# Control of Quantum Systems Despite Feedback Delay

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#### **Abstract**

Feedback control (based on the quantum continuous measurement) of quantum systems inevitably suffers from estimation delays. In this paper we give a delay-dependent stability criterion for a wide class of nonlinear stochastic systems including quantum spin systems. We utilize a semi-algebraic problem approach to incorporate the structure of *density matrices*. To show the effectiveness of the result, we derive a globally stabilizing control law for a quantum spin-1/2 systems in the face of feedback delays.

#### Index Terms

Quantum control, Delay systems, Sum of squares, Filtering, Spin systems

### I. Introduction

Quantum systems substantially differ from classical (i.e., non-quantum) systems in that state variables are represented by noncommutative operators acting on a Hilbert space; see e.g., [1]. Such noncommutativity imposes some critical constraints on the structure of a quantum controller. This makes it difficult to analyze/synthesize feedback control systems for quantum systems. However, *quantum filtering theory* [2], [3], [4], [5] has clarified that a number of quantum control problems can be formulated and solved within the framework of standard classical stochastic control theory [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

A brief description of the filter-based approach to quantum control is as follows. The plant dynamics are given by a *quantum stochastic differential equation*, where the state is a noncommutative random variable [16]. The dynamics are partially monitored by means of a continuous measurement that allows us to construct an estimator of the plant variables. The resulting filter is a classical stochastic differential equation called the *Belavkin equation* or *stochastic master equation*. Our objective is to synthesize an effective controller such that the filter shows a desirable behavior.

For this problem, two types of control law have been proposed. The first one is a simple proportional feedback of the output signal. The second one is a feedback of the estimate of the plant variables, which we call the *filter-based* 

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controller. A more detailed description of these two controllers will be given in the next section, but we here note that, for the implementation of the filter-based controller, a non-negligible computation time is required to process the estimation [17]. Therefore, from a practical point of view, a filter-based feedback controller should be considered taking the feedback delay into explicit account. For example, Steck et al. have numerically studied the issue of delay in the case of feedback stabilization of atomic motion [18]. However, to the authors' best knowledge, there have been no theoretical means to perform such investigations in the quantum case.

In this paper we study the effect of the delay in quantum systems with the full use of several techniques for analyzing stability of stochastic delay differential systems; see e.g., [19], [20], [21] and references therein. In particular, we focus on the control problem of a quantum spin system, which has also been studied in [8], [10], [13]. This system is very important, since it is one of the most basic components in quantum information processing [22].

This paper is organized as follows. Section II reviews quantum filtering and control. In particular, we discuss delay in this feedback control scheme. Section III is the main part of this paper. Theorem 1 gives a delay-dependent stability criterion for a class of nonlinear stochastic systems including some quantum spin systems. The effectiveness of the result is then verified by deriving a stabilizing controller for the spin-1/2 particle case.

**Notation** For  $z \in \mathbb{R}^n$  and  $M \in \mathbb{R}^{n \times n}$ ,  $||z||_M^2 := z^T M z$ . The subscript is omitted when M is the identity matrix. A function  $F : D \to \mathbb{R}$  is said to be *negative* (resp. *positive*) in D if  $F(z) \le 0$  (resp.  $F(z) \ge 0$ ) for any  $z \in D$ . A subset C in  $\mathbb{R}^n$  is said to be *semi-algebraic* if

$$C := \{x \in \mathbb{R}^n : p_i(x) < 0, i = 1, 2, \dots, l\}$$

with polynomials  $p_i$ .

Let  $C_{\mathsf{C}}^h$  be the set of C-valued uniformly continuous functions on [-h,0]. This is a Banach space equipped with  $\|\tilde{x}\|_{C_{\mathsf{C}}^h} := \sup_{\theta \in [-h,0]} \|\tilde{x}(\theta)\|_{\mathsf{C}}$ . Given a probability measure, the probability and expectation are denoted by  $\mathbb{P}$  and  $\mathbb{E}$ . We say an event  $\Omega$  occurs almost surely if  $\mathbb{P}\left\{\Omega\right\} = 1$ . If it exists, the *infinitesimal generator* of a function V along a Markov process  $\tilde{x}_t$  is denoted by  $\mathscr{A}V$  i.e.,  $\mathscr{A}V(\tilde{x}) := \lim_{t \to 0} \frac{\mathbb{E}^{\tilde{x}}[\tilde{x}_t] - \tilde{x}}{t}$  where  $\mathbb{E}^{\tilde{x}}$  represents the expectation with respect to paths which start at  $\tilde{x}_0 = \tilde{x}$ ; see [20], [19], [21] for a formula.

