Reduction Principles and the Stabilization of Closed Sets for Passive Systems

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Abstract

In this paper we explore the stabilization of closed invariant sets for passive systems, and present conditions under which a passivity-based feedback asymptotically stabilizes the goal set. Our results rely on novel reduction principles allowing one to extrapolate the properties of stability, attractivity, and asymptotic stability of a dynamical system from analogous properties of the system on an invariant subset of the state space.

1 Introduction

The problem of stabilizing equilibrium points has been at the centre of much research in linear and nonlinear control theory ever since the inception of the field, and continues to receive attention today. Instead, the more general problem of stabilizing *sets* has received comparatively less attention. The set stabilization problem has intrinsic interest because many complex control specifications can be naturally formulated as set stabilization requirements. The synchronization problem [1], which entails making the states of two or more dynamical systems converge to each other, can be formulated as the problem of stabilizing the diagonal subspace in the state space of the coupled system. The observer design problem can be viewed in the same manner. The control of oscillations in a dynamical system [2] can be thought of as the stabilization of a set homeomorphic to the unit circle or, more generally, to the k-torus. The maneuver regulation or path following problem, which entails making the output of a dynamical system approach and follow a specified path in the output space of a control system [3], can be thought of as the stabilization of a certain invariant subset of the state space, compatible with the motion on the path.

Recently, significant progress has been made toward a Lyapunov characterization of set stabilizability. Albertini and Sontag showed, in [4], that uniform asymptotic controllability to a closed, possibly non-compact set is equivalent to the existence of a continuous control-Lyapunov function. Kellett and Teel in [5], [6], and [7], proved that for a locally Lipschitz control system, uniform global asymptotic controllability to a closed, possibly non-compact set is equivalent to the existence of a

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locally Lipschitz control Lyapunov function (see also a related result by Rifford in [8]). Moreover, they were able to use this result to construct a semiglobal practical asymptotic stabilizing feedback.

A geometric approach to a specific set stabilization problem for single-input systems was taken by Banaszuk and Hauser in [9]. The authors gave conditions for dynamics transversal to an open-loop invariant periodic orbit to be feedback linearizable. This is referred to as transverse feedback linearization. In [10], Nielsen and one of the authors generalized Banaszuk and Hauser's results to more general controlled-invariant sets. In [11], the same authors developed a theory of transverse feedback linearization for multi-input systems. The latter can be viewed as a geometric approach to set stabilization.

The notion of passivity for state space representations of nonlinear systems, pioneered by Willems in the early 1970's, [12, 13], was instrumental for much research on nonlinear equilibrium stabilization. Key contributions in this area were made in the early 1980's by Hill and Moylan in [14, 15, 16, 17], and later by Byrnes, Isidori, and Willems, in their landmark paper [18]. More recently, in a number of papers [19, 20, 21], Shiriaev and Fradkov addressed the problem of stabilizing compact invariant sets for passive nonlinear systems. Their work is a direct extension of the equilibrium stabilization results by Byrnes, Isidori, and Willems in [18].

The passivity paradigm is particularly successful for stabilization because it provides a useful interpretation of the control design process in terms of energy exchange, a view which makes the control design more intuitive, and allows one to naturally handle interconnections of dynamical systems. This view is at the centre of much research on stabilization of Euler-Lagrange control systems and, more generally, port-Hamiltonian systems; we refer the reader to the books by Ortega *et al.* [22], A.J. van der Schaft [23], and the paper [24].

In this paper we develop a theory of set stabilization for passive systems which generalizes the equilibrium theory of [18], as well as the results in [19, 20, 21]. We investigate the stabilization of a closed set Γ, not necessarily compact, which is open-loop invariant and contained in the zero level set of the storage function. Our results answer this question: *when is it that a passivity-based controller makes Γ stable, attractive, or asymptotically stable for the closed-loop system?* Even in the special case when Γ is an equilibrium, our theory yields novel results, among them necessary and sufficient conditions for the passivity-based asymptotic stabilization of the equilibrium in question without imposing that the storage function be positive definite. The theory in [18], and [19, 20, 21] does not handle this situation.

