choose the initial condition of the original system to ensure the boundedness of the trajectory $x(t)$. Taking the initial condition and observer gain as $x(0) = (-0.4, 1, -0.7, -0.5)^T$, $\hat{x}(0) = 0$, $L = (l_1, l_2)$ with $l_1 = (4, 4, 0, 0)^T$, $l_2 = (0, 0, 4, 4)^T$, we have the simulation result in Fig. 1. Simulation demonstrates local error dynamics stability and the ease of implementation of the observer using multiple time scales.

V. CONCLUSION

Time scaling of the multi-output observer form for uncontrolled nonlinear continuous-time systems is considered in this note. Necessary and sufficient existence conditions of a time-scaled observer form are provided. Numerical examples show the construction of the state and time scaling transformations, and the implementation of an observer with multiple time scaling transformations.

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Time Complexity of Decentralized Fixed-Mode Verification

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***Abstract*—Given an interconnected system, this note is concerned with the time complexity of verifying whether an unrepeated mode of the system is a decentralized fixed mode (DFM). It is shown that checking the decentralized fixedness of any distinct mode is tantamount to testing the strong connectivity of a digraph formed based on the system. It is subsequently proved that the time complexity of this decision problem using the proposed approach is the same as the complexity of matrix multiplication. This work concludes that the identification of distinct DFMs (by means of a deterministic algorithm, rather than a randomized one) is computationally very easy, although the existing algorithms for solving this problem would wrongly imply that it is cumbersome. This note provides not only a complexity analysis, but also an efficient algorithm for tackling the underlying problem.**

***Index Terms*—Computational complexity, decentralized control, graph theory, stabilization.**

I. INTRODUCTION

An interconnected system consists of a number of interacting subsystems, which could be homogeneous or heterogeneous. It is evident that many real-world systems can be modeled as interconnected systems, some of which are communication networks, large space structures, power systems, and chemical processes [1]–[5]. The classical control techniques often fail to control such systems, in light of some well-known practical issues such as computation or communication constraints. This has given rise to the emergence of the decentralized control area that aims to design non-classical structurally constrained controllers [6]. More precisely, a (conventional) decentralized controller comprises a set of non-interacting local controllers corresponding to different subsystems.

The notion of decentralized fixed modes (DFMs) was introduced in [7] to characterize those modes of the system which cannot be moved using a linear time-invariant (LTI) decentralized controller. Several methods have been proposed in the literature to find the DFMs of a system [8]–[11]. For instance, an algebraic characterization of DFMs was provided in [8]. The method given in [9], on the other hand, characterizes the DFMs of a system in terms of its transfer function. It was also shown in [12] that the DFMs of any system can be found by checking the transmission zeros of a set of artificial systems derived from the original system. An algorithm was presented in [10] to identify the DFMs of a system by computing the rank of a set of matrices. Unfortunately, the number of systems whose transmission zeros need to be checked in [12] and the number of matrices whose ranks are to be computed in [10] depend exponentially on the number of subsystems of the original system.

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On the other hand, a method is delineated in [13] stating that in order to numerically find the DFMs of a system, it is sufficient to apply a randomly generated static decentralized controller to the system, and then verify what modes of the system are still fixed. Nevertheless, this method is often inaccurate for large-scale systems. More precisely, calculating the eigenvalues of a large-size matrix is normally associated with some errors (especially when the matrix has complex eigenvalues), which makes it impossible to distinguish the fixed modes from the approximate fixed modes [14]. Another issue is that a generic static decentralized controller may not be able to sufficiently displace a mode so that it is recognized as not being a DFM.

A graph-theoretic method was proposed in the recent work [11], which constructs a bipartite graph corresponding to each unrepeated mode of the system. This work states that the mode is a DFM if and only if the graph contains a bipartite subgraph satisfying two specific properties. Although this method seems to be far simpler than the other available deterministic methods, it is not clear how to systematically verify the existence of such a subgraph.

Consider a decision problem, whose answer to be found is "yes" or "no". An algorithm provided for this problem is said to be *efficient* if its time complexity is satisfactory. Informally speaking, the time complexity measures the number of machine instructions executed during the running time of the algorithm (as a function of the size of the input). It is well-understood in computer science that if there exists an efficient randomized algorithm for a decision problem, the existence of a deterministic algorithm with a similar complexity is expected. In other words, randomized algorithms cannot be far more efficient than deterministic algorithms. Regarding the decentralized question posed here (i.e., finding the DFMs of a system), the work [13] shows that there exists an efficient randomized algorithm, whereas the available deterministic algorithms have high time complexities. Based on the above-mentioned discussion, one would conjecture that there exists an efficient deterministic algorithm for the underlying decentralized problem. Finding such an algorithm and investigating its properties are central to the current work.

