Robust and Adaptive Observer-Based Partial Stabilization for a Class of Nonlinear Systems[1]

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Abstract— **The problem of adaptive stabilization with respect to a set for a class of nonlinear systems in the presence of external disturbances is considered. A novel adaptive observer-based solution for the case of noisy measurements is proposed. The efficiency of proposed solution is demonstrated via example of swinging a pendulum with unknown parameters.**

I. INTRODUCTION

The problem of nonlinear adaptive control got a number of solutions during the last decade [2], [11], [14], [17], [22], [24]. Most of the existing solutions are tailored to achieve such goals as regulation or tracking, where the system trajectory converges to a point or to a curve. In these cases the goal functionals possess radial unboundedness with respect to the whole state vector of the controlled system. However, in a number of applications the plant stabilization with respect to *a part of variables* (i.e. with respect to a set) is needed. For example, such problems arise when stabilizing the desired energy level for physical or mechanical systems, synchronization, etc. [10], [11], [23].

An additional requirement may consist in boundedness of control signal [3], [11]. In the presence of parametric uncertainty the dependence of bounded control law on adjusted parameters leads to the problem of adaptation with nonlinear parameterization of the controller. This fact prevents from applying previously mentioned results. The problem is to design an output feedback control for an unknown plant, providing stabilization of the given set or its vicinity in the presence of disturbances and measurement errors. An important additional requirement to theoretic results is that the guaranteed bound for limit set of the closed loop system should approach the given set as the level of disturbances and errors approach zero. Such a statement of the problem looks natural when the level of disturbances is unknown, though bounded. Among numerous examples of such situations are energy control problems and synchronization problems.

A number of problems of the above class were solved previously by the speed-gradient method under assumption of passivity or passifiability [10], [11], [27], [28]. However, many systems of interest, e.g. those

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having relative degree greater than one cannot be made passive. Solutions for nonpassifiable systems were suggested in [12] based on a special nonlinear observer structure proposed by V. Nikiforov [11], [12], [24].

In the previous works of the authors [5], [6],[7], [8], [9] solutions for such sort of problems were proposed for output synchronization, observation, I-O stabilization. This paper is devoted to the robust and adaptive partial stabilization. Partial stabilization is considered with respect to a function and the goal set is a surface in the state space. We stress, that consideration of partial stability as the set stability is only one of the possible notions of partial stability; under some circumstances, indeed, more than one measure is requested to formulate the property in a suitable way. A new solution is proposed below for a class of nonlinear affine in inputs systems in an observer canonical form.

II. Preliminaries

Let us consider dynamical systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}),\ \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector; $\mathbf{u} \in R^m$ is the input vector; $\mathbf{y} \in R^p$ is the output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0,0) = 0$. Euclidean norm will be denoted as $|\mathbf{x}|$, and $\|\mathbf{u}\|_{[t_0,t]}$ denotes the L_∞^m norm of the input ($\mathbf{u}(t)$ is Lebesgue measurable and locally essentially bounded function $\mathbf{u}: R_+ \to R^m$, $R_+ = \{\tau \in R : \tau \ge 0\}$):

$$\|\mathbf{u}\|_{[t_0,T]} = ess\sup\{|\mathbf{u}(t)|,\ t \in [t_0, T]\},$$

if $T = +\infty$ then we will simply write $\|\mathbf{u}\|$. We will denote as $\mathcal{M}_{R^m}$ the set of all such Lebesgue measurable inputs $\mathbf{u}$ with property $\|\mathbf{u}\| < +\infty$. For initial state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ let $\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$ if all other arguments of solution are clear from the context; $\mathbf{y}(t, \mathbf{x}_0, \mathbf{u}) = \mathbf{h}(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u}))$), which is defined on some finite interval $[0, T)$; if for every initial state $\mathbf{x}_0 \in R^n$ and $\mathbf{u} \in \mathcal{M}_{R^m}$ the solutions are defined for all $t \ge 0$, then system is called forward complete. It is said that system (1) has unboundedness observability (UO) property, if for each state $\mathbf{x}_0 \in R^n$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ such that $T < +\infty$ necessarily

$$\limsup_{t \to T} |\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| = +\infty .$$

In other words it is possible to observe any unboundedness of the state by detecting an unboundedness of the output. The necessary and sufficient conditions for forward completeness and UO properties were investigated in [1]. Distance in R^n from given point $\mathbf{x}$ to set $\mathcal{A}$ is denoted as $|\mathbf{x}|_{\mathcal{A}} = \mathrm{dist}(\mathbf{x}, \mathcal{A}) = \inf_{\boldsymbol{\eta}\in\mathcal{A}} |\mathbf{x} - \boldsymbol{\eta}|$. As usual, a continuous function $\sigma : R_+ \to R_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; additionally it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded; a continuous function $\beta : R_+ \times R_+ \to R_+$ is from class $\mathcal{KL}$, if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in R_+$, and $\beta(s, \cdot)$ is strictly decreasing to zero for each $s \in R_+$.

System (1) is called bounded-input-bounded-state (BIBS) stable if for all $\mathbf{x}_0 \in R^n$, $\mathbf{u} \in \mathcal{M}_{R^m}$, $t \ge 0$ the property $|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})| \le \max\{\mu(|\mathbf{x}_0|), \mu(\|\mathbf{u}\|)\}$ holds for $\mu \in \mathcal{K}$.

D e f i n i t i o n 1 [13], [29]. *UO system (1) is input-to-output stable (IOS), if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such, that inequality* $|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| \le \beta(|\mathbf{x}(t_0)|, t - t_0) + \gamma(\|\mathbf{u}\|_{[t_0, t)})$, $t \ge t_0 \ge 0$ *holds for all* $\mathbf{x}(t_0) \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_{R^m}$. □

D e f i n i t i o n 2 [9]. *Forward complete system (1) is called integral input-to-state stable (iISS) with respect to closed invariant set* $\mathcal{A}$ *if there exist functions* $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$ *such, that for any* $\mathbf{x}_0 \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_{R^m}$ *the inequality holds* $\alpha(|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})|_{\mathcal{A}}) \le \beta(|\mathbf{x}_0|_{\mathcal{A}}, t) + \int_0^t \gamma(|\mathbf{u}(\tau)|) d\tau$, $t \ge 0$. □

A. Robust stabilization with respect to a set via passification approach

Let us consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})[\mathbf{u} + \mathbf{v}], \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{y} \in R^m$ are state, input and output vectors correspondingly; $\mathbf{v} \in R^m$ is external disturbances vector; $\mathbf{f}$, $\mathbf{h}$ and columns of matrix $\mathbf{G}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0) = 0$.