## II. CONTROL SCHEME BASED ON QUANTUM FILTERING

### A. Quantum filtering

We here provide a brief summary of quantum filtering theory [2], [3], [4]. For a more detailed description, see [5].

In the framework of quantum filtering, a plant dynamics is described in a similar form to a general classical stochastic differential equation. For example, when using a homodyne detector [23], a single state variable  $X_t$ 

satisfies

$$dX_t = f(X_t)dt + g(X_t)dW_t,$$
  

$$dY_t = (h(X_t) + h(X_t)^*)dt + dV_t,$$
(1)

where f,g, and h are smooth functions with specific structures. However, unlike the classical case, the state variable  $X_t$ , the output  $Y_t$ , and the stochastic noises  $W_t, V_t$  are *observables*, i.e., Hermitian operators that act on a certain Hilbert space (\* denotes the self-adjoint operation). Thus, in general they do not commute with each other. Note that any noncommutative random variables cannot take their realization values on a same probabilistic space. This implies that the classical stochastic control theory is not directly applicable, because we cannot define the conditional expectation  $\pi(X_t) := \mathbf{E}(X_t | \mathcal{Y}_t)$ , and consequently the optimal filter. Here,  $\mathcal{Y}_t$  denotes the set of  $Y_s$  ( $0 \le s \le t$ ). Quantum filtering theory identifies systems free from these difficulties, i.e., systems satisfying the *nondemolition properties*  $[Y_s, Y_t] = 0$  ( $\forall s, t$ ) and  $[X_t, Y_s] = 0$  ( $\forall s \le t$ ), where [A, B] := AB - BA. Fortunately, in many important cases, especially in quantum optics, we can build such systems. The filter is then given by

$$d\pi(X_t) = \pi(f(X_t, u_t))dt + \left(\pi(X_t h(X_t) + h(X_t)^* X_t) - \pi(X_t)\pi(h(X_t) + h(X_t)^*)\right) \times (dY_t - \pi(h(X_t) + h(X_t)^*)dt).$$
(2)

Surprisingly, this is the same form as the classical filtering equation except the symmetrized terms. We now introduce a *density matrix*  $\rho$ ; in a finite-dimensional case, it belongs to the convex set

$$S := \{ \rho \in \mathbb{C}^{N \times N} : \rho = \rho^* \ge 0, \operatorname{tr} \rho = 1 \}, \tag{3}$$

where N is determined from the system. The statistics of the measurement results of an observable X is completely characterized by  $\rho$ . For example, the k-th moment of the outcomes is given by  $\operatorname{tr}(X^k\rho)$ . Thus the conditional expectation  $\pi(X_t)$  should also be represented in terms of a time-dependent density matrix  $\rho_t$  as  $\pi(X_t) = \operatorname{tr}(X\rho_t)$ , which together with (2) leads to the time-evolution of  $\rho_t$ . In particular, when the homodyne detection scheme is used, the most simple form of it is given by the following *stochastic master equation*:

$$d\rho_{t} = \mathcal{L}^{*}(\rho_{t}, u_{t})dt + \left(L\rho_{t} + \rho_{t}L^{*} - \operatorname{tr}(L\rho_{t} + \rho_{t}L^{*})\rho_{t}\right)$$

$$\times \left(dY_{t} - \operatorname{tr}(L\rho_{t} + \rho_{t}L^{*})dt\right),$$

$$\mathcal{L}^{*}(\rho, u) := i[H, \rho] + L\rho L^{*} - \frac{1}{2}L^{*}L\rho - \frac{1}{2}\rho L^{*}L.$$
(4)

Here, H is an observable called Hamiltonian, representing the energy of the system. The measurement operator L determines how the system interacts with the measurement apparatus (e.g. a laser field; see Figure 1).