Given an LTI interconnected system realized in a canonical form, consider a distinct mode of the system. The primary objective of this note is to determine the time complexity of deciding whether this mode is a DFM of the system. To tackle this decision problem, a digraph is constructed by means of an algorithm, whose time complexity is the same as the complexity of matrix multiplication. It is then shown that the answer to the posed decision problem is affirmative if and only if the digraph is not strongly connected. The time complexity of the latter problem (i.e., checking the strong connectivity) is quadratic with respect to the number of subsystems of the system. It is eventually concluded that the time complexity of the original decision problem is the same as that of matrix multiplication, being at most equal to $O(n^{2.376})$, where n denotes the order of the given system. Note that it is unlikely to find another algorithm for this decentralized problem whose complexity is less than the one provided in the present work, due to the fact that any possible algorithm is not permitted to use ordinary matrix operations such as *multiplication, inversion, determinant, rank*, etc. over arbitrary unstructured matrices. This signifies that the obtained complexity order is believed to be the best possible one.

The note is organized as follows. The problem is formulated in Section II, where some preliminaries are provided. The main results are developed in Section III, followed by a numerical example in Section IV. Later on, some concluding remarks are drawn in Section V. A few classical definitions in graph theory are finally given in the Appendix .

II. Preliminaries and Problem Formulation

Consider an LTI interconnected system $\mathcal{S}$ consisting of ν subsystems $S_1, S_2, \ldots, S_\nu$, represented by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{j=1}^{\nu} B_j u_j(t)\\ y_i(t) &= C_i x(t) + \sum_{j=1}^{\nu} D_{ij} u_j(t), \quad i \in \boldsymbol{\nu} := \{1, 2, \ldots, \nu\}\end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state, and $u_i(t) \in \Re^{m_i}$ and $y_i(t) \in \Re^{r_i}, i \in \boldsymbol{\nu}$, are the input and output of the i^{th} subsystem, respectively. Define

$$\begin{aligned} B &:= [\,B_1 \quad \cdots \quad B_\nu\,], \quad C := \begin{bmatrix} C_1 \\ \vdots \\ C_\nu \end{bmatrix}\\ D &:= \begin{bmatrix} D_{11} & \cdots & D_{1\nu} \\ \vdots & \ddots & \vdots \\ D_{\nu 1} & \cdots & D_{\nu\nu} \end{bmatrix}\\ m &:= \sum_{i=1}^{\nu} m_i, \quad r := \sum_{i=1}^{\nu} r_i. \end{aligned} \tag{2}$$

A (conventional) decentralized controller for the system $\mathcal{S}$ is composed of a set of ν local controllers, where the i^{th} local controller, $i \in \boldsymbol{\nu}$, observes only the local output $y_i(t)$ to construct the local input $u_i(t)$ of the i^{th} subsystem. The following definition was presented in [7] for strictly proper systems and generalized in [10] for proper systems.

Definition 1: A mode σ is said to be a DFM of the system $\mathcal{S}$ if there exists no static decentralized controller to displace this mode. In other words, σ is a DFM of the system $\mathcal{S}$ if the relation

$$\sigma \in \operatorname{sp}\left(A + BK(I - DK)^{-1}C\right) \tag{3}$$

holds for every block diagonal matrix K whose i^{th} block entry, $i \in \boldsymbol{\nu}$, is a matrix of dimension $m_i \times r_i$, where $\operatorname{sp}(\cdot)$ stands for the matrix spectrum.

It is noteworthy that as shown in [10], a DFM is fixed with respect to not only static decentralized controllers but also all types of LTI decentralized controllers. In what follows, different methods for finding the DFMs of a system are outlined. Note that for a better understanding of this work, some useful concepts in graph theory are presented in an appendix given at the end of the note.