D e f i n i t i o n 3 [4], [11], [31]. *It is said that system (2) is passive with continuously differentiable storage function* $V : R^n \to R_+$ *if for all* $\mathbf{x} \in R^n$ *and* $\mathbf{u} \in R^m$, $\mathbf{v} \in R^m$ *it holds that* $\dot{V} \le \mathbf{y}^T[\mathbf{u} + \mathbf{v}]$. □

The passification method [11], [27], [28] is based on a feedback design making the closed loop system passive. It allows one to solve partial stabilization problem for system (2) with respect to the zero level set of storage function. The key property for this approach to partial stabilization is detectability assumption [26],

[27], [28] described in the following definition.

Definition 4. *It is said that passive system (2) with storage function* $V: R^n \to R_+$ *is V-detectable with respect to output* $\mathbf{y}$ *if for all* $\mathbf{x}_0 \in R^n$ *it holds*:

$$\mathbf{y}(t,\mathbf{x}_0,0) \equiv 0\,,\ t \geq 0 \ \Rightarrow \lim_{t\to+\infty} V(\mathbf{x}(t,\mathbf{x}_0,0)) = 0\,. \qquad \square$$

The following result [7], [9] gives conditions of iISS with respect to set stabilization by passification.

Theorem 1. *Let the system (2) be passive with continuously differentiable storage function* $V: R^n \to R_+$ *and a non decreasing function* $\varphi: R^m \to R^m$, $\varphi(0) = 0$ *have the property* $\mathbf{y}^T\varphi(\mathbf{y}) > 0$ *for all* $\mathbf{y} \in R^m/\{0\}$, $\lim_{|\mathbf{x}|_{\mathcal{V}_0}\to\infty} |\mathbf{h}(\mathbf{x})|/V(\mathbf{x}) < \infty$. *Let, additionally, there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that for all* $\mathbf{x} \in R^n$ *inequalities* $\alpha_1(|\mathbf{x}|_{\mathcal{V}_0}) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|_{\mathcal{V}_0})$ *are satisfied, where* $\mathcal{V}_0 = \{\mathbf{x} \in R^n : V(\mathbf{x}) = 0\}$ *is a compact set. Then the system (2) with control* $\mathbf{u} = -\varphi(\mathbf{y})$ *has iISS property with respect to set* $\mathcal{V}_0$ *if the system is V-detectable with respect to the output* $\mathbf{y}$. $\square$

B. Positivity in the average

Identification ability of adaptation algorithms is one of the most attractive problems in the adaptive control theory. The solution of this problem is closely connected with persistent excitation (PE) property. There exist several closely related definitions of PE property [10], [18], [19], [22]. Here we will use the following one.

Definition 5. *Function* $a: R_+ \to R$ *is called positive in the average (PA) if there exists some* $\Delta > 0$ *and* $r > 0$ *such that for all* $t \geq 0$ *and* $\delta \geq \Delta > 0$

$$\int_t^{t+\delta} a(\tau)\,d\tau \geq r\,\delta\,. \qquad \square$$

Importance of PA property is explained in the following lemma, which slightly modified version was proven in [8].

Lemma 1. *Let us consider time-varying linear dynamical system* $\dot{p} = -a(t)p + b(t)$, $t_0 \geq 0$, *where* $p \in R$, $p(t_0) \in R$ *and functions* $a: R_+ \to R$, $b: R_+ \to R$ *are Lebesgue measurable,* b *is locally essentially bounded, function* a *is PA for some* $r > 0$, $\Delta > 0$ *and essentially bounded from below, i.e. there exists* $A \in R_+$ *such, that* $\operatorname{ess\,inf}\{a(t),\ t \geq t_0\} \geq -A$. *Then solutions of the system are defined for all* $t \geq t_0$ *and admit the estimate*

$$|p(t)| \le \begin{cases} |p(t_0)|\, e^{-r(t-t_0)+(A+r)\Delta} + \|b\| \max\{A^{-1}e^{At_0}, r^{-1}e^{-rt_0}\}, & A \ne 0; \\ |p(t_0)|\, e^{-r(t-t_0)+r\Delta} + \|b\| \max\{\Delta, r^{-1}e^{-rt_0}\}, & A = 0. \end{cases} \qquad \square$$

It is possible to show that PA property is *equivalent* to some versions of PE property. However, PA is more convenient for quantitative analysis. Standard sufficient conditions for PE that can be interpreted for PA can be found, e.g. in [22].

C. Adaptive observer design

Let us consider the following system with signal and parametric uncertainties:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{d}_1,\ \mathbf{y} = \mathbf{C}\mathbf{x},\ \mathbf{y}_d = \mathbf{y} + \mathbf{d}_2, \tag{3}$$

where $\mathbf{x} \in R^n$ is state vector; $\mathbf{y} \in R^m$ is output vector; $\boldsymbol{\theta} \in \Omega_\theta \subset R^p$ is vector of uncertain parameters, which values belong to compact set Ω_θ; $\mathbf{d}_1 \in \mathcal{M}_{R^n}$, $\mathbf{d}_2 \in \mathcal{M}_{R^m}$ are vector signals of external disturbances and measurement noise, $\mathbf{d} = [\mathbf{d}_1^T\ \mathbf{d}_2^T]^T$; $\mathbf{y}_d$ is vector of noisy measurements of the system (3) output. Vector function φ and columns of matrix functions $\mathbf{A}$ and $\mathbf{B}$ are locally Lipschitz continuous, $\mathbf{C}$ is some constant matrix of appropriate dimension.

The problem is to design an adaptive observer, which in the absence of disturbances $\mathbf{d}$ provides partial estimates of unmeasured components of vector $\mathbf{x}$ and estimates of unknown components of vector $\boldsymbol{\theta}$. For any $\mathbf{d} \in \mathcal{M}_{R^{n+m}}$ the observer should ensure boundedness of the system solutions. In works [6], [8], [12] a solution of this problem is proposed under the following suppositions.