## B. Implementation of filter-based controller

In a typical situation, the Hamiltonian term is a function of the control input  $u_t$ ; H = H(u). Our goal is to design  $u_t$  such that the filter of Eq. (4) has a desirable behavior. Note that, as in the classical case, the last term

 $dw_t := dY_t - tr(L\rho_t + \rho_t L^*)dt$  is a classical Wiener increment. This implies that Eq. (4) is a classical stochastic differential equation to which several techniques developed in control theory can be applied.

The proportional output feedback controller  $u_t = k dY_t/dt$  ( $k \in \mathbb{R}$  is the gain) is often considered [15], [11], [12] and was implemented in the experimental setup of spin-squeezing control [24], [25]. On the other hand, note that we can compute  $\rho_t$  by using the past output sequence  $\{Y_s\}_{s \le t}$  by Eq. (5). If it is possible to perform this computation on-line, we can implement controller of the state feedback form  $u_t = u(\rho_t)$ , i.e., the filter-based controller. With this control the target state is limited to the *eigenstates* of the measurement operator L unlike the proportional feedback case where the target can be to some extent changed flexibly [26], [27], but we can instead take much wider variety of designing methods of the filter-based controller. In fact, it has been proven that the Lyapunov theory was successfully employed to show the global stability of the filter for some systems [8], [10], [13], [28]. Moreover, it is known that the optimal controller for a general type of quantum optimal control problem is given by a filter-based controller. This is known as the *separation theorem* [29].

However, in general, the time required to compute  $\rho_t$  is not negligible compared to the time-constants associated with the dynamics of a nano-mechanical system. In other words, from a practical point of view,  $\rho_t$  cannot be used to determine  $u_t$ . In view of this we should consider the delayed feedback control input  $u_t = u(\rho_{t-\tau})$ , where  $\tau > 0$  denotes the delay length. Note that this formulation is able to handle further delays, for example input delays. Such input delays occur because the control input  $u_t$  must be physically implemented by means of actuators. The purpose of this paper is to propose a rigorous methodology for analyzing the behavior of quantum control systems in the face of feedback delay.

## III. STABILIZATION OF QUANTUM SPIN SYSTEMS IN THE FACE OF DELAY

### A. The physical model and control problem

In this section we consider a cold atomic ensemble trapped in an optical cavity [24], [8], [10], [11], [13], as depicted in Figure 1. The total angular momentum operator  $F_i$  of the atom around the *i*-axis (i = y, z) is given by

$$F_y := \frac{i}{2} \begin{bmatrix} 0 & c_1 \\ -c_1 & 0 & c_2 \\ & \ddots & \ddots & \ddots \\ & & -c_{N-2} & 0 & c_{N-1} \\ & & & -c_{N-1} & 0 \end{bmatrix},$$

$$c_m := \sqrt{(N-m)m}, \quad m = 1, 2, \dots, N-1,$$

$$F_z := \frac{1}{2} \operatorname{diag}\{N-1, N-3, \dots, -(N-3), -(N-1)\},$$

where N-1 represents the number of atoms. The system interacts with a laser field oriented along the z-axis at a homodyne-type photo detector, which implies  $L = F_z$ . The system also interacts with an external magnetic field,

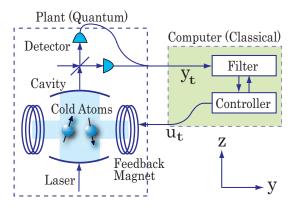


Fig. 1. Quantum plant and classical controller. The filter needs a finite time  $\tau > 0$  to compute the control input  $u_t = u(\rho_{t-\tau})$ .

which is oriented along the y-axis,  $H(u_t) = u_t F_y$ , where the control input  $u_t$  corresponds to the magnetic field strength, which can be modified in time. As a result, the controlled filter equation (4) becomes

$$d\rho_t = i[u_t F_y, \rho_t] dt - \frac{1}{2} [F_z, [F_z, \rho_t]] dt$$

$$+ \sqrt{\eta} (F_z \rho_t + \rho_t F_z - 2 \operatorname{tr}(F_z \rho_t) \rho_t) dw_t,$$
(5)

where  $\eta \in (0,1]$  represents the measurement efficiency. Note that the Wiener process  $w_t$  contains the measurement data  $y_t$ .