A. Matrix Rank Checking

According to [10], a (centralized) controllable and observable mode σ is a DFM of the system $\mathcal{S}$ if and only if there exist a permutation of $\{1, 2, \ldots, \nu\}$ denoted by $(i_1, i_2, \ldots, i_\nu)$ and an integer $p \in \{1, 2, \ldots, \nu - 1\}$ such that the rank of the following matrix is less than n:

$$\begin{bmatrix} A - \sigma I_n & B_{i_1} & B_{i_2} & \ldots & B_{i_p} \\ C_{i_{p+1}} & D_{i_{p+1}i_1} & D_{i_{p+1}i_2} & \ldots & D_{i_{p+1}i_p} \\ C_{i_{p+2}} & D_{i_{p+2}i_1} & D_{i_{p+2}i_2} & \ldots & D_{i_{p+2}i_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{i_\nu} & D_{i_\nu i_1} & D_{i_\nu i_2} & \ldots & D_{i_\nu i_p} \end{bmatrix}. \tag{4}$$

The number of matrices in the above form is equal to $(\nu - 1) \times \nu!$. However, it is not required to compute the rank of all such matrices, as those ones which can be converted to each other via re-ordering rows and columns have the same rank. In fact, one needs to find the rank of only $2^\nu - 2$ matrices, which is still an exponential number. This clearly signifies that computing the DFMs of the system $\mathcal{S}$ using this method is cumbersome.

B. Randomized Algorithm

Pick a matrix $K \in \Re^{r\times m}$ at random and consider the matrices A and $A + BK(I - DK)^{-1}C$. The works [13] and [10] state that the DFMs of the system are, almost surely, the common eigenvalues of these two matrices. This gives rise to a randomized algorithm that almost always works correctly. As explained in the introduction, this method suffers from some numerical issues. Nonetheless, this technique indicates that there exists a simple randomized algorithm for finding DFMs, whose complexity is much lower than the deterministic one explained earlier (i.e., testing the ranks of an exponential number of matrices).

C. Derandomization

The work [10] proposes a derandomization technique for the randomized algorithm given in [13] (discussed above). Observe that a decentralized gain matrix K has $\sum_{i=1}^{\nu} m_i r_i$ free parameters, sitting on the block diagonal of this matrix. For every nonnegative integer j satisfying the inequality $j \leq \sum_{i=1}^{\nu} m_i r_i$, pick j of these free parameters, give arbitrary nonzero values to them, and set the remaining free parameters to zero. This leads to a structured decentralized gain matrix. Repeating this procedure for all possible combinations yields p block-diagonal matrices $K_1, K_2, \ldots, K_p$, where

$$p = 2^{\sum_{i=1}^{\nu} m_i r_i}. \tag{5}$$

The derandomized algorithm says that the DFMs of the system are the common eigenvalues of the matrices $A + BK_i(I - DK_i)^{-1}C$, $i = 1, 2, \ldots, p$. Note that even though this algorithm requires that some elements of K_i be given *arbitrary* nonzero values, this method is not really a randomized algorithm. The reason is that those elements need not be chosen using a pseudo-random number generator, and can all be simply considered equal to 1. Observe that although the randomized algorithm mentioned in the preceding subsection runs in polynomial time, its derandomized counterpart runs in exponential time.

D. Graph-Theoretic Approach

Assume that σ is an eigenvalue of A with multiplicity 1, which is also a (centralized) observable and controllable mode of the system $\mathcal{S}$. With no loss of generality, suppose that the matrix A is in the canonical form

$$A = \begin{bmatrix} \sigma & 0 \\ 0 & \mathbf{A} \end{bmatrix} \tag{6}$$

where $\mathbf{A}$ is a matrix of appropriate dimension (this can be achieved using a proper similarity transformation, if need be). Define

$$M(\sigma) := \mathbf{C}(\sigma I_{n-1} - \mathbf{A})^{-1}\mathbf{B} + D \tag{7}$$

where:
- $\mathbf{C}$ is derived from C by eliminating its first column;
- $\mathbf{B}$ is obtained from B by removing its first row.

Denote the (i, j) block entry of $M(\sigma)$ with $M_{ij}(\sigma) \in \Re^{r_i\times m_j}$, for every $i, j \in \boldsymbol{\nu}$.

Definition 2: Let $\mathcal{G}(\sigma)$ be a bipartite graph constructed as follows.
- Consider two sets of vertices, namely set 1 and set 2, with ν vertices in each of them.
- For every $i, j \in \boldsymbol{\nu},\ i \neq j$, connect vertex i in set 1 to vertex j in set 2 if all of the following conditions are satisfied:
  — the first column of C_i is zero;
  — the first row of B_j is zero;
  — $M_{ij}(\sigma)$ is a zero matrix.

We proposed the next result in [11] to verify whether or not σ is a DFM of the system $\mathcal{S}$.

