A Partial Solution of the Aizerman Problem for Second-Order Systems With Delays

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***Abstract*—This paper considers the Aizerman problem for second-order systems with delays. It is proved that for retarded systems with a single delay the Aizerman conjecture is true. For systems with multiple delays, a delay-dependent class of systems is found, for which the Aizerman conjecture is true. The proof is based on the Popov's frequency-domain criterion for absolute stability.**

***Index Terms*—Absolute stability, Aizerman problem, delay systems, frequency-domain methods.**

I. Introduction

The Aizerman problem has a very long history. For systems without delays, the matter is completely settled: the Aizerman conjecture is true for second-order systems and, generally, false for systems of order three and higher [1]. For systems with delays, the problem is unsolved [2] except that Rasvan himself proved that the Aizerman conjecture is true for first-order systems with a single delay, independently of the delay [3].

In this paper, we consider the second-order retarded system described by the scalar delay-differential equation

$$\ddot{x}(t) + a_1\dot{x}(t) + \varphi(x) + b_1\dot{x}(t-\tau) + bx(t-\tau) = 0. \tag{1}$$

It is assumed that the function $\varphi(x)$, hereafter called the nonlinearity, satisfies the sector condition

$$0 < \frac{\varphi(x)}{x} \le \mu. \tag{2}$$

For the linear terms in (1), we can define a transfer function

$$W(s) = \left[s^2 + a_1 s + (b_1 s + b)e^{-\tau s}\right]^{-1}. \tag{3}$$

In proving the results of this paper, we are going to rely extensively on the Popov's frequency-domain stability criterion: the zero solution of (1) is globally asymptotically stable (GAS) if there exists a constant β, such that for all values of ω, including infinity, the following inequality holds:

$$\mu^{-1} + \mathrm{Re}[(1 + i\omega\beta)W(i\omega)] > 0. \tag{4}$$

In addition to (1), we are also going to consider the linear equation

$$\ddot{x}(t) + a_1\dot{x}(t) + ax + b_1\dot{x}(t-\tau) + bx(t-\tau) = 0. \tag{5}$$

The problem under investigation requires comparing the values of μ, for which the zero solution of (1) is GAS, with the values of a, for which such solution of (5) is GAS. The Aizerman conjecture states that these values are the same. The question is if this conjecture is true.

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The first step in answering this question is to determine stability conditions for (5). This will be carried out in Section II. Delay not involving derivatives $(b_1 = 0)$ is considered in Section III. Delay involving the first derivative is investigated in Section IV. Finally, in Section V, we extend some of the results of Section III to systems with multiple delays.

II. Linear Systems

For the system described by (5), we define the transfer function

$$W_L(s) = \left[P(s) + Q(s)e^{-\tau s}\right]^{-1}. \tag{6}$$

In this equation

$$P(s) = s^2 + a_1 s + a; \quad Q(s) = b_1 s + b. \tag{7}$$

It is well known that the zero solution of (5) is GAS if and only if all the poles of $W_L(s)$ have negative real parts. An immediate consequence of the results of Pontryagin [4] is that the following inequality constitutes a necessary condition for this to be true:

$$|b| < |a|. \tag{8}$$

If this inequality holds, then the necessary and sufficient condition for the zero solution of (5) to be GAS for all nonnegative values of the delay τ is that the following two conditions are met [5].

1) The real parts of all the roots of the polynomial $P(s)$ are negative. This is true if and only if both coefficients a_1 and a are positive.
2) For any $\omega > 0$

$$|Q(i\omega)| < |P(i\omega)|. \tag{9}$$

These stability conditions can be reduced as follows. Both of the following inequalities are the necessary conditions:

$$|b| < a; \quad |b_1| < a_1. \tag{10}$$

If (10) are satisfied, then the necessary and sufficient condition for stability is that that *one* of the following inequalities is satisfied:

$$|b| < \frac{a_1^2 - b_1^2}{2}; \quad a > \frac{4b^2 + \left(a_1^2 - b_1^2\right)^2}{4\left(a_1^2 - b_1^2\right).} \tag{11}$$

These conditions can be represented graphically in the plane of the parameters a and $|b|$ shown in Fig. 1.