Our goal is to design a feedback control law  $u_t = u(\rho_{t-\tau})$  that achieves the deterministic convergence of  $\rho_t$  to a prescribed target state. This problem was solved in [8], [10], [13], for the case of no delay. Note that controlled filter equation (5) shows a significant dependence on the delay, through the input  $u_t = u(\rho_{t-\tau})$ . Therefore the control problem is much more difficult than the previous one.

# B. Delay-dependent stability criteria

The system of Eq. (5) is described by  $\rho_t \in \mathbb{C}^{N \times N}$ . By concatenating the real and imaginary part of all elements of  $\rho_t$  into a column vector, we can rewrite Eq. (5) as a  $\mathbb{R}^n$ -valued nonlinear stochastic delay system. It is important to note that the resulting system has the following features:

- The drift and diffusion terms are polynomials in the state variable.
- The bounded semi-algebraic set determined by S is positively invariant; see also [8, Proposition 1].
- The control input, which possibly suffers from delays, is applied only to the drift term.

We here do not limit our attention to the specifically structured dynamics of Eq. (5), but rather consider a wide class of nonlinear stochastic systems with the above properties. A delay-dependent stability criterion is given in Theorem 1<sup>1</sup>.

<sup>&</sup>lt;sup>1</sup>Throughout this section, the symbols  $\tilde{x}_{\bullet}$  (resp.  $x_{\bullet}$ ) are used to represent functions (resp. vectors). These symbols with (resp. without) the time index denote the solution to Eq. (6) (resp. any functions or vectors).

Theorem 1: Let  $f(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ ,  $g(\cdot):\mathbb{R}^n\to\mathbb{R}^n$ , be polynomials and C a bounded semi-algebraic set in  $\mathbb{R}^n$  such that for any initial condition  $\tilde{x}_i\in C_\mathsf{C}^\tau$  the solution to the delay differential stochastic equation

$$dx_t = f(x_t, x_{t-\tau})dt + g(x_t)dw_t$$
 (6)

$$x_{\theta} = \tilde{x}_{i}(\theta) \in \mathsf{C}, \quad \theta \in [-\tau, 0]$$
 (7)

does not exit C almost surely. Suppose there exist a polynomial  $V_*(\cdot)$  which is positive in C, n-variable polynomials  $V_i$   $(i=0,1), S \in \mathbb{R}^{2n\times n}$ , and positive-definite matrices  $R, T \in \mathbb{R}^{n\times n}$  such that  $\Upsilon$  defined below is negative in  $C \times C \times \mathbb{R}^{2n}$ :

$$F(x, x_{d}) := \left(\frac{\partial V_{0}(x)}{\partial x}\right)^{\mathsf{T}} f(x, x_{d})$$

$$+ \frac{1}{2} g(x)^{\mathsf{T}} \frac{\partial}{\partial x} \left(\frac{\partial V_{0}(x)}{\partial x}\right)^{\mathsf{T}} g(x)$$

$$+ V_{1}(x) - V_{1}(x_{d}) + V_{*}(x) + \tau \|g(x)\|_{T}^{2}$$

$$+ 2 \left[ x^{\mathsf{T}} x_{d}^{\mathsf{T}} \right] S(x - x_{d}) + \tau \|f(x, x_{d})\|_{R}^{2}$$

$$\Upsilon(x, x_{d}, y) := F(x, x_{d})$$

$$+ \left[ x \atop x_{d} \atop y \right]^{\mathsf{T}} \left[ 0 \quad S \quad \tau S \atop \tau S^{\mathsf{T}} \quad -T \quad 0 \atop \tau S^{\mathsf{T}} \quad 0 \quad -\tau R \right] \left[ x \atop x_{d} \atop y \right] .$$

$$(8)$$

Then,  $V_*(x_t)$  converges to 0 almost surely for any initial condition  $\tilde{x}_i \in C_{\mathsf{C}}^{\tau}$ .