Theorem 1: The mode σ is a DFM of the system $\mathcal{S}$ if and only if the graph $\mathcal{G}(\sigma)$ contains a subgraph $\mathcal{G}_o(\sigma)$ with the following properties.
- It is complete bipartite.
- If $(i_1, i_2, \ldots, i_p)$ and $(j_1, j_2, \ldots, j_q)$ represent the sets of vertices of $\mathcal{G}_o(\sigma)$ (i.e., set 1 and set 2 of $\mathcal{G}_o(\sigma)$), then $(i_1, \ldots, i_p, j_1, \ldots, j_q)$ is a permutation of the set $\boldsymbol{\nu}$.

Although this method seems to be far simpler than the deterministic methods outlined above, it is not clear how to verify the existence of such a subgraph $\mathcal{G}_o(\sigma)$ systematically.

E. Objective of This Work

This work develops the result of [11] under the assumption that the matrix A is in the canonical form (6). The objective is twofold. First, it is desired to propose a simple deterministic algorithm to check whether σ is a DFM of the system $\mathcal{S}$. Second, it is aimed to study the time complexity of this decision problem using deterministic algorithms.

III. Main Results

Assume that the quantities m, r, ν are all less than or equal to n. This realistic assumption is made so that the complexity of computing the DFMs of $\mathcal{S}$ can be written only in terms of n. The following definitions turn out to be convenient in proceeding with the development of the note.

Definition 3: Define $\tilde{\mathcal{G}}(\sigma)$ to be a directed graph (digraph) constructed as follows:
- Consider ν vertices, labeled as $1, 2, \ldots, \nu$.
- For every $i, j \in \boldsymbol{\nu},\ i \neq j$, connect vertex i to vertex j by means of a directed edge if any of the conditions given below is satisfied:
  — the first column of C_i is a nonzero vector;
  — the first row of B_j is a nonzero vector;
  — $M_{ij}(\sigma)$ is not a zero matrix.

It is well known from graph theory that $\tilde{\mathcal{G}}(\sigma)$ can be uniquely decomposed as a union of strongly connected components, namely $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$, such that:
- $\mathcal{C}_i$, $i = 1, 2, \ldots, k$, is a strongly connected induced subgraph of $\tilde{\mathcal{G}}(\sigma)$;
- for every $i, j \in \{1, 2, \ldots, k\},\ i < j$, there is no directed edge going from $\mathcal{C}_i$ to $\mathcal{C}_j$.

This fact will be exploited in the sequel to present one of the main results of the note.

Theorem 2: The mode σ is a DFM of the system $\mathcal{S}$ if and only if the digraph $\tilde{\mathcal{G}}(\sigma)$ is not strongly connected.

Proof of Sufficiency: Assume that the digraph $\tilde{\mathcal{G}}(\sigma)$ is not strongly connected. In light of the discussion given prior to this theorem, the set $\{1, 2, \ldots, \nu\}$ can be partitioned as $\{i_1, i_2, \ldots, i_p\}$ and $\{j_1, j_2, \ldots, j_q\}$ such that there exists no directed edge from vertex i_α to vertex j_β in the digraph $\tilde{\mathcal{G}}(\sigma)$, for all $\alpha \in \{1, 2, \ldots, p\}$ and $\beta \in \{1, 2, \ldots, q\}$. Consider vertices $i_1, i_2, \ldots, i_p$ in set 1 and vertices $j_1, j_2, \ldots, j_q$ in set 2 of the bipartite graph $\mathcal{G}(\sigma)$. Denote with $\mathcal{G}_o(\sigma)$ the bipartite subgraph induced by these vertices.
- Set 1 of $\mathcal{G}_o(\sigma)$ consists of vertices $i_1, i_2, \ldots, i_p$ in set 1 of $\mathcal{G}(\sigma)$.
- Set 2 of $\mathcal{G}_o(\sigma)$ consists of vertices $j_1, j_2, \ldots, j_q$ in set 2 of $\mathcal{G}(\sigma)$.
- The edges of this subgraph have been induced from the original graph $\mathcal{G}(\sigma)$.

For every $\alpha \in \{1, 2, \ldots, p\}$ and $\beta \in \{1, 2, \ldots, q\}$, since vertex i_α is not connected to vertex j_β in the digraph $\tilde{\mathcal{G}}(\sigma)$, it follows from Definitions 2 and 3 that vertex i_α in set 1 of the bipartite graph $\mathcal{G}(\sigma)$ is connected to vertex j_β in set 2 of this graph. This leads to the first observation that the induced subgraph $\mathcal{G}_o(\sigma)$ is complete bipartite. On the other hand

$$\{1, 2, \ldots, \nu\} = \{i_1, i_2, \ldots, i_p\} \cup \{j_1, j_2, \ldots, j_q\}. \tag{8}$$

Now, it follows immediately from Theorem 1 that the mode σ is a DFM, due to the fact that the graph $\mathcal{G}(\sigma)$ contains a subgraph $\mathcal{G}_o(\sigma)$ with the required properties.

