^d$ by

$$\text{ctl}(x_{\text{smpld}}, y) = \text{Centroid}(\mathcal{X}(x_{\text{smpld}}, y)) - x_{\text{smpld}},$$

where $\mathcal{X}(x, y) = Q \cap \bar{B}\left(x, \frac{r}{2}\right) \cap \left(\bigcap_{p \in y, p \neq \texttt{null}} H_{x,p}\right)$ and $H_{x,p}$ is the half-space $\{q \in \mathbb{R}^d \mid \|q - x\|_2 \leq \|q - p\|_2\}$. One can show that Q^n is a positively invariant set for this control law.

The following theorem on the centroid control and communication law summarizes the known results about the asymptotic properties and the novel results on the complexity of this law. In characterizing complexity, we assume $\text{diam}(Q)$ is independent of n, r, and ε. As for the circumcenter law, we provide complete time-complexity results for the case $d = 1$.

Theorem IV.1 (Time Complexity of Centroid Law): For $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, consider the network $\mathcal{S}_{r\text{-LD}}$ with initial conditions in Q. The following statements hold:

1) for $d \in \mathbb{N}$, the law $\mathcal{CC}_{\text{centrd}}$ achieves the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$;
2) for $d = 1$ and $\phi = 1, \text{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(n^3 \log(n\varepsilon^{-1}))$. •

Proof: Fact 1) is proved in [9] for $d \in \{1, 2\}$; the same proof technique can be generalized to any dimension. In what follows, we sketch the proof of Fact 2). For $d = 1, Q$ is a compact interval on $\mathbb{R}$, say $Q = [q_-, q_+]$.

We start with a brief discussion about connectivity. In the r-limited Delaunay graph, two agents that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Additionally, we claim that, if agents i and j are neighbors at time instant ℓ, then $|\text{Centroid}(\mathcal{X}^{[i]}(\ell)) - \text{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq r$. To see this, assume without loss of generality that $x^{[i]}(\ell) \leq x^{[j]}(\ell)$. Let us consider the case where the agents have neighbors on both sides (the other cases can be treated analogously). Let $x_-^{[i]}(\ell)$ (respectively, $x_+^{[j]}(\ell)$) denote the position of the neighbor of agent i to the left (respectively, of agent j to the right). Now,

$$\text{Centroid}\left(\mathcal{X}^{[i]}(\ell)\right) = \frac{1}{4}\left(x_-^{[i]}(\ell) + 2x^{[i]}(\ell) + x^{[j]}(\ell)\right),$$
$$\text{Centroid}\left(\mathcal{X}^{[j]}(\ell)\right) = \frac{1}{4}\left(x^{[i]}(\ell) + 2x^{[j]}(\ell) + x_+^{[j]}(\ell)\right).$$

Therefore, $|\text{Centroid}(\mathcal{X}^{[i]}(\ell)) - \text{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq \frac{1}{4}(|x_-^{[i]}(\ell) - x^{[i]}(\ell)| + 2|x^{[i]}(\ell) - x^{[j]}(\ell)| + |x^{[j]}(\ell) - x_+^{[j]}(\ell)|) \leq r$. This implies that agents i and j belong to the same connected component of the r-limited Delaunay graph at time instant $\ell + 1$.

Next, let us consider the case when $E_{r\text{-LD}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[n]}(0) = (x_0)_n$. We distinguish the following three cases depending on the proximity of the leftmost and rightmost agents 1 and n, respectively, to the boundary of the environment: case (a) both agents are within a distance $\frac{r}{2}$ of ∂Q; case (b) none of the two is within a distance $\frac{r}{2}$ of ∂Q; case (c) only one of the agents is within a distance $\frac{r}{2}$ of ∂Q. Here is an important observation: from one time instant to the next one, the network configuration can fall into any of the cases described previously. However, because of the discussion on connectivity, transitions can only occur from case (b) to either case (a) or (c) and from case (c) to case (a). As we show in the following, for each of these cases, the network evolution under $\mathcal{CC}_{\text{centrd}}$ can be described as a discrete-time linear dynamical system which respect to agents' ordering.