Assumption 1. *For all* $\mathbf{x}_0 \in \Omega_x$, $\boldsymbol{\theta} \in \Omega_\theta$, $\mathbf{d} \in \mathcal{M}_{R^{n+m}}$ *system (3) is BIBS:*

$$\left|\mathbf{x}(t, \mathbf{x}_0, \boldsymbol{\theta}, \mathbf{d})\right| \le \sigma_0(|\mathbf{x}_0|) + \sigma_0(\|\mathbf{d}\|),\ \sigma_0 \in \mathcal{K},\ t \ge 0. \qquad \square$$

The rest suppositions deal with stabilizability by output feedback of the linear part of system (3).

Assumption 2. *There exist matrix* $\mathbf{L}$, *locally Lipschitz continuous matrix function* $\mathbf{K}: R^m \to R^{n\times m}$ *and continuously differentiable function* $V: R^n \to R_+$ *satisfying relations* $|\mathbf{C}\mathbf{x}| \le |\mathbf{L}\mathbf{x}|$,

$$\alpha_1(|\mathbf{x}|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{x}|);\ \partial V(\mathbf{x})/\partial \mathbf{x}\ \mathbf{G}(\mathbf{y}_d)\mathbf{x} \le -\alpha_3 |\mathbf{L}\mathbf{x}|^2,$$

for all $\mathbf{y}_d \in R^m$, $\mathbf{x} \in R^n$, *where* α_1, α_2 *are from class* $\mathcal{K}_\infty$ *and* $\alpha_3 > 0$, $\mathbf{G}(\mathbf{y}_d) = \mathbf{A}(\mathbf{y}_d) - \mathbf{K}(\mathbf{y}_d)\mathbf{C}$. $\square$

Assumption 2 ensures uniform asymptotic stability with respect to variable $\mathbf{L}\mathbf{s}$ [11], [25] for the system

$$\dot{\mathbf{s}} = \mathbf{G}(\mathbf{y}_d)\mathbf{s} + \mathbf{r} \tag{4}$$

coupled with the system (3) and uniform stability property with respect to variable $\mathbf{s} \in R^n$ for the case $\mathbf{r} = 0$.

The next assumption requires bounded input-bounded state stability of the auxiliary system (4).

Assumption 3. *For all initial conditions* $\mathbf{s}_0 \in R^n$ *and inputs* $\mathbf{r} \in \mathcal{M}_{R^n}$, $\mathbf{y}_d \in \mathcal{M}_{R^m}$ *the system (4) is BIBS uniformly with respect to signal* $\mathbf{y}_d$:

$$|\mathbf{s}(t, \mathbf{s}_0, \mathbf{r}, \mathbf{y}_d)| \le \sigma_1(|\mathbf{s}_0|) + \sigma_1(\|\mathbf{r}\|),\ \sigma_1 \in \mathcal{K},\ t \ge 0. \qquad \square$$

Consider the following equations of adaptive observer:

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y}_d)\mathbf{z} + \varphi(\mathbf{y}_d) + \mathbf{B}(\mathbf{y}_d)\hat{\boldsymbol{\theta}} + \mathbf{K}(\mathbf{y}_d)(\mathbf{y}_d - \hat{\mathbf{y}}),\ \hat{\mathbf{y}} = \mathbf{C}\mathbf{z}; \tag{5}$$

$$\dot{\boldsymbol{\eta}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\eta} - \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}}; \tag{6}$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\Omega} + \mathbf{B}(\mathbf{y}_d); \tag{7}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T(\mathbf{y}_d - \hat{\mathbf{y}} + \mathbf{C}\boldsymbol{\eta}), \tag{8}$$

where $\mathbf{z} \in R^n$ is the vector of variable $\mathbf{x}$ estimates; vector $\boldsymbol{\eta} \in R^n$ and matrix $\boldsymbol{\Omega} \in R^{n\times p}$ are auxiliary variables, which help to overcome high relative degree obstruction for system (3); $\hat{\boldsymbol{\theta}} \in R^p$ is vector of $\boldsymbol{\theta}$ estimates; $\gamma > 0$ defines rate of adaptation.

Theorem 2 [6]. *Let assumptions 1–3 hold and minimum singular value* $a(t)$ *of matrix function* $\mathbf{C}^T\boldsymbol{\Omega}^T(t)$ *be PA. Then solutions of system (3), (5)–(8) are bounded for any initial conditions and* $\mathbf{d} \in \mathcal{M}_{R^{n+m}}$ *and any* $\gamma > 0$, *in the absence of disturbance* $\mathbf{d}$ *the following relations hold*:

$$\lim_{t\to+\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta},\ \lim_{t\to+\infty} \mathbf{L}\mathbf{x}(t) - \mathbf{L}\mathbf{z}(t) = 0. \qquad \square$$

III. MAIN RESULT

Let us consider uncertain nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{R}(\mathbf{y})[\mathbf{u} + \mathbf{d}_3] + \mathbf{d}_1, \tag{9}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x},\ \mathbf{y}_d = \mathbf{y} + \mathbf{d}_2,$$

where (as for the system (3) previously) $\mathbf{x} \in R^n$ is the state vector; $\mathbf{y} \in R^m$ is the output vector; $\boldsymbol{\theta} \in \Omega_\theta \subset R^p$ is the vector of unknown parameters with values from set Ω_θ; $\mathbf{u} \in R^q$ is the control; $\mathbf{d}_1 \in \mathcal{M}_{R^n}$, $\mathbf{d}_2 \in \mathcal{M}_{R^m}$, $\mathbf{d}_3 \in \mathcal{M}_{R^q}$ are vector signals of external disturbances and measurement noise, $\mathbf{d} = [\mathbf{d}_1^T\ \mathbf{d}_2^T\ \mathbf{d}_3^T]^T$; $\mathbf{y}_d$ is noisy output vector of the system (9). Vector function φ and matrix functions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{R}$ are continuous and locally Lipschitz, $\mathbf{C}$ is a matrix of appropriate dimension.