Proof of Necessity: Assume that σ is a DFM. It is to be proved that the digraph $\tilde{\mathcal{G}}(\sigma)$ is not strongly connected. To this end, one can utilize Theorem 1 to deduce that the graph $\mathcal{G}(\sigma)$ possesses a bipartite subgraph $\mathcal{G}_o(\sigma)$ with the two properties mentioned earlier. Denote the sets of vertices of this bipartite subgraph with $\{i_1, i_2, \ldots, i_p\}$ and $\{j_1, j_2, \ldots, j_q\}$. Due to the properties of $\mathcal{G}_o(\sigma)$, not only is the relation (8) satisfied, but the following are true for every $\alpha \in \{1, 2, \ldots, p\}$ and $\beta \in \{1, 2, \ldots, q\}$:

- the first column of C_α is zero;
- the first row of B_β is zero;
- $M_{\alpha\beta}(\sigma)$ is a zero matrix.

This means that there exists no directed edge from vertex i_α to vertex j_β in the digraph $\tilde{\mathcal{G}}(\sigma)$. As a result of this observation and the relation (8), one can conclude that the set of vertices of $\tilde{\mathcal{G}}(\sigma)$ can be partitioned into two subsets $\{i_1, i_2, \ldots, i_p\}$ and $\{j_1, j_2, \ldots, j_q\}$ such that there exists no directed edge from vertices $i_1, i_2, \ldots, i_p$ to vertices $j_1, j_2, \ldots, j_q$. Hence, the digraph $\tilde{\mathcal{G}}(\sigma)$ is not strongly connected. ■

Theorem 2 states that checking the decentralized fixedness of σ reduces to testing the strong connectivity of the digraph $\tilde{\mathcal{G}}(\sigma)$. Fortunately, the latter problem is a very simple combinatorial problem, for which several methods have been developed. For instance, one can use Kosaraju's algorithm, which has been regarded as the simplest method for this graph problem [15]. This algorithm performs two complete traversals of the graph and the idea behind it is a depth-first search. Alternatively, Tarjan's algorithm can be employed, whose efficiency is better than Kosaraju's algorithm [16]. Another efficient algorithm, which is mostly suitable for dense graphs, is the Cheriyan/Mehlhorn/Gabow algorithm [17]. Tarjan's algorithm will be adopted in this note to check the strong connectivity of $\tilde{\mathcal{G}}(\sigma)$. Note that this algorithm has been implemented in the Bioinformatics Toolbox of MATLAB.

Lemma 1: Given the matrix $M(\sigma)$, the complexity of constructing the graph $\tilde{\mathcal{G}}(\sigma)$ is at most equal to $O(n^2)$.

Proof: Define the following:

- T is a binary $\nu \times \nu$ matrix whose (i, j) entry is equal to 0 iff the matrix $M_{ij}(\sigma)$ is equal to 0, for all $i, j \in \boldsymbol{\nu}$.
- T_C is a vector of order v whose i-th entry is equal to 0 iff the first column of C_i is equal to 0, for all $i \in \boldsymbol{\nu}$.
- T_B is a vector of order v whose j-th entry is equal to 0 iff the first row of B_j is equal to 0, for all $j \in \boldsymbol{\nu}$.

It is evident that the complexity of forming the abovementioned matrices is equal to that of reading off the entries of the matrix $M(\sigma)$, the first column of C and the first row of B. This leads to the complexity order $O(n^2) + O(r) + O(m) = O(n^2)$. In the course of setting up the graph $\tilde{\mathcal{G}}(\sigma)$, the (i, j) entry of T, the i-th entry of T_C and the j-th entry of T_B are to be checked, for every $i, j \in \boldsymbol{\nu}$, in order to determine if there should be an edge connecting vertex i to vertex j. The complexity of this step is at most equal to $O(\nu^2)$. The proof follows from the fact that $O(\nu^2) + O(n^2) = O(n^2)$. ■

Let the time complexity of matrix multiplication in $\Re^{n\times n}$ be denoted by $O(n^\omega)$, where ω is a positive real. The following theorem presents one of the main results of this work.

Theorem 3: Consider the decision problem "whether or not the mode σ is a DFM". This problem can be solved in $O(n^\omega)$ time by first computing the matrix $M(\sigma)$, and then testing the strong connectivity of its associated digraph $\tilde{\mathcal{G}}(\sigma)$.