The stability region is the shaded area, bounded by the abscissa axis, the diagonal $|b| = a$, and the curve given by the equation

$$a = \frac{4b^2 + \left(a_1^2 - b_1^2\right)^2}{4\left(a_1^2 - b_1^2\right)}. \tag{12}$$

at the point of tangency $|b| = (a_1^2 - b_1^2)/2$.

This provides a complete answer to the delay-independent stability problem for the second-order linear systems with a single delay (with the exception of neutral systems). The next task is to compare these stability conditions with those for nonlinear systems.

III. Delay Not Involving Derivatives

In case of a single delay not involving derivatives, the inequalities (11) simplify to

$$|b| < \frac{a_1^2}{2}; \quad a > \frac{4b^2 + a_1^4}{4a_1^2}. \tag{13}$$

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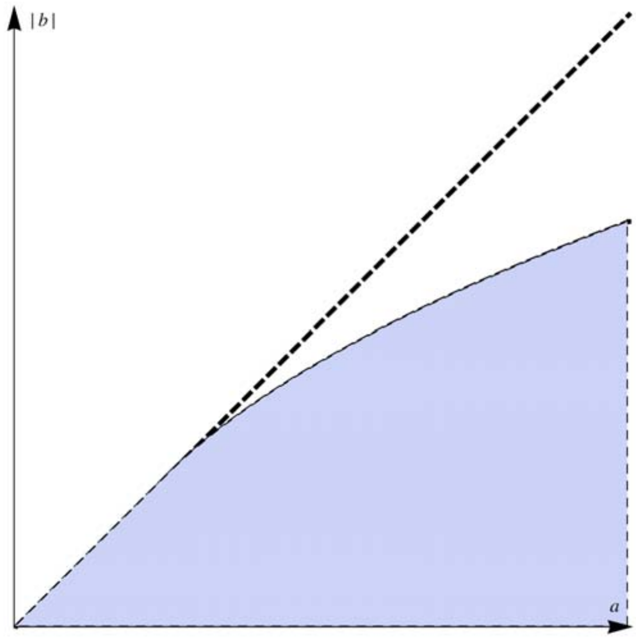

Fig. 1. Stability boundaries for the linear system (5).

If the first of these inequalities holds, then $a > |b|$ is a necessary and sufficient condition for stability of the zero solution of (5). Therefore, the nonlinearity $\varphi(x)$ must lie in the sector $(|b|, +\infty)$ and it makes sense to define the function $f(x)$ by

$$f(x) = \varphi(x) - |b|x. \tag{14}$$

The new nonlinearity $f(x)$ satisfies the sector condition (2) with $\mu^{-1} = 0$ and the transfer function of the linear terms becomes

$$W(s) = \left[s^2 + a_1 s + |b| + be^{-\tau s}\right]^{-1}. \tag{15}$$

Expansion of (4) shows that it holds for all $\mu > 0$ if

$$|b| + (a_1\beta - 1)\omega^2 - b\beta\omega \sin\omega\tau + b\cos\omega\tau \geq 0. \tag{16}$$

Using a well-known trigonometric identity, this inequality can be rewritten in the form

$$|b| + (a_1\beta - 1)\omega^2 + \sqrt{(b\beta\omega)^2 + b^2}\sin(\omega\tau + \phi) \geq 0. \tag{17}$$

It is easy to show that as long as the first of the inequalities (13) holds, (17) holds for all values of ω if the constant β is chosen to satisfy

$$\frac{a_1}{|b|} - \sqrt{\frac{a_1^2 - 2|b|}{b^2}} < \beta < \frac{a_1}{|b|} + \sqrt{\frac{a_1^2 - 2|b|}{b^2}}. \tag{18}$$

This shows that the Aizerman conjecture is true in this case.

Of course, if $b = 0$, we have a system without delays, and the Aizerman conjecture is known to be true.