Assumption 4. *There exist locally Lipschitz continuous functions* $\mathbf{u}: R^{m+k+p} \to R^q$, $\boldsymbol{\psi}: R^n \to R^r$ *and matrix* $\mathbf{L}$ *with dimension* $(k \times n)$ *such, that control*

$$\mathbf{u} = \mathbf{u}(\mathbf{y}, \mathbf{L}\,\mathbf{x}, \boldsymbol{\theta}) \tag{10}$$

guarantees for system (9) forward completeness and one of the following properties:

A. *IOS from input* $\mathbf{d}$ *to output* $\boldsymbol{\psi}$.

B. *iISS with respect to set* $\mathcal{Z} = \{\mathbf{x} : \boldsymbol{\psi}(\mathbf{x}) = 0\}$ *for input* $\mathbf{d}$. □

Starting from control (10), depending on unmeasured variables $\mathbf{L}\,\mathbf{x}$ and vector of uncertain parameters of the system $\boldsymbol{\theta}$, it is necessary to design a new control using only measured signal $\mathbf{y}_d$. The control should provide boundedness of the closed loop system solutions for $\mathbf{d} \in \mathcal{M}_{R^{n+m+q}}$ and for the case $\mathbf{d} = 0$ it should ensure asymptotic convergence to zero of output $\boldsymbol{\psi}$ or attractiveness of the set $\mathcal{Z}$ (depending which part of Assumption 4 is satisfied).

It is worth to stress, that two output functions $\mathbf{y}$ and $\boldsymbol{\psi}$ have been introduced. The first one defines the measured variables of the system (9), the second one characterizes the distance to the goal set. Although the vector of unknown parameters $\boldsymbol{\theta}$ appears in a linear fashion in the right hand side of system (9), the right hand side of the closed loop system (9), (10) may nonlinearly depend on $\boldsymbol{\theta}$ since Assumption 4 does not specify the form of function $\mathbf{u}$ dependence on its arguments.

The form of the system (9) is similar to the system (3) (observer canonical form) for which it is possible to design adaptive observer (5)–(8). Substituting in control (10) the estimates of vectors $\mathbf{L}\mathbf{x}$ and $\boldsymbol{\theta}$ provided by the observer, it is possible to solve the posed problem (we assume that matrixes $\mathbf{L}$ in assumptions 2 and 4 are identical). The principal difference of the solved problem from the problem of adaptive observer design as in Theorem 2 consists in appearance of control $\mathbf{u}$ in the right hand side of system (9). Generally speaking in the absence of control (10) system can possess unbounded solutions (Assumption 1 fails). Fortunately this obstacle does not prevent from the design of the observer similarly to (5)–(8):

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y}_d)\mathbf{z} + \boldsymbol{\varphi}(\mathbf{y}_d) + \mathbf{B}(\mathbf{y}_d)\hat{\boldsymbol{\theta}} + \mathbf{R}(\mathbf{y}_d)\mathbf{u} + \mathbf{K}(\mathbf{y}_d)(\mathbf{y}_d - \hat{\mathbf{y}}),\ \hat{\mathbf{y}} = \mathbf{C}\mathbf{z}; \tag{11}$$

$$\dot{\boldsymbol{\eta}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\eta} - \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}}; \tag{12}$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\Omega} + \mathbf{B}(\mathbf{y}_d); \tag{13}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \gamma \boldsymbol{\Omega}^T \mathbf{C}^T (\mathbf{y}_d - \hat{\mathbf{y}} + \mathbf{C}\boldsymbol{\eta}),\ \gamma > 0, \tag{14}$$

where all symbols have the same meaning, $\gamma > 0$ is adaptation gain. Since matrix function $\mathbf{R}$ depends on the output vector only and control $\mathbf{u}$ is produced by the controller, their appearance does not change dynamics of state estimation error $\mathbf{e}=\mathbf{x}-\mathbf{z}$ and auxiliary error $\boldsymbol{\delta}=\mathbf{e}+\boldsymbol{\eta}-\boldsymbol{\Omega}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})$:

$$\begin{aligned}\dot{\mathbf{e}}=&\,\mathbf{G}(\mathbf{y}_d)\mathbf{e}+\varphi(\mathbf{y})-\varphi(\mathbf{y}_d)+\mathbf{B}(\mathbf{y})\boldsymbol{\theta}-\mathbf{B}(\mathbf{y}_d)\hat{\boldsymbol{\theta}}-\\&-\mathbf{K}(\mathbf{y}_d)\mathbf{d}_2(t)+\mathbf{R}(\mathbf{y})\mathbf{d}_3+\left[\mathbf{R}(\mathbf{y})-\mathbf{R}(\mathbf{y}_d)\right]\mathbf{u}+\\&+\mathbf{d}_1(t)+\left[\mathbf{A}(\mathbf{y})-\mathbf{A}(\mathbf{y}_d)\right]\mathbf{x}\ ,\end{aligned}\tag{15}$$

$$\begin{aligned}\dot{\boldsymbol{\delta}}=&\,\mathbf{G}(\mathbf{y}_d)\boldsymbol{\delta}+\varphi(\mathbf{y})-\varphi(\mathbf{y}_d)+\left[\mathbf{B}(\mathbf{y})-\mathbf{B}(\mathbf{y}_d)\right]\boldsymbol{\theta}-\\&-\mathbf{K}(\mathbf{y}_d)\mathbf{d}_2(t)+\mathbf{R}(\mathbf{y})\mathbf{d}_3+\left[\mathbf{R}(\mathbf{y})-\mathbf{R}(\mathbf{y}_d)\right]\mathbf{u}+\\&+\mathbf{d}_1(t)+\left[\mathbf{A}(\mathbf{y})-\mathbf{A}(\mathbf{y}_d)\right]\mathbf{x}\ .\end{aligned}\tag{16}$$

For the case of the absence of the disturbances $\mathbf{d}=0$ systems (15), (16) can be rewritten as follows

$$\dot{\mathbf{e}}=\mathbf{G}(\mathbf{y})\mathbf{e}+\mathbf{B}(\mathbf{y})[\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}],\tag{17}$$

$$\dot{\boldsymbol{\delta}}=\mathbf{G}(\mathbf{y})\boldsymbol{\delta}\ .\tag{18}$$

Form of equations (17), (18) are the same as calculated for the observer (5)–(8) and, therefore, the convergence proof for the observer (11)–(14) follows from the proof of Theorem 2 with minimal modifications dealing with a-prior absence of Assumption 1 for system (9). In the presence of noise $\mathbf{d}_2$ the dependence of right hand sides of (15), (16) on vectors $\mathbf{u}$ and $\mathbf{x}$ makes difficulties for employing of the proof of Theorem 2. This is the reason why this case will be considered under special conditions below.