Proof: Denote the number of edges of $\tilde{\mathcal{G}}(\sigma)$ with η. It is well-known that Tarjan's algorithm runs in $O(\nu + \eta)$ time in order to test the strong connectivity of $\tilde{\mathcal{G}}(\sigma)$ [16]. Since η is less than or equal to $\nu(\nu - 1)$, the complexity of checking the connectivity of the graph $\tilde{\mathcal{G}}(\sigma)$ is at most $O(\nu^2)$. On the other hand, it is known that matrix inversion and matrix multiplication have identical time complexity exponent [18]. Since $M(\sigma)$ is computed by one matrix inversion and two matrix multiplications, $M(\sigma)$ can be found in $O(n^\omega)$ time. Moreover, the complexity of matrix multiplication over $\Re^{n\times n}$ is at least $O(n^2)$, because there are n^2 entries in the matrix which must be part of any computation. Hence, the quantity ω is at least equal to 2. These results along with Lemma 1 lead to the conclusion that checking the decentralized fixedness of σ can be accomplished in $O(n^2)+O(\nu^2)+O(n^\omega) = O(n^\omega)$ time (note that $\nu \leq n$, by assumption). ■

Remark 1: If the standard method of matrix multiplication is used to compute $M(\sigma)$, the time complexity of checking the decentralized fixedness of σ turns out to be $O(n^3)$, in light of Theorem 3. However, one can employ the Coppersmith-Winograd algorithm for matrix multiplication to reduce this complexity to $O(n^{2.376})$ [19].

Corollary 1: Consider the decision problem "whether or not the mode σ is a DFM". If there exists an algorithm for this decision problem which runs in $O(n^{\bar{\omega}})$ where $\bar{\omega} < \omega$, the algorithm must not use any of the following operations over arbitrary unstructured matrices of approximate dimension $n \times n$: *matrix multiplication, matrix inversion, determinant, LUP-decomposition, computing the characteristic polynomial, orthogonal basis transformation, matrix rank*.

Proof: The proof follows from Theorem 3 and the fact that the operations pointed out in the corollary have (complexity) exponents greater than or equal to that of matrix multiplication [18]. ■

Remark 2: In this note, the conventional decentralized control structure has been considered in which each local controller receives the output of one subsystem in order to generate the input signal of the same subsystem. However, there are numerous applications for which each local controller can receive the outputs of more than one subsystem (based on a given information flow structure). This control structure is usually referred to as *structurally constrained control* or *decentralized overlapping control*. The modes of the system $\mathcal{S}$ which are fixed with respect to every LTI decentralized overlapping controller are said to be decentralized overlapping fixed modes (DOFMs) [20]. It is shown in the recent paper [20] that the DOFMs of the system $\mathcal{S}$ are identical to the DFMs of a virtual system. Hence, the method proposed in this note can be employed to obtain the complexity of finding the DOFMs (by working on the DFMs of the virtual system).

IV. Numerical Example

Let the system $\mathcal{S}$ be composed of ten single-input single-output interconnected subsystems, with the state-space matrices given in Section IV of [21], where the matrix A is already in the canonical form (6) for $\sigma = 1$. The goal is to check whether the mode $\sigma = 1$ is a DFM of the system. To this end, the matrix $M(\sigma)$ introduced in (7) should be computed first. The digraph $\tilde{\mathcal{G}}(\sigma)$ constructed in terms of this matrix is depicted in Fig. 1. This graph has 10 vertices and 47 edges. Tarjan's Algorithm can be employed to traverse all these edges and vertices in order to find the strongly connected components of this graph. This is carried out in the Bioinformatics Toolbox of MATLAB using the command "graphconncomp". This software has detected two strongly connected components: one containing vertices 1, 2, 3, and the other one containing the remaining vertices. Consequently, the digraph is not strongly connected, and hence $\sigma = 1$ is a DFM. Note that even though the graph $\tilde{\mathcal{G}}(\sigma)$ seems to be complex, its connectivity