Let us consider case (a). In this case, we have

$$\begin{aligned}
x^{[1]}(\ell+1) &= \frac{1}{4}\left(x^{[1]}(\ell) + x^{[2]}(\ell)\right) + \frac{1}{2}q_-, \\
x^{[2]}(\ell+1) &= \frac{1}{4}\left(x^{[1]}(\ell) + 2x^{[2]}(\ell) + x^{[3]}(\ell)\right), \\
&\vdots \\
x^{[n-1]}(\ell+1) &= \frac{1}{4}\left(x^{[n-2]}(\ell) + 2x^{[n-1]}(\ell) + x^{[n]}(\ell)\right), \\
x^{[n]}(\ell+1) &= \frac{1}{4}\left(x^{[n-1]}(\ell) + x^{[n]}(\ell)\right) + \frac{1}{2}q_+.
\end{aligned}$$

Equivalently, we can write $x(\ell+1) = A_{(a)} \cdot x(\ell) + b_{(a)}$, where the $n \times n$-matrix $A_{(a)}$ and the vector $b_{(a)}$ are given by

$$A_{(a)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad b_{(a)} = \begin{bmatrix} \frac{1}{2}q_- \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}q_+ \end{bmatrix}.$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]} = q_- + \frac{1}{2n}(1 + 2(i-1))(q_+ - q_-), \qquad i \in I,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case (a)). We can, therefore, write $(x(\ell) - x_*) = A_{(a)}(x(\ell-1) - x_*)$. Now, note that $A_{(a)} = \mathrm{ATrid}_n^-\left(\frac{1}{4}, \frac{1}{2}\right)$. Reference [1, Theorem A.4, Case 2)] implies that $\lim_{\ell \to +\infty}(x(\ell) - x_*) = \mathbf{0}$ and that the maximum time required for $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^n$) is $\Theta(n^2 \log \varepsilon^{-1})$. It is not obvious, but it can be verified, that the initial condition providing the lower bound in the time complexity estimate does indeed have the property of respecting the agents' ordering; this fact holds for all three cases (a)–(c).

The case (b) can be treated in the same way. The network evolution takes now the form $x(\ell+1) = A_{(b)} \cdot x(\ell) + b_{(b)}$, where the $n \times n$-matrix $A_{(b)}$ and the vector $b_{(b)}$ are given by

$$A_{(b)} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad b_{(b)} = \begin{bmatrix} -\frac{1}{4}r \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}r \end{bmatrix}.$$

In this case, a (nonunique) equilibrium network configuration respecting the ordering of the agents is of the form

$$x_*^{[i]} = ir - \frac{1+n}{2}r, \qquad i \in I.$$

Note that this is a $\frac{r}{2}$-centroidal Voronoi configuration [under the assumption of case (b)]. We can, therefore, write $(x(\ell) - x_*) = A_{(b)}(x(\ell-1) - x_*)$. Now, note that $A_{(b)} = \mathrm{ATrid}_n^+\left(\frac{1}{4}, \frac{1}{2}\right)$. We compute $x_{\mathrm{ave}} = \frac{1}{n}\mathbf{1}^T(x_0 - x_*) = \frac{1}{n}\mathbf{1}^T x_0$. With this calculation, [1, Theorem A.4, Case 1)] implies that $\lim_{\ell \to +\infty}(x(\ell) - x_* - x_{\mathrm{ave}}\mathbf{1}) = \mathbf{0}$, and that the maximum time required for $\|x(\ell) - x_* - x_{\mathrm{ave}}\mathbf{1}\|_2 \leq \varepsilon \|x(0) - x_* - x_{\mathrm{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^n$) is $\Theta(n^2 \log \varepsilon^{-1})$.

Case (c) needs to be handled differently. Without loss of generality, assume that agent 1 is within distance $\frac{r}{2}$ of ∂Q and agent n is not (the other case is treated analogously). Then, the network evolution takes now the form $x(\ell+1) = A_{(c)} \cdot x(\ell) + b_{(c)}$, where the $n \times n$-matrix $A_{(c)}$ and the vector $b_{(c)}$ are given by

$$A_{(c)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad b_{(c)} = \begin{bmatrix} \frac{1}{2}q_- \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}r \end{bmatrix}.$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]} = q_- + \frac{1}{2}(2i-1)r, \qquad i \in I,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case (c)). In order to analyze $A_{(c)}$, we recast the n-dimensional discrete-time dynamical system as a $2n$-dimensional one. To do this, we define a $2n$-dimensional vector y by

$$y^{[i]} = x^{[i]}, \quad i \in I \text{ and } y^{[n+i]} = x^{[n-i+1]}, \quad i \in I. \tag{6}$$

Now, one can see that the network evolution can be alternatively described in the variables $(y^{[1]}, \ldots, y^{[2n]})$ as a linear dynamical system determined by the $2n \times 2n$-matrix $\mathrm{ATrid}_{2n}^-\left(\frac{1}{4}, \frac{1}{2}\right)$. Using [1, Theorem A.4, Case 2)], and exploiting the chain of equalities (6), we can infer that, in case (c), the maximum time required for $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^n$) is $\Theta(n^2 \log \varepsilon^{-1})$.