Suppose that  $V_*(x)$  represents a distance between x and a given target state. Then, this theorem states that  $x_t$  converges to the target state if a *semi-algebraic problem* is feasible; see also Subsection III.C. Semi-algebraic problems are in general NP-hard. However, if the degrees of polynomials have been decided, *sums of squares* (SOS) relaxation enables us to solve the problem efficiently [31], [32]. In the numerical example in the next subsection, we utilized MATLAB SOSTOOLS [33], [34].

Remark 1: In Theorem 1,  $\Upsilon$  is required to be negative only in  $C \times C \times \mathbb{R}^{2n}$ , not globally (i.e., in  $\mathbb{R}^{4n}$ ). This is the reason why Theorem 1 can incorporate the structure of density matrices which is useful for reducing the conservativeness. Similar criteria for some modified problem formulations (i.e., time-varying delay or delay-independent stability) can be obtained straightforwardly.

We prove Theorem 1 by using the following Lyapunov-Krasovskii type argument:

Proposition 1: Let  $x_t$  be the solution of the stochastic delay differential equations (6) and (7). Define

$$\tilde{x}_t(\theta) := x_{t+\theta}, \ \theta \in [-2\tau, 0]$$

for  $t \geq \tau$ . Suppose that there exists a positive functional V defined in  $C_{\mathsf{C}}^{2\tau}$  such that

$$\mathbb{E}\left[\mathscr{A}V(\tilde{x}_t) + V_*(x_t)\right] \le 0\tag{9}$$

for any  $t \ge \tau$ . Then,  $V_*(x_t)$  converges to 0 in the same sense as in Theorem 1.

*Proof:* Recall that  $x_t$  evolves only in the bounded domain C. Hence Fubini's theorem yields

$$\mathbb{E}\left[\int_{\tau}^{t} \mathscr{A}V(\tilde{x}_{s}) ds\right] = \int_{\tau}^{t} \mathbb{E}\left[\mathscr{A}V(\tilde{x}_{s})\right] ds.$$

By combining this equality, Eq. (9), and Dynkin's formula [20], [36], we obtain

$$\mathbb{E}\left[V(\tilde{x}_t)\right] - V(\tilde{x}_\tau) = \mathbb{E}\left[\int_{\tau}^{t} \mathscr{A}V(\tilde{x}_s) ds\right]$$

$$\leq -\int_{\tau}^{t} \mathbb{E}\left[V_*(x_t)\right] ds \leq 0.$$

Therefore we conclude that  $V(\tilde{x}_t)$  is a nonnegative super-martingale. The remainder of the proof is the same as the standard Lyapunov-Krasovsii argument; see e.g. Theorems 6.1 and 6.2 in [36] and their proofs. This completes the proof.

Now we are ready to prove Theorem 1.

Proof of Theorem 1: It suffices to show that V

$$V(\tilde{x}) := V_0(\tilde{x}(0)) + \int_{-\tau}^{0} V_1(\tilde{x}(\theta)) d\theta + \int_{-\tau}^{0} \int_{v}^{0} \left\{ \|f(\tilde{x}(\theta), \tilde{x}(-\tau + \theta))\|_{R}^{2} + \|g(\tilde{x}(\theta))\|_{T}^{2} \right\} d\theta dv$$
(10)

satisfies the assumptions made in Proposition 1.

The polynomials  $V_i$  (i = 0, 1) are bounded from below on C due to the continuity of polynomials and the boundedness of the domain. Note that adding any constant to  $V_i$  does not affect  $\Upsilon$ . Therefore, without loss of generality we can assume that V is positive.