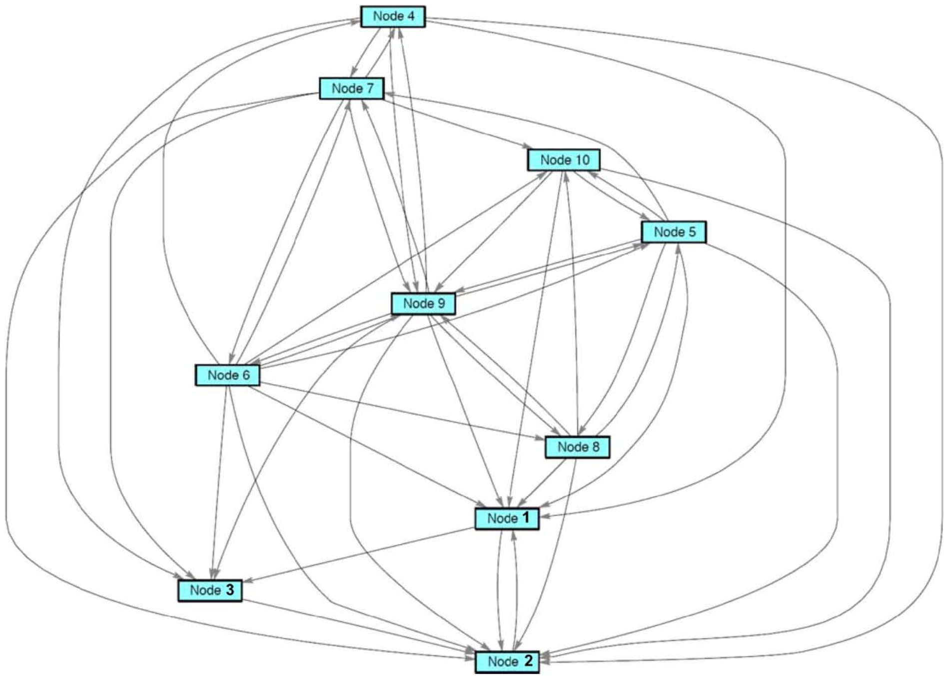

Fig. 1. Digraph $\tilde{\mathcal{G}}(\sigma)$ corresponding to the system used in the numerical example.

verification is an easy job which can be accomplished in quadratic running time (in terms of ν). In this regard, it is worth mentioning that computing the matrix $M(\sigma)$ is more involved than testing the connectivity of $\tilde{\mathcal{G}}(\sigma)$. This clearly shows the simplicity of the method proposed here.

V. Conclusions

This note deals with the time complexity of checking the existence of a stabilizing linear time-invariant (LTI) decentralized controller for a given LTI interconnected system. In particular, the complexity of computing the DFMs of a system is studied. It is well-known that the existing deterministic methods for this problem are computationally intractable, whereas the available randomized numerical method is fairly simple. To investigate the true complexity of solving this problem using a deterministic algorithm, it is shown that checking whether a certain (unrepeated) mode of the system is a DFM amounts to testing the strong connectivity of some digraph. This gives rise to proving that the verification of the decentralized fixedness of a distinct mode of the system has the same time complexity as matrix multiplication and inversion.

Appendix

This part presents some classical definitions in the field of graph theory that have been used in the present note. A *graph* G can be defined as a pair (V, E), where V is a set of vertices and E is a set of edges between the vertices such that $E \subseteq \left\{\{u, v\} | u, v \in V\right\}$. A graph $G' = (V', E')$ is a *subgraph* of G if

$$V' \subseteq V, \quad \left(E' \subseteq E\right) \wedge \left((u, v) \in E' \rightarrow u, v \in V'\right). \qquad (9)$$

A subgraph G' of the graph G is said to be *induced* if, for every pair of vertices u and v in G', the pair (u, v) is an edge of G' if and only if it is an edge of G. A *connected component* of the graph $\mathcal{G}$ is a subgraph in which there exists a path between every two vertices, and to which no more vertices can be added while preserving the path connectivity property. If the edges of G are ordered pairs of vertices, the graph is said to be *directed* (digraph); otherwise, it is called *undirected*. A connected component of a digraph is said to be *strongly connected* if there exists a directed path from every vertex to every other vertex within that component (subgraph). The graph G is *bipartite* if its vertices can be divided into two sets of vertices U_1 and U_2 (referred to as set 1 and set 2) such that each edge of the graph connects a vertex of U_1 to a vertex of U_2. If every vertex of U_1 is connected to all vertices of U_2, the corresponding bipartite graph is said to be *complete bipartite*.