Theorem 3. *Let for system (9) Assumption 2 hold and Assumption 3 be satisfied for any Lebesgue measurable signal* $\mathbf{y}$*; minimum singular value* $a(t)$ *of matrix function* $\mathbf{C}^T\boldsymbol{\Omega}^T(t)$ *be PA;* $|\mathbf{B}(\mathbf{y}_d(t))|\le B$, $|\mathbf{R}(\mathbf{y}_d(t))|\le R$ *for all* $t\ge 0$, $B,R\in R_+$. *Then the control law* $\mathbf{u}=\mathbf{u}(\mathbf{y}_d,\mathbf{L}\mathbf{z},\hat{\boldsymbol{\theta}})$ *ensures forward completeness of system (9) , boundedness of the system (11)–(14) solutions and boundedness of variable* $\psi(\mathbf{x}(t))$ *for all initial conditions,* $\mathbf{d}\in\mathcal{M}_{R^{n+m+q}}$ *and any* $\gamma>0$ *provided that at least one of the following additional suppositions is valid*:

1) *Assumption 4.A holds, control* $\mathbf{u}=\mathbf{u}(\mathbf{y}_d,\mathbf{L}\mathbf{z},\hat{\boldsymbol{\theta}})$ *is globally Lipschitz function with respect to the last two arguments and* $\mathbf{d}_2(t)\equiv 0$ *for all* $t\ge 0$;

2) *Assumption 4.A holds control* $\mathbf{u}=\mathbf{u}(\mathbf{y}_d,\mathbf{L}\mathbf{z},\hat{\boldsymbol{\theta}})$ *is globally Lipschitz function, function* φ *is globally Lipschitz,* $\mathbf{A}(\mathbf{y})=\mathbf{A}$, $\mathbf{B}(\mathbf{y})=\mathbf{B}$ *and* $\mathbf{R}(\mathbf{y})=\mathbf{R}$;

3) *Assumption 4.B holds and* $\mathbf{d}(t)\equiv 0$ *for all* $t\ge 0$.

Additionally, if $\mathbf{d}(t)\equiv 0$ *for all* $t\ge 0$, *then limit relations* $\lim_{t\to+\infty}\psi(t)=0$ *(Assumption 4.A) or*

$\lim_{t\to+\infty} |\mathbf{x}(t)|_Z = 0$ *(Assumption 4.B) hold.*

P r o o f. At first let us consider the case $\mathbf{d}_2(t)\equiv 0$, $t\geq 0$ under Assumption 4.A. Then differential equations (15), (16) take form:

$$\dot{\mathbf{e}} = \mathbf{G}(\mathbf{y})\mathbf{e} + \mathbf{B}(\mathbf{y})[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + \mathbf{d}_1(t), \tag{19}$$

$$\dot{\boldsymbol{\delta}} = \mathbf{G}(\mathbf{y})\boldsymbol{\delta} + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + \mathbf{d}_1(t). \tag{20}$$

Equations (20) and (13) have form (4) with bounded inputs, thus, according to Assumption 3 variables $\boldsymbol{\delta}$ and $\boldsymbol{\Omega}$ are bounded. Let us consider the time derivative of Lyapunov function $W(\hat{\boldsymbol{\theta}}) = \gamma^{-1}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$, which for system (14) has form:

$$\dot{W} = -2(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T\boldsymbol{\Omega}^T\mathbf{C}^T\left[\,\mathbf{C}\left(\,\boldsymbol{\delta} + \boldsymbol{\Omega}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\,\right) + \mathbf{d}_2\,\right] \leq -\gamma a(t)W + |\mathbf{C}\boldsymbol{\delta} + \mathbf{d}_2|^2. \tag{21}$$

For bounded input $\mathbf{C}\boldsymbol{\delta} + \mathbf{d}_2$ the boundedness of error $\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ follows from Lemma 1 and PA property of signal a. Having in mind this conclusion it is possible to transform equation (19) to the form (4) with bounded inputs. Applying again Assumption 3 one can substantiate boundedness of $\mathbf{e}$. Variable $\boldsymbol{\eta}$ is a part of error $\boldsymbol{\delta}$, where all other parts and $\boldsymbol{\delta}$ are bounded. Hence $\boldsymbol{\eta}$ is also bounded. Let us substitute the control into equation (9):

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \boldsymbol{\varphi}(\mathbf{y}) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{R}(\mathbf{y})[\mathbf{u}(\mathbf{y}, \mathbf{L}\,\mathbf{x}, \boldsymbol{\theta}) + \mathbf{d}_3] + \mathbf{R}(\mathbf{y})\mathbf{e}_u + \mathbf{d}_1, \tag{22}$$

where $\mathbf{e}_u = \mathbf{u}(\mathbf{y}, \mathbf{L}\,\mathbf{z}, \hat{\boldsymbol{\theta}}) - \mathbf{u}(\mathbf{y}, \mathbf{L}\,\mathbf{x}, \boldsymbol{\theta})$ is the error of control (10) realization. By conditions control $\mathbf{u} = \mathbf{u}(\mathbf{y}_d, \mathbf{L}\,\mathbf{z}, \hat{\boldsymbol{\theta}})$ is globally Lipschitz continuous and errors $\mathbf{e}$ and $\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ are bounded. Therefore, there exists a constant $L_u > 0$ such, that for all $t \geq 0$ inequality $|\mathbf{e}_u(t)| \leq L_u[|\mathbf{L}\,\mathbf{e}(t)| + |\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)|]$ holds and error $\mathbf{e}_u$ is bounded. According to Assumption 4, control (10) ensures for system (9) forward completeness and IOS properties, that implies boundedness of function $\boldsymbol{\psi}$.