In summary, for all three cases a)–c), our calculations show that, in time $O(n^2 \log \varepsilon^{-1})$, the error 2-norm satisfies the contraction inequality $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$. We convert this inequality on 2-norms into an appropriate inequality on ∞-norms as follows. Note that $\|x(0) - x_*\|_\infty = \max_{i \in I} |x^{[i]}(0) - x_*^{[i]}| \leq (q_+ - q_-)$. For ℓ of order $n^2 \log \eta^{-1}$, we have

$$\begin{aligned} \|x(\ell) - x_*\|_\infty &\leq \|x(\ell) - x_*\|_2 \leq \eta \|x(0) - x_*\|_2 \\ &\leq \eta\sqrt{n}\|x(0) - x_*\|_\infty \leq \eta\sqrt{n}(q_+ - q_-). \end{aligned}$$

This means that ε-r-deployment is achieved for $\eta\sqrt{n}(q_+ - q_-) = \varepsilon$, that is, in time $O(n^2 \log \eta^{-1}) = O(n^2 \log(n\varepsilon^{-1}))$.

Up to here, we have proved that, if the graph $(I, E_{r\text{-LD}}(x_0))$ is connected, then $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(n^2 \log(n\varepsilon^{-1}))$. If $(I, E_{r\text{-LD}}(x_0))$ is not connected, note that along the network evolution there can only be a finite number of time instants, at most $n-1$, where a merging of two connected components occurs. Therefore, the time complexity is at most $O(n^3 \log(n\varepsilon^{-1}))$. ■

Remark IV.2 (Congestion Effects): Note that the proof of Theorem IV.1 holds verbatim if, motivated by wireless congestion considerations, we take the communication range r to be a monotone nonincreasing function $r : \mathbb{N} \to]0, 2\pi[$ of the number of robotic agents n. •

V. Conclusion

Building on the framework for robotic networks proposed in the companion paper [1], we have formalized various motion coordination algorithms as follows: 1) the move-toward-average law and the circumcenter laws that achieve the rendezvous task and 2) the centroid law that achieves the deployment task. We have computed the time complexity of these algorithms, providing upper and lower bounds as the number of agents grows. To obtain these complexity estimates, we have relied on analysis methods involving linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices. These results demonstrate the usefulness of the proposed formal model.

The complexity bounds reported in this and the companion paper are of low polynomial order and are comparable to those found in the literature on distributed algorithms and on stochastic matrices, e.g., see [17], [20], and [21]. None of the algorithms has an exponential complexity. From a practical viewpoint, what level of complexity (logarithmic, linear, polynomial) is acceptable will depend on the specific application considered and we leave this question to future work.

The analysis presented in this paper is useful for robotic network applications because it provides a rigorous assessment of the performance of the aforementioned coordination algorithms. Given a desired task, our vision is that the combination of coordination algorithms with the best scalability properties will enable the synthesis of efficient cooperative strategies. Once a catalog of example coordination tasks and algorithms have been carefully understood, one could envision the design of more complex strategies building on this knowledge. It is also our hope that the kind of analysis performed here will help characterize the complex tradeoffs between computation, communication, and motion control in robotic networks.

A number of research avenues look now promising including the following: 1) time complexity analysis in higher dimensions, 2) communication complexity analysis for unidirectional and omnidirectional models of communication, 3) analysis of other known algorithms for flocking, cohesion, formation, and motion planning, and 4) complexity analysis results for coordination tasks, as opposed to for algorithms.

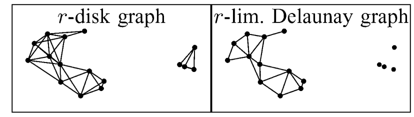


Fig. 5. The r-disk and r-limited Delaunay graphs in $\mathbb{R}^2$.

Appendix
Basic Geometric Notions

Here, we present various geometric concepts used throughout this paper. Let $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be compact. *The circumcenter of* S, denoted by $\mathrm{Circum}(S)$, is the center of the smallest radius sphere in $\mathbb{R}^d$ enclosing S. Given an integrable function $\phi : S \to \mathbb{R}_{>0}$, the mass of S is $\mathrm{Mass}(S) = \int_S \phi(q)\, dq$, and the *centroid* of S is

$$\mathrm{Centroid}(S) = \frac{1}{\mathrm{Mass}(S)} \int_S q\phi(q)\, dq.$$

A *partition* of S is a collection of subsets of S with disjoint interiors and whose union is S. Given a set of n distinct points $\mathcal{P} = \{p_1, \dots, p_n\}$ in S, the *Voronoi partition of* S generated by $\mathcal{P}$ (with respect to the Euclidean norm) is the collection of sets $\{V_1(\mathcal{P}), \dots, V_n(\mathcal{P})\}$ defined by $V_i(\mathcal{P}) = \{q \in S \mid \|q - p_i\|_2 \leq \|q - p_j\|_2, \text{ for all } p_j \in \mathcal{P}\}$. We usually refer to $V_i(\mathcal{P})$ as V_i. For a detailed treatment of Voronoi partitions, we refer to [22] and [19].