A direct computation yields

$$0 \le \tau e^{\mathsf{T}} X e - \int_{-\tau}^{0} e^{\mathsf{T}} X e \mathrm{d}s$$

$$0 = (2 - 2) \cdot e^{\mathsf{T}} S \left\{ \tilde{x}(0) - \tilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s) \mathrm{d}s \right\}$$

$$\le 2e^{\mathsf{T}} S(\tilde{x}(0) - \tilde{x}(-\tau)) - \int_{-\tau}^{0} 2e^{\mathsf{T}} S \underline{f}(s) \mathrm{d}s$$

$$+ e^{\mathsf{T}} S T^{-1} S^{\mathsf{T}} e + \left\| \tilde{x}(0) - \tilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s) \mathrm{d}s \right\|_{T}^{2}$$

where  $e := \begin{bmatrix} \tilde{x}(0)^\mathsf{T} & \tilde{x}(-\tau)^\mathsf{T} \end{bmatrix}^\mathsf{T}$ ,  $\underline{f}(s) := f(\tilde{x}(s), \tilde{x}(-\tau+s))$ , and  $X := SR^{-1}S^\mathsf{T} \geq 0$ . Combining these inequalities and

$$\mathscr{A}V(\tilde{x}) = \left(\frac{\partial V_0(x)}{\partial x}\Big|_{\tilde{x}(0)}\right)^{\mathsf{T}} f(\tilde{x}(0), \tilde{x}(-\tau))$$

$$+ \frac{1}{2} g(\tilde{x}(0))^{\mathsf{T}} \frac{\partial}{\partial x} \left(\frac{\partial V_0(x)}{\partial x}\right)^{\mathsf{T}}\Big|_{\tilde{x}(0)} g(\tilde{x}(0))$$

$$+ V_1(\tilde{x}(0)) - V_1(\tilde{x}(-\tau)) + \tau(\|\underline{f}(0)\|_R^2 + \|g(\tilde{x}(0))\|_T^2)$$

$$- \int_{-\tau}^0 \left\{ \|\underline{f}(s)\|_R^2 + \|g(\tilde{x}(s))\|_T^2 \right\} \mathrm{d}s,$$

we obtain

$$\mathscr{A}V(\tilde{x}) + V_*(\tilde{x}(0)) \le \tilde{\Upsilon}(\tilde{x}(0), \tilde{x}(-\tau)) - G_1(\tilde{x}) - G_2(\tilde{x})$$

with

$$\widetilde{\Upsilon}(x, x_{d}) := F(x, x_{d}) + \left\| \begin{bmatrix} x^{\mathsf{T}} & x_{d}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \right\|_{\tau X + ST^{-1}S^{\mathsf{T}}}^{2}$$

$$G_{1}(\widetilde{x}) := \int_{-\tau}^{0} \left\| \begin{bmatrix} e^{\mathsf{T}} & \underline{f}(s)^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \right\|_{\Xi}^{2} ds \ge 0$$

$$G_{2}(\widetilde{x}) := \int_{-\tau}^{0} \|g(\widetilde{x}(s))\|_{T}^{2} ds$$

$$- \left\| \widetilde{x}(0) - \widetilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s) ds \right\|_{T}^{2},$$

$$\Xi := \begin{bmatrix} X & S \\ S^{\mathsf{T}} & R \end{bmatrix} \ge 0.$$

Let us take the expectation after substituting  $\tilde{x} = \tilde{x}_t$ . We can show

$$\mathbb{E}\left[G_2(\tilde{x}_t)\right] = 0$$

by using the Itô isometry. We thus have

$$\mathbb{E}\left[\mathscr{A}V(\tilde{x}_t) + V_*(x_t)\right] \le \mathbb{E}\left[\tilde{\Upsilon}(x_t, x_{t-\tau})\right].$$

Finally, by the assumption on  $\Upsilon$  and defining

$$\bar{y} := \begin{bmatrix} T^{-1} \\ R^{-1} \end{bmatrix} S^{\mathsf{T}} \begin{bmatrix} x_t \\ x_{t-\tau} \end{bmatrix} \in \mathbb{R}^{2n},$$

we obtain

$$\tilde{\Upsilon}(x_t, x_{t-\tau}) = \Upsilon(x_t, x_{t-\tau}, \bar{y}) \le 0.$$

Therefore Eq. (9) follows. This completes the proof.