Let us turn our attention to the case when

$$a > \frac{4b^2 + a_1^4}{4a_1^2}. \tag{19}$$

Instead of (14), we now have

$$f(x) = \varphi(x) - \frac{4b^2 + a_1^2}{4a_1^2}x. \tag{20}$$

Similarly, (15) is replaced with

$$W(s) = \left[s^2 + a_1 s + \frac{4b^2 + a_1^4}{4a_1^2} + be^{-\tau s}\right]^{-1}. \tag{21}$$

Expansion of (4) shows that it holds for all μ if

$$a_1^4 + 16b^2 + 4a_1^2(a_1\beta - 1)\omega^2 + 4a_1^2 b(\cos\omega\tau - \beta\omega\sin\omega\tau) \geq 0. \tag{22}$$

Following the same procedure as in the previous case, it can be shown that this inequality holds for all values of ω if β is chosen to satisfy both of the following inequalities:

$$\frac{16}{a_1} + \sqrt{\frac{128}{a_1^2} + \frac{a_1^6}{b^4} + \frac{24a_1^2}{b^2}} + \frac{a_1^3}{b^2} > 4\beta \tag{23}$$

$$\sqrt{\frac{128}{a_1^2} + \frac{a_1^6}{b^4} + \frac{24a_1^2}{b^2}} + 4\beta > \frac{16}{a_1} + \frac{a_1^3}{b^2}. \tag{24}$$

Therefore, in this case the Aizerman conjecture is true as well. This proves that it is true for all second-order systems with a single delay not involving derivatives.

IV. Delay Involving the First Derivative

For systems with a single delay involving first derivative, the situation is somewhat more complicated. If the first of the inequalities (11) holds, then the inequalities (12) are necessary and sufficient conditions for stability of the zero solution of (5). Once again, we can define the function $f(x)$ by (14). The transfer function of the linear terms becomes

$$W(s) = \left[s^2 + a_1 s + |b| + (b + b_1 s)e^{-\tau s}\right]^{-1}. \tag{25}$$

Expansion of (4) shows that it holds for all $\mu > 0$ if

$$|b| + (a_1\beta - 1)\omega^2 + (b_1 - b\beta)\omega\sin\omega\tau + \left(b + b_1\beta\omega^2\right)\cos\omega\tau \geq 0. \tag{26}$$

It can be shown by the same process as in the previous section that as long as the inequalities (10) and the first of the inequalities (11) hold, (26) holds for all values of ω if the constant β is chosen to satisfy

On the other hand, if $b > 0$, we choose β to satisfy

$$\frac{a_1}{|b|} - \sqrt{\frac{a_1^2 - 2|b| - b_1^2}{b^2}} < \beta < \frac{a_1}{|b|} + \sqrt{\frac{a_1^2 - 2|b| - b_1^2}{b^2}}. \tag{27}$$

If $b = 0$, we replace (26) with

$$(a_1\beta - 1)\omega^2 + b_1\omega\sin\omega\tau + b_1\beta\omega^2\cos\omega\tau \geq 0. \tag{28}$$

Following the same procedure as before, we find that (28) holds for all values of ω if we choose β to satisfy

$$1 + b_1^2 - a_1^2\beta > 0. \tag{29}$$

This proves the Aizerman conjecture for the case of a single delay involving the first derivative if the first of the inequalities (11) holds.

If the second of the inequalities (11) holds, then we define

$$B = \frac{4b^2 + \left(a_1^2 - b_1^2\right)^2}{4\left(a_1^2 - b_1^2\right)}, \tag{30}$$

Instead of (26), we have

$$B + (a_1\beta - 1)\omega^2 + (b_1 - b\beta)\omega\sin\omega\tau + \left(b + b_1\beta\omega^2\right)\cos\omega\tau \geq 0. \tag{31}$$

It is easy to see that $B \geq b$. Therefore, if (26) holds, then (31) holds as well.

Therefore, the Aizerman conjecture is true for all second-order systems with a single delay involving the first derivative.