For $I = \{1, \dots, n\}$ and $S \subset \mathbb{R}^d$, a *proximity edge map* is a map of the form $E : S^n \to 2^{I \times I}$. For $r \in \mathbb{R}_{>0}$, we define the r-disk proximity edge map $E_{r\text{-disk}} : (\mathbb{R}^d)^n \to 2^{I\times I}$ and the r-limited Delaunay proximity edge map $E_{r\text{-LD}} : (\mathbb{R}^d)^n \to 2^{I \times I}$ as follows. An edge $(i,j) \in I \times I$ belongs to $E_{r\text{-disk}}(x_1, \dots, x_n)$ if and only if $i \neq j$ and $\|x_i - x_j\|_2 \leq r$. An edge $(i,j) \in I \times I$ belongs to $E_{r\text{-LD}}(x_1, \dots, x_n)$ if and only if $i \neq j$ and

$$\left(V_i \cap \bar{B}\left(x_i, \frac{r}{2}\right)\right) \cap \left(V_j \cap \bar{B}\left(x_j, \frac{r}{2}\right)\right) \neq \emptyset,$$

where $\{V_1, \dots, V_n\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x_1, \dots, x_n\}$. Illustrations of these concepts are given in Fig. 5.

As proved in [9], the r-limited Delaunay graph and the r-disk graph have the same connected components. Additionally, the r-limited Delaunay graph is "computable" on the r-disk graph in the following sense: any node in the network can compute the set of its neighbors in the r-limited Delaunay graph if it is given the set of its neighbors in the r-disk graph. This implies that any control and communication law for a network with communication graph $E_{r\text{-LD}}$ can be implemented on a analogous network with communication graph $E_{r\text{-disk}}$.

REFERENCES

[1] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, "On synchronous robotic networks—Part I: Models, tasks, and complexity," *IEEE Trans. Autom. Control*, vol. 52, no. 12, pp. 2199–2213, Dec. 2007.

[2] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Trans. Robot. Autom.*, vol. 15, no. 5, pp. 818–828, Oct. 1999.

[3] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous oblivious robots with limited visibility," *Theoretical Comput. Sci.*, vol. 337, no. 1–3, pp. 147–168, 2005.

[4] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem - the asynchronous case," in *Proc. IEEE Conf. Decision Control*, Paradise Island, Bahamas, Dec. 2004, pp. 1926–1931.

[5] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. Autom. Control*, vol. 51, no. 8, pp. 1289–1298, Aug. 2006.

[6] J. A. Marshall, M. E. Broucke, and B. A. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Trans. Autom. Control*, vol. 49, no. 11, pp. 1963–1974, Nov. 2004.

[7] S. L. Smith, M. E. Broucke, and B. A. Francis, "A hierarchical cyclic pursuit scheme for vehicle networks," *Automatica*, vol. 41, no. 6, pp. 1045–1053, 2005.

[8] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robot. Autom.*, vol. 20, no. 2, pp. 243–255, Apr. 2004.

[9] J. Cortés, S. Martínez, and F. Bullo, "Spatially-distributed coverage optimization and control with limited-range interactions," *ESAIM Contro, Optim. Calculus Variat.*, vol. 11, pp. 691–719, 2005.

[10] V. Gazi and K. M. Passino, "Stability analysis of swarms," *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 692–697, Apr. 2003.

[11] P. Ögren, E. Fiorelli, and N. E. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1292–1302, Aug. 2004.

[12] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[13] E. W. Justh and P. S. Krishnaprasad, "Equilibria and steering laws for planar formations," *Syst. Control Lett.*, vol. 52, no. 1, pp. 25–38, 2004.

[14] V. Sharma, M. Savchenko, E. Frazzoli, and P. Voulgaris, "Transfer time complexity of conflict-free vehicle routing with no communications," *Int. J. Robot. Res.*, vol. 26, no. 3, pp. 255–272, Mar. 2007.

[15] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[16] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[17] N. A. Lynch, *Distributed Algorithms*. San Mateo, CA: Morgan Kaufmann, 1997.