## C. Numerical example: Control of a spin-1/2 system

This subsection focuses on a *spin-1/2* model such that the system is composed of only a single particle. In this case, the density matrix  $\rho_t$  is in  $\mathbb{C}^{2\times 2}$ . The filter equation (5) without the input (i.e.,  $u_t=0$ ) shows the following probabilistic convergence:

$$\rho_t \to \rho_{\uparrow} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \rho_t \to \rho_{\downarrow} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This phenomenon is known as quantum state reduction [30]. Here  $\rho_{\uparrow}$  (resp.  $\rho_{\downarrow}$ ) denotes the eigenstate (of  $L=F_z$ ) for which the monitored spin state of the atom is deterministically up (resp. down). Note that when  $u_t=0$ , these two matrices are the only equilibrium points of Eq. (5). Our goal is to design a feedback control law  $u_t=u(\rho_{t-\tau})$  that achieves the deterministic convergence of  $\rho_t$  to the prescribed target  $\rho_f$ , which is either  $\rho_{\downarrow}$  or  $\rho_{\uparrow}$ , as we choose.

It is shown in [8] that the control input  $u_t = u(\rho_t)$  with

$$u(\rho) := k_1(1 - \operatorname{tr}(\rho \rho_f)) + k_2 \operatorname{tr}(i[F_y, \rho]\rho_f)$$
(11)

achieves the control objective  $\rho_t \to \rho_f$  when both  $k_1$  and  $k_2$  are chosen appropriately<sup>2</sup>. In this subsection, we derive a sufficient condition for this control law to globally stabilize the spin-1/2 system in the face of feedback delay.

Let us rewrite Eq. (5) in terms of the regulation error

$$\begin{bmatrix} x_t^{\text{(1)}} & x_t^{\text{(2)}} + \mathrm{i} x_t^{\text{(3)}} \\ x_t^{\text{(2)}} - \mathrm{i} x_t^{\text{(3)}} & -x_t^{\text{(1)}} \end{bmatrix} := \begin{cases} \rho_\mathrm{f} - \rho_t, & \text{if } \rho_\mathrm{f} = \rho_\uparrow \\ \rho_t - \rho_\mathrm{f}, & \text{if } \rho_\mathrm{f} = \rho_\downarrow. \end{cases}$$

It can easily be verified that  $\rho_t \to \rho_f$  is equivalent to  $x_t := \begin{bmatrix} x_t^{(1)} & x_t^{(2)} \end{bmatrix}^\mathsf{T} \to 0$ . When we apply the control input  $u_t = u(\rho_{t-\tau})$  with  $u(\cdot)$  given by Eq. (11), the dynamics of  $x_t$  are independent of  $x_t^{(3)}$  and are given by Eq. (6) with

$$f(x, x_{d}) := \begin{bmatrix} -kx_{d}x^{(2)} \\ kx_{d}\left(x^{(1)} - \frac{1}{2}\right) - \frac{1}{2}x^{(2)} \end{bmatrix},$$

$$g(x) := \sqrt{\eta} \begin{bmatrix} 2x^{(1)}(x^{(1)} - 1) \\ (2x^{(1)} - 1)x^{(2)} \end{bmatrix},$$

$$k := \begin{bmatrix} k_{1} & k_{2} \end{bmatrix}.$$

Note that  $\rho_t \geq 0$  means  $x_t$  is in the circular domain C

$$\mathsf{C} := \left\{ \left[ \begin{array}{c} x^{\scriptscriptstyle (1)} \\[1mm] x^{\scriptscriptstyle (2)} \end{array} \right] \in \mathbb{R}^2 : \Psi(x) := x^{\scriptscriptstyle (1)}(x^{\scriptscriptstyle (1)}-1) + x^{\scriptscriptstyle (2)}{}^2 \leq 0 \right\}.$$

It can be verified that, independently of  $u_t$ , the solution of Eq. (6) does not exit C almost surely. In summary, according to Theorem 1, if the following SOS decomposition problem has a solution, then the control objective is achieved:

Problem 1: With the definitions above, let  $V_*(x) := ||x||^2$ . Then, find  $S \in \mathbb{R}^{4\times 2}$ , positive-definite matrices  $R, T \in \mathbb{R}^{2\times 2}$ , and polynomials  $V_i$   $(i = 0, 1), h, h_d$  such that

$$-\Upsilon(x, x_{d}, y) - h(x, x_{d}, y)\Psi(x) - h_{d}(x, x_{d}, y)\Psi(x_{d}),$$

$$h(x, x_{d}, y),$$

$$h_{d}(x, x_{d}, y)$$

are the sum of squares of polynomials in  $x, x_d \in \mathbb{R}^2$  and  $y \in \mathbb{R}^4$ .

We provide a numerical example to illustrate the effectiveness of Theorem 1. Decision polynomials are restricted to quadratic functions. Let  $k_1 = 1.0$  and  $k_2 = 4.0$  which gives the control law whose stabilizing effect for the

<sup>&</sup>lt;sup>2</sup> The interpretation of this control law is as follows. The second term (containing  $k_2 > 0$ ) locally stabilizes  $\rho_f$ . Unfortunately, both  $\rho_{\uparrow}$  and  $\rho_{\downarrow}$  are equilibria of the closed-loop system. Hence, when  $\rho_t$  is close to the eigenstate that is not the regulation point,  $\rho_t$  must be prevented from converging to it. This is done by the first term. See [35] for a discussion on the effect of delays when a switching control law is employed instead.

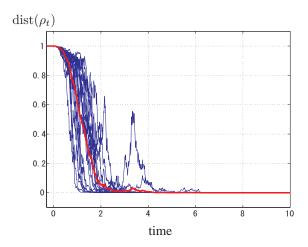


Fig. 2. Time responses of sample paths (thin blue lines) and their average process (thick red line).

delay-free case was examined in [8, subsection IV.G]. Other parameters are chosen to be  $\eta=0.9$  and  $\tau=0.3$ . In this case, Problem 1 has a solution; that is, the target state in the controlled system is shown to be stable. It took 3.01 seconds to check the feasibility of Problem 1 using a computer with a Pentium 4 3.2GHz processor and 2 GB memory.

By setting the target state  $\rho_f := \rho_{\uparrow}$ , we performed a numerical simulation. Time responses of the function

$$\operatorname{dist}(\rho) := 1 - \operatorname{tr}(\rho \rho_{\mathrm{f}}) : \mathcal{S} \to [0, 1]$$

are shown in Figure 2 (30 sample paths and their average). This function gives the distance from the target state, i.e.,  $\operatorname{dist}(\rho) = 0$  (resp.  $\operatorname{dist}(\rho) = 1$ ) if and only if  $\rho = \rho_f$  (resp.  $\rho = \rho_{\downarrow}$ ). The initial state is given by  $\rho_t \equiv \rho_{\downarrow}$  for  $-\tau \leq t \leq 0$ . From Figure 2 it can be seen that stability is achieved.

Remark 2: In principle, the numerical approach introduced in this subsection is applicable to the stability analysis of the general multi-spin system despite time-delays. The computational complexity grows quickly with the dimension. Very high dimensional problems are therefore computationally intractable. On the other hand, there exist some *analytical* results for the N-dimensional delay-free case [10], [28]. The authors are currently investigating computational approaches which combine the aforementioned numerical and analytical methods, in order to overcome this computational issue.

## IV. CONCLUSION

From a practical point of view, filter-based quantum control problems should be formulated taking feedback delay into explicit account. A delay-dependent stability criterion was derived for a class of nonlinear stochastic systems including some quantum spin control systems. A semi-algebraic approach was shown to be useful for incorporating the structure of density matrices.

Theorem 1 was motivated by quantum spin control systems. Theorem 1 can deal with any stochastic delay system having the three properties listed above it. Many finite-dimensional quantum systems satisfy these properties. Hence Theorem 1 is applicable to a wide class of finite-dimensional quantum systems.

This paper is a first attempt to analyze quantum systems which suffer from feedback delays. Hence, many important and interesting problems are left unsolved. The research topic mentioned in Remark 2 is one of them.

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