Assume now the presence of noise $\mathbf{d}_2$ under structure restrictions $\mathbf{A}(\mathbf{y}) = \mathbf{A}$, $\mathbf{B}(\mathbf{y}) = \mathbf{B}$ and $\mathbf{R}(\mathbf{y}) = \mathbf{R}$. Obviously, that in this case $\mathbf{K}(\mathbf{y}) = \mathbf{K}$. Equations (15), (16) can be rewritten as follows:

$$\dot{\mathbf{e}} = \mathbf{G}(\mathbf{y}_d)\mathbf{e} + \boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{y}_d) + \mathbf{B}[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] - \mathbf{K}\,\mathbf{d}_2(t) + \mathbf{R}\,\mathbf{d}_3 + \mathbf{d}_1(t),$$

$$\dot{\boldsymbol{\delta}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\delta} + \boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{y}_d) - \mathbf{K}\,\mathbf{d}_2(t) + \mathbf{R}\,\mathbf{d}_3 + \mathbf{d}_1(t).$$

Since $\boldsymbol{\varphi}$ is globally Lipschitz continuous, applying Assumption 3 we justify boundedness of variables $\boldsymbol{\delta}$, $\boldsymbol{e}$ and $\boldsymbol{\Omega}$. Analyzing properties of function W we obtain boundedness of variable $\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. Boundedness of all other variables of the system can be proven in the same way as in the previous case.

In the absence of disturbances ($\mathbf{d}=0$) system (15), (16) takes form (17), (18). It follows from Assumption 2 that the variable $\boldsymbol{\delta}$ is bounded and system is asymptotically stable with respect to the part of variables $\mathbf{L}\boldsymbol{\delta}$ [25]. Assumption 3 gives boundedness of variable $\boldsymbol{\Omega}$. In this case for time derivative of function W Lemma 1 provides asymptotic convergence to zero of variable $\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}(t)$. According to assumptions 2 and 3 the system (17) is asymptotically stable with respect to variable $\mathbf{Le}$ and has bounded solutions. Since signals $\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}(t)$ and $\mathbf{Le}(t)$ converge to zero, the error $\mathbf{e}_u(t)$ also converges to zero. Having in mind the properties of control (10) from Assumption 4.A we obtain convergence to zero of variable $\boldsymbol{\psi}(t)$. In such case error $\mathbf{e}_u(t)$ is integrally bounded and we can apply Assumption 4.B. Proof is completed. ■

R e m a r k . Globally Lipschitz property requirement for control (9) naturally holds for bounded controls with bounded partial derivative (like, for example, for *tanh*() used in the next section). □

For Assumption 4.A the theorem provides convergence conditions in the presence of external disturbances and noise. The noisy case needs additional structural restrictions.

For Assumption 4.B Theorem 3 does not propose constructive conditions, which ensure operating of the system in the presence of disturbances $\mathbf{d}$. Robust properties of control (10) in this case are oriented on parametric uncertainty and partial state measurements compensation. It is possible to weaken requirements of Theorem 3 for the case of Assumption 4.B, supposing boundedness and asymptotic convergence to zero of disturbance $\mathbf{d}$. The proof of Theorem 3 remains valid with minimal modifications. However, even under such restrictive conditions Assumption 4.B is the most important for practical applications since this part allows to design adaptive controls for Hamiltonian systems.

Theorem 1 presents results for iISS stabilization of passive systems with respect to a set (energy levels (Hamiltonian levels) stabilization for mechanical systems). If parameters of the plant are unknown, then Hamiltonian can depend on vector of uncertain parameters $\boldsymbol{\theta}$ in complex nonlinear fashion, that prevents application of conventional adaptation techniques oriented on convex parameterization of the system equations. Combining results of theorems 1 and 3 it is possible to propose a solution of this problem.

C o r o l l a r y 1 . *Assume that:*

1. *System (9) for* $\mathbf{d}=0$ *is passive with respect to output* $\boldsymbol{\psi}=L_{\mathbf{R}(\mathbf{y})}W(\mathbf{x})^T$ *and input* $\mathbf{u}$ *with smooth storage function* $W:R^n\to R_+$, $\alpha_1(|\mathbf{x}|_{\mathcal{W}_0})\le W(\mathbf{x})\le\alpha_2(|\mathbf{x}|_{\mathcal{W}_0})$, $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, *where* $\mathcal{W}_0=\{\mathbf{x}:W(\mathbf{x})=0\}$ *is a compact*

set; system (9) is W-detectable for output ψ, $\lim_{|\mathbf{x}|_{\mathcal{W}_0}\to\infty} |\psi(\mathbf{x})|/W(\mathbf{x})<\infty$.

2. *For system (9) Assumption 2 holds and Assumption 3 is satisfied for any Lebesgue measurable signal* $\mathbf{y}$*; minimum singular value* $a(t)$ *of matrix function* $\mathbf{C}^T\mathbf{\Omega}^T(t)$ *is PA;* $|\mathbf{B}(\mathbf{y}_d(t))|\le B$, $|\mathbf{R}(\mathbf{y}_d(t))|\le R$ *for all* $t\ge 0$, $B,R\in R_+$.

3. *Smooth function* $\varphi: R^q\to R^q$ *for all* $\psi\in R^q/\{0\}$ *possesses inequality* $\psi^T\varphi(\psi)>0$ *and* $\mathbf{u}=-\varphi(\psi(\mathbf{x}))=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})$.

Then control law $\mathbf{u}=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{z},\hat{\boldsymbol{\theta}})$ *provides for system (9), (11)–(14) global boundedness of solutions for the case* $\mathbf{d}=0$ *and any* $\gamma>0$*, additionally* $\lim_{t\to+\infty}|\mathbf{x}(t)|_{\mathcal{W}_0}=0$.

P r o o f. The first and the third parts of conditions provide implementation of Theorem 1 in this case. In such situation control $\mathbf{u}=-\varphi(\psi(\mathbf{x}))=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})$ ensures iISS property with respect to compact set $\mathcal{W}_0$ and input $\mathbf{d}_3$ for system (9). Due to compactness property of the set, the system is also forward complete. Therefore, all conditions of Assumption 4.B are satisfied and taking in mind other conditions of the corollary the result of Theorem 3 holds. ■

Further let us consider example of Corollary 1 results application.