V. Multiple Delays

Let us extend some of our results to systems with multiple delays. We are only going to consider the case of delays not involving derivatives

$$\ddot{x}(t) + a_1\dot{x}(t) + \varphi(x) + \sum_{j=1}^{m} b_j x(t-\tau_j) = 0. \tag{32}$$

We are not going to investigate in depth the stability of the corresponding linear system except to note that the necessary condition (8) now becomes [5]

$$\sum_{j=1}^{m} |b_j| < |a|\,. \tag{33}$$

Therefore, instead of (14), we have

$$f(x) = \varphi(x) - x\sum_{j=1}^{m} |b_j|. \tag{34}$$

Instead of (16), we now obtain

$$\sum_{j=1}^{m} |b_j| + (a_1\beta - 1)\,\omega^2 - \beta\omega\sum_{j=1}^{m} b_j \sin\omega\tau_j + \sum_{j=1}^{m} b_j\cos\omega\tau_j \geq 0. \tag{35}$$

We can take advantage of the easily verified estimate $\chi\sin\alpha\chi \leq \alpha\chi^2$, valid for $\alpha > 0$, and state that (35) holds for all values of ω if there exists $\beta > 0$ such that the following inequality holds for all values of ω:

$$\sum_{j=1}^{m} \left(|b_j| + b_j\cos\omega\tau_j\right) + \left[\left(a_1 - \sum_{j=1}^{m} |b_j|\,\tau_j\right)\beta - 1\right]\omega^2 \geq 0. \tag{36}$$

This can be assured by choosing $\beta > 0$ to satisfy

$$\left(a_1 - \sum_{j=1}^{m} |b_j|\,\tau_j\right)\beta > 1. \tag{37}$$

Clearly, this can be done if and only if the following inequality is true

$$a_1 > \sum_{j=1}^{m} |b_j|\,\tau_j. \tag{38}$$

Thus, in this case we have identified a delay-dependent class of systems, for which the Aizerman conjecture is true.

VI. Conclusion

The results obtained can be summarized as follows. For retarded systems with a single delay, the Aizerman problem is solved completely—the conjecture is proved to be true. For systems with multiple delays, the frequency-domain inequality yields a delay-dependent stability criterion.

The problem is still open for neutral systems. Another open question is the possibility of improving the result in Section V since the estimate used in the derivation is rather coarse. Indeed, if we set $m = 1$, the resulting stability criterion is much weaker than the one obtained in Section III.

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Reversibility and Poincaré Recurrence in Linear Dynamical Systems

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***Abstract*—In this paper, we study the Poincaré recurrence phenomenon for linear dynamical systems, that is, linear systems whose trajectories return infinitely often to neighborhoods of their initial condition. Specifically, we provide several equivalent notions of Poincaré recurrence and review sufficient conditions for nonlinear dynamical systems that ensure that the system exhibits Poincaré recurrence. Furthermore, we establish necessary and sufficient conditions for Poincaré recurrence in linear dynamical systems. In addition, we show that in the case of linear systems the absence of volume-preservation is equivalent to the absence of Poincaré recurrence implying irreversibility of a dynamical system. Finally, we introduce the notion of output reversibility and show that in the case of linear systems, Poincaré recurrence is a sufficient condition for output reversibility.**

***Index Terms*—Irreversibility, Lagrangian and Hamiltonian systems, output reversibility, Poincaré recurrence, volume-preserving flows.**

I. Introduction

The Poincaré recurrence theorem states that every finite-dimensional, isolated dynamical system with volume-preserving flow and bounded trajectories will return arbitrarily close to its initial state infinitely many times. This theorem was proven by Poincaré [1] and further studied by Birkhoff [2] for Lagrangian systems and Halmos [3] for ergodic systems. Poincaré recurrence has been the main source for the long and fierce debate between the microscopic and macroscopic points of view of thermodynamics [4]. In thermodynamic models predicated on statistical mechanics, an isolated dynamical system will return arbitrarily close to its initial state of molecular positions and velocities infinitely often. If the system entropy is determined by the

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