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The Carleman Approximation Approach to Solve a Stochastic Nonlinear Control Problem

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***Abstract*—This note investigates the optimal linear quadratic control problem in the discrete-time framework, for stochastic systems affected by disturbances generated by a nonlinear stochastic exosystem. The application of the maximum principle to nonlinear optimal control problems does not admit, in general, implementable solutions. Therefore, it is worthwhile to look for finite-dimensional approximation schemes. The approach followed in this note is based on the ν-degree Carleman approximation of a stochastic nonlinear system applied to the exosystem and provides a real-time algorithm to design an implementable control law. Simulations support theoretical results and show the improvements when the approximation index ν is increased.**

***Index Terms*—Kalman filtering, stochastic optimal control, stochastic systems.**

I. Introduction

This note considers the finite-horizon optimal control problem for the following linear stochastic system in the discrete-time framework:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + M_x(k)z(k) + N_x(k) \\ x(0) &= x_0, \quad k = 0, \dots, N-1 \\ y(k) &= C(k)x(k) + M_y(k)z(k) + N_y(k) \end{aligned} \tag{1}$$

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where $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in \mathbb{R}^p$ is the control input, $\{N_x(k)\}$ is a sequence of zero-mean independent random vectors taking values in $\mathbb{R}^n$, $y(k) \in \mathbb{R}^q$ is the measured output, $\{N_y(k)\}$ is a sequence of zero-mean independent random vectors taking values in $\mathbb{R}^q$, independent of $\{N_x(k)\}$; $z(k) \in \mathbb{R}^m$ is a persistent disturbance generated by the following nonlinear stochastic exogenous system (the *exosystem*):

$$z(k+1) = f\Big(k, z(k)\Big) + N_z(k), \quad z(0) = z_0 \tag{2}$$

with $f : \mathbb{Z}^+ \times \mathbb{R}^m \mapsto \mathbb{R}^m$ a smooth nonlinear map and $\{N_z(k)\}$ a sequence of zero-mean independent random vectors taking values in $\mathbb{R}^m$, independent of the state and output noise sequences $\{N_x(k)\}$ and $\{N_y(k)\}$. x_0, z_0 are a pair of independent random vectors, independent of all the noise sequences, taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. According to the dimension of the involved vectors, the system matrices are such that: $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times p}$, $C(k) \in \mathbb{R}^{q\times n}$, $M_x(k) \in \mathbb{R}^{n\times m}$, $M_y(k) \in \mathbb{R}^{q\times m}$.

In the case of output regulation, coping with the task of tracking an assigned trajectory and/or rejecting persistent disturbances, the control problem is standardly stated in a deterministic framework and important results in the designing of a suitable control law have been reached, even in the case of uncertainties affecting the model and/or the exosystem. If the dynamics of the exosystem is not known, but it is known that it belongs to a prescribed family of functions, the so called *internal-model principle* allows to reconstruct in some way this lack of information (see [1] and [2] for linear and nonlinear systems, respectively). For instance, an internal-model based control is able to cope with uncertainties affecting the amplitude and phase of an exogenous sinusoid, but it requires the knowledge of the frequency; in order to overcome this limitation, in [3] an adaptation mechanism has been used so that the natural frequencies of the internal model are tuned to match those of the unknown exosystem. Among recent publications on this topic in the continuous-time framework, see [4]–[6], and references therein. As far as the discrete-time framework, in [7] a combination of the regulation theory [8] and fuzzy-modeling is applied to synthesize a robust regulator for an uncertain nonlinear system.

In this note a stochastic framework has been considered to state the finite-horizon optimal control problem of minimizing the quadratic cost functional shown in

$$\begin{aligned} J\Big(u(\cdot)\Big) = \frac{1}{2}\mathbb{E}\Big\{ & x^T(N)Sx(N) \\ & + \sum_{k=0}^{N-1}\Big(x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\Big)\Big\} \end{aligned} \tag{3}$$

with S, $\{Q(k)\}$ symmetric positive semidefinite matrices, and $\{R(k)\}$ symmetric positive definite matrices, for any k. According to (2), the system (1) is driven by a colored noise, whose distribution depends on the distribution of the state of the exosystem $z(k)$. Such a distribution is needed to estimate the state of the system to design the optimal control law. Unfortunately, the state estimation task for nonlinear stochastic systems has not a finite-dimensional solution in the general case. Moreover, even neglecting the stochastic entries $N_x(k)$, $N_y(k)$, $N_z(k)$, the application of the maximum principle to the nonlinear system obtained by coupling (1) with (2) does not ensure an analytical solution to the resulting nonlinear *two-point boundary-value* (TPBV) problem [9].

In the last decades a great deal of literature has been developed with the aim to obtain implementable control schemes from the maximum principle optimality conditions, mainly in a deterministic continuous-time framework [10]. As far as the discrete-time framework, recently, a *successive-approximation* approach has been adopted to pro-