IV. Adaptive swinging a pendulum

Consider the problem of energy stabilization for a pendulum with partial observations and parametric uncertainty:

$$\dot{x}_1=x_2,\ y=x_1,\ \dot{x}_2=-\omega^2\sin(x_1)+u,$$

where $\mathbf{x}=[x_1\, x_2]^T$ is state vector; ω is unknown natural frequency, $\theta=\omega^2$. It is required to stabilize the desired value H^* of pendulum energy $H(x_1,x_2)=0.5x_2^2+\omega^2(1-\cos(x_1))$. The system is passive with respect to output $\psi=x_2\,[H(x_1,x_2)-H^*]$ with positive and smooth storage function $W(x_1,x_2)=0.5[H(x_1,x_2)-H^*]^2$. The system is W-detectable with respect to the output [26]. If $H^*\le 2\omega^2$, then the zero level set of the storage function is compact. The value $H^*=2\omega^2$ corresponds to stabilization of the upper equilibrium of the pendulum.

In [11] the energy control law $u=-\varphi(\psi)$ was proposed, and successfully tested by simulation for

$\varphi(\psi) = \tanh(\psi)$. Let us show that such control law and storage function satisfy conditions of Corollary 1. The equations (11)–(14) take form:

$$\dot{z}_1 = z_2 + K(x_1 - z_1); \; K > 0;$$
$$\dot{z}_2 = K(x_1 - z_1) - \hat{\theta}\sin(x_1) -$$
$$- \varphi(z_2 \; [0.5 z_2^2 + \hat{\theta}(1 - \cos(x_1)) - 2\hat{\theta}]);$$
$$\dot{\Omega}_1 = -K\Omega_1 + \Omega_2; \qquad \dot{\eta}_1 = -K\eta_1 + \eta_2 - \Omega_1\dot{\hat{\theta}};$$
$$\dot{\Omega}_2 = -K\Omega_1 - \sin(x_1); \qquad \dot{\eta}_2 = -K\eta_1 - \Omega_2\dot{\hat{\theta}};$$
$$\dot{\hat{\theta}} = \gamma\Omega_1(x_1 - z_1 + \eta_1).$$

To test PA property of signal $\mathbf{C}^T\mathbf{\Omega}^T(t) = \Omega_1(t)$ it is enough to establish PE property of signal $v(t) = \sin(x_1(t))$ or PA property of $v'(t) = \sin^2(x_1(t))$. Indeed, $v(t)$ is the single input of stable linear filter (13). Clearly, that forced part of solution (proportional to $v(t)$) defines properties of signal $\Omega_1(t)$ (transient motions converge to zero asymptotically). The PA property of signal $v'(t)$ implies, that the system trajectories do not converge and do not stay into the points $x_1 = \pm n\pi$, $n = 0,1,2,...$. This convergence is possible only in the equilibriums of the system $(\pm n\pi, 0)$, $n = 0,1,2,...$, but linearization of the pendulum dynamics closed by the proposed control is unstable in these equilibriums for $0 < H^* < 2\omega^2$, since these equilibriums are not the desired final positions of the system. Moreover, the simulation below show, that even for the case $H^* = 2\omega^2$ the algorithm keeps its identification abilities.

The proposed observer with control

$$u = -\varphi(z_2 \; [0.5 z_2^2 + \hat{\theta}(1 - \cos(x_1)) - 2\hat{\theta}])$$

provides stabilization of the upper equilibrium of the pendulum (in this case $H^* = 2\omega^2 = 2\hat{\theta}$). The simulation results are shown in Fig. 1 for $\omega = K = \gamma = 1$ and zero initial conditions (except $x_1(0) = 0.1$). Trajectories in the state space of the pendulum (solid line) and in the coordinate space of the adaptive observer (z_1, z_2) (dotted line) are shown in Fig. 1,*a*. The observation error is presented in Fig. 1,*b* separately. In figures 1,*c* and 1,*d* plots of variables $\hat{\theta}(t)$ and $H(t)$ are shown.

Note that solutions from papers, [10], [16], ,[23], [30] can not be applied in this example due to boundedness of control (high gain feedbacks, which suppress nonlinearities, are not possible) or since output stabilization is required here.

V. Conclusion

In this paper the previous results of the authors [5], [6], [7], [8], [9] obtained for output synchronization, observation, I-O stabilization are extended to the robust and adaptive partial stabilization problems for a class of nonlinear systems affine in control and disturbances. Note that the existing results on stabilization with respect to a part of variables [25] are not applicable since in this paper the partial stabilization is considered with respect to a function and the goal set is a surface in the state space (e.g. energy surface). Applicability conditions of the algorithms are established in the presence of external disturbances and partial observations with measurement noise. In the absence of measurement noise for any disturbance the proposed control algorithms ensure plant identification and estimation with bounded errors, while in the absence of disturbances the algorithms provide output stabilization with exact identification and estimation. This result also holds in the presence of measurement noise under some special structural conditions imposed on the plant equations.

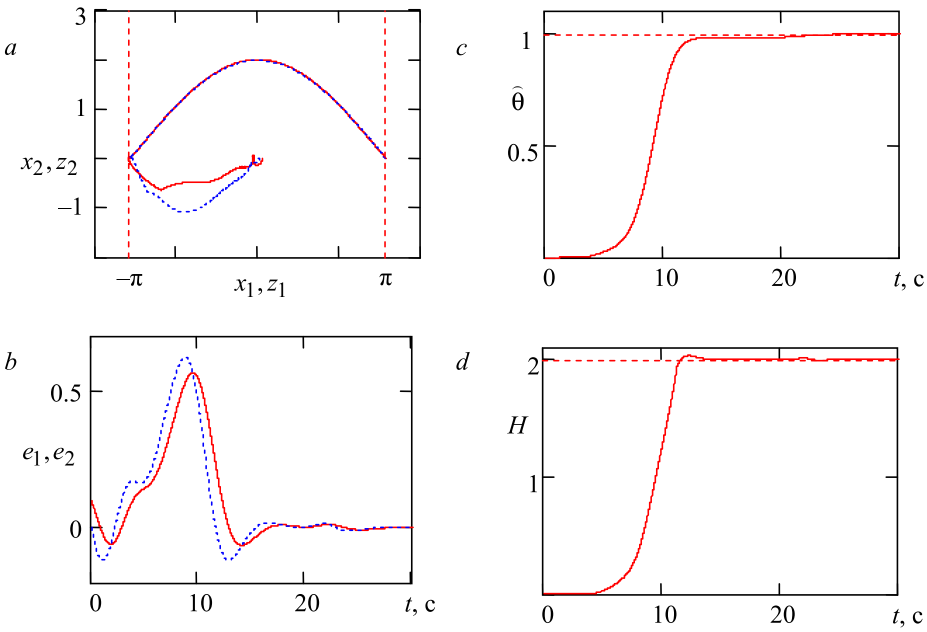

Fig. 1. Adaptive swinging up of the pendulum.