[18] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, no. 6–7, pp. 1226–1229, 1995.

[19] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, ser. Probability and Statistics, 2nd ed. New York: Wiley, 2000.

[20] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Belmont, MA: Athena Scientific, 1997.

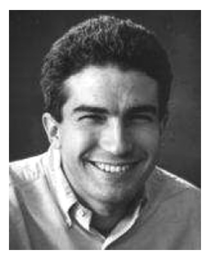
[21] H. J. Landau and A. M. Odlyzko, "Bounds for eigenvalues of certain stochastic matrices," *Linear Algebra Appl.*, vol. 38, pp. 5–15, 1981.

[22] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational Geometry: Algorithms and Applications*, 2nd ed. New York: Springer-Verlag, 2000.

Sonia Martínez (S'02–M'04) received the Licenciatura degree in mathematics from the Universidad de Zaragoza, Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from the Universidad Carlos III de Madrid, Madrid, Spain, in 2002.

From September 2002 to September 2003, she was a Visiting Assistant Professor of Applied Mathematics at the Technical University of Catalonia, Spain. She has been a Postdoctoral Fulbright Fellow at the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL (from October 2003 to August 2004), and at the Center for Control, Dynamical Systems, and Computation, University of California, Santa Barbara, CA (from September 2004 to November 2005). Currently, she is an Assistant Professor at the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA. Her main research interests include systems and information theory, nonlinear control theory, and robotics. Her current research focuses on the development of distributed coordination algorithms for the deployment of sensor networks and highly autonomous vehicle systems.

Dr. Martínez received the Best Student Paper Award at the 2002 IEEE Conference on Decision and Control for her work on the control of underactuated mechanical systems. She is the recipient of a 2007 National Science Foundation (NSF) CAREER award.



Francesco Bullo (S'95–M'03–SM'03) received the Laurea degree in electrical engineering from the University of Padova, Padova, Italy, in 1994, and the Ph.D. degree in control and dynamical systems from the California Institute of Technology, Pasadena, CA, in 1999.

From 1998 to 2004, he was an Assistant Professor at the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL. Currently, he is an Associate Professor at the Department of Mechanical Engineering, University of California, Santa Barbara, CA. He is the coauthor (with A. D. Lewis) of the book *Geometric Control of Mechanical Systems* (New York: Springer Verlag, 2004). His research interests include motion planning and coordination for autonomous vehicles, motion coordination for multiagent networks, and geometric control of mechanical systems.

Dr. Bullo is currently serving on the editorial board of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and the *SIAM Journal of Control and Optimization*.

Jorge Cortés (M'04–SM'06) received the Licenciatura degree in mathematics from the Universidad de Zaragoza, Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from the Universidad Carlos III de Madrid, Madrid, Spain, in 2001.

He was a Postdoctoral Research Associate at the Systems, Signals and Control Department, University of Twente, Enschede, The Netherlands (from January to June 2002), and at the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL (from August 2002 to September 2004). From 2004 to 2007, he was an Assistant Professor at the Department of Applied Mathematics and Statistics, University of California, Santa Cruz, CA. Currently, he is an Assistant Professor at the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA. He is the author of the book *Geometric, Control and Numerical Aspects of Nonholonomic Systems* (New York: Springer Verlag, 2002). His current research interests focus on mathematical control theory, distributed motion coordination for groups of autonomous agents, and geometric mechanics and geometric integration.

Dr. Cortés is currently an Associate Editor for the *European Journal of Control*. He is the recipient of the 2006 Spanish Society of Applied Mathematics Young Researcher Prize.

Emilio Frazzoli (S'99–M'03) received the Laurea degree in aerospace engineering from the University of Rome, "La Sapienza," Rome, Italy, in 1994, and the Ph.D. degree in navigation and control systems from the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, in 2001.

Currently, he is an Associate Professor of Aeronautics and Astronautics at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology. Between 1994 and 1997, he worked as an Officer in the Italian Navy and as a Spacecraft Dynamics Specialist for the European Space Agency, Darmstadt, Germany, and Telespazio, Rome, Italy. From 2001 to 2004, he was an Assistant Professor of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL. From 2004 to 2006, he was an Assistant Professor of Mechanical and Aerospace Engineering at the University of California, Los Angeles, CA. His current research interests include algorithmic, computational and geometric approaches to control design for mobile robotic networks. Application areas include distributed cooperative control of multiple vehicle systems, guidance and control of agile vehicles, high-confidence software engineering for high-performance dynamical systems, verification of hybrid systems.

Dr. Frazzoli was the recipient of a 2002 National Science Foundation (NSF) CAREER award.