References

[1] Angeli, D. and E.D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations". *Systems and Control Letters*, **38**, 1999, pp. 209 – 217.

[2] Astolfi A., Ortega R. Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems. *IEEE Trans. Aut. Contr.*, **48**, 2003, pp. 590 – 606.

[3] Bullo F. Stabilization of relative equilibria for underactuated systems on Riemannian manifolds. *Automatica*, **36**, 2000, pp. 1819–1834.

[4] Byrnes C.I. Isidori A. Willems J.C. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Aut. Contr*., 1991, **36** (11) pp. 1228 – 1240.

[5] Efimov D.V., Fradkov A.L. Adaptive partial observers with application to time-varying chaotic systems. IUTAM Symposium on Chaotic Dynamics and Control of Systems and Processes in Mechanics. (Ser. Solid Mechanics and Its Applications, Vol. 122), Kluwer, 2005. pp. 27-35..

[6] Efimov D.V. Robust Adaptive Nonlinear Partial Observers for Time-Varying Chaotic Systems. Proc. IEEE CDC 2004, Atlantis, Paradise Island, Bahamas, 2004, pp. 2059–2064.

[7] Efimov, D.V. *Robust and adaptive control of nonlinear oscillations*. St.Petersburg: Nauka, 2005.

[8] Efimov, D.V., Fradkov A.L. Adaptive nonlinear partial observers with application to time-varying chaotic systems. *Proc. IEEE Conf. Control Applications*, Istanbul, WdM1-2, 2003.

[9] Efimov D.V., Fradkov A.L. Input-to-Output Stabilization of Nonlinear Systems via Backstepping. *Proc. IEEE DCD – ECC 2005*, Seville, Spain, 2005, pp. 6128–6133.

[10] Fradkov A.L., Pogromsky A.Yu., Introduction to control of oscillations and chaos. World Scientific Publishers, Singapore, 1998,

[11] Fradkov, A.L., I.V. Miroshnik and V.O. Nikiforov, *Nonlinear and adaptive control of complex systems*. Kluwer Acad. Publishers, 1999.

[12] Fradkov A.L., Nikiforov V.O., Andrievsky B.R. Adaptive observers for nonlinear nonpassifiable systems with application to signal transmittion. *Proc. 41th IEEE Conf. Dec. Control*, 2002, pp.4706-4711.

[13] Ingalls, B. and Y. Wang, “On input-to-output stability for systems not uniformly bounded”. *In Proc. NOLCOS’01*, Saint-Petersburg, 2001.

[14] Ioannou, P. A., Sun, J. *Robust adaptive control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[15] Isidori A. *Nonlinear control systems: An Introduction.* 2nd ed. — Berlin: Springer-Verlag, 1989.

[16] Karsenti, L., F. Lamnabhi-Lagarrige and Bastin G.. Adaptive control of nonlinear systems with nonlinear parameterization. *System and Control Letters*, 1996, **27**, pp. 87–97.

[17] Krstić M., Kanellakopoulos I. and P.V. Kokotović, *Nonlinear and Adaptive Control Design*. Wiley & Sons, Inc., 1995.

[18] Loria A, Kelly R, Teel A.R. Uniform parametric convergence in the adaptive control of mechanical systems. Europ. J. Contr. **11** (2), 2005 , pp. 87-100.

[19] A. Loría E. Panteley, D. Popović, and A. Teel, “δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity,” *in Proc. 41st. IEEE Conf. Decision Control,* 2002.

[20] Marino, R. and Tomei P. Global adaptive output-feedback control of nonlinear systems, part II: Nonlinear parameterization. *IEEE Trans. Automatic Control*, 1993, **38**(1), pp. 33–48.

[21] Marino, R. and Tomei P. *Nonlinear Control Design — Geometric, Adaptive and Robust*. London, U.K.: Prentice-Hall, 1995.

[22] Narendra K. S., Annaswamy A. M. *Stable adaptive systems*. New Jersey: Prentice-Hall, Inc., 1989.

[23] Nijmeijer H. A dynamical control view on synchronization Physica D **154** (3-4) pp. 219-228, 2001

[24] Nikiforov, V.O., Robust High-order Tuner of Simplified Structure, Automatica **35,** (8), pp. 1409–1415, 1999.

[25] Rumyantsev V.V., Oziraner A.S. *Stability and stabilization of motions with respect to part of variables*. M.: Nauka, 1987.

[26] Shiriaev A.S. The notion of V-detectability and stabilization of controlled systems. *Systems & Control Letters*, 2000, **39**, pp. 327 – 338.

[27] Shiriaev, A.S. and A.L. Fradkov, "Stabilization of invariant sets for nonlinear non-affine systems". *Automatica*, **36**, 2000, pp. 1709 – 1715.

[28] Shiriaev, A.S. and A.L. Fradkov, "Stabilization of invariant sets for nonlinear systems with application to control of oscillations". *Intern. J. Robust and Nonlinear Control*, **11**, 2001, pp. 215 – 240.

[29] Sontag, E.D. and Y. Wang, "Notions of input to output stability". *Systems and Control Letters*, **38**, 1999, pp. 235 – 248.

[30] Tyukin I.Yu. Algorithms in finite forms for nonlinear dynamical plants. *Automatics and Remote Control*, 2003, **64**, pp. 951–974.

[31] Willems J.C. Dissipative dynamical systems – Part I: General theory. *Arch. Rational Mechanics and Analysis*, 1972, **45**, pp. 321 – 351.