At the heart of the solution of the set stabilization problem lies the following *reduction problem* for a dynamical system $\Sigma : \dot{x} = f(x)$: *Consider two closed sets Γ and $\mathcal{O}$, with $\Gamma \subset \mathcal{O}$, which are invariant for Σ; suppose that Γ is stable, attractive, or asymptotically stable for the restriction of Σ to $\mathcal{O}$. When is it that Γ is stable, attractive, or asymptotically stable with respect to the whole state space?*

The above reduction problem, originally formulated by Seibert and Florio in 1969-1970 [25], [26], is fundamentally important in control theory, as it often arises whenever one wants to infer stability properties of a control system based on its properties on a subset of the state space. The investigation of the stability of cascade-connected systems (see [27, Theorem 3.1], [28, Corollary 5.2], [29, Theorem 10.3.1, Corollaries 10.3.2, 10.3.3]) and the development of a separation principle in output feedback control are examples of situations where one faces the same kind of question. Seibert and Florio in [30] investigated the general setting of dynamical systems on metric spaces, and found reduction principles for stability and asymptotic stability of *compact* sets. The second main contribution of this paper is the development of a new reduction principle for attractivity, and the extension, in the finite-dimensional setting, of Seibert and Florio's reduction principles to the case of closed, but not necessarily compact, sets.

The paper is organized as follows. In Section 2 we present stability definitions and review basic notions used in this paper, in particular the concept of prolongational limit set. We then present the stabilization and reduction problems solved in this paper. In Section 3 we present novel reduction principles for attractivity and asymptotic stability (Theorems 3.1 and 3.2) of closed sets. As a corollary of these results, we give a stability criterion for cascade-connected systems which generalizes well-known results in the literature (Corollary 3.3). In Section 4 we review previous results on passivity-based stabilization and apply our reduction principles to the set stabilization problem for passive systems, giving conditions for asymptotic stabilization (Theorem 4.14). Our result relies on a notion of "Γ-detectability" which encompasses detectability notions used in [18] and [19, 20, 21]. The relationships between these notions are explored in Lemmas 4.9 and 4.10. In Proposition 4.12 we present sufficient conditions for Γ-detectability. Finally, in the appendix we present the proofs of several technical results.

2 Preliminaries and Problem Statement

In this paper we consider control-affine systems described by

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{aligned} \tag{1}$$

with state space $\mathcal{X} \subset \mathbb{R}^n$, set of input values $\mathcal{U} \subset \mathbb{R}^m$ and set of output values $\mathcal{Y} \subset \mathbb{R}^m$. The set $\mathcal{X}$ is assumed to be either an open subset or a smooth submanifold of $\mathbb{R}^n$. We assume that f and g_i, $i = 1, \ldots m$, are smooth vector fields on $\mathcal{X}$, and that $h : \mathcal{X} \to \mathcal{Y}$ is a smooth mapping.

2.1 Notation

Let $\mathbb{R}^+$ denote the positive real line $[0, +\infty)$. Given either a smooth feedback $u(x)$ or a piecewise-continuous open-loop control $u(t) : \mathbb{R}^+ \to \mathcal{U}$, we denote by $\phi_u(t, x_0)$ the unique solution of (1) with initial condition x_0. By $\phi(t, x_0)$ we denote the solution of the open-loop system $\dot{x} = f(x)$ with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, we denote by $\phi_u(I, S)$ the set $\phi_u(I, S) := \{\phi_u(t, x_0) : t \in I, x_0 \in S\}$. The set $\phi(I, S)$ is defined analogously.

Given a closed nonempty set $S \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$, and a vector norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, the point-to-set distance $\|x\|_S$ is defined as $\|x\|_S := \inf\{\|x - y\| : y \in S\}$. Given two subsets S_1 and S_2 of $\mathcal{X}$, the maximum distance of S_1 to S_2, $d(S_1, S_2)$, is defined as $d(S_1, S_2) := \sup\{\|x\|_{S_2} : x \in S_1\}$. The state space $\mathcal{X}$, being a subset of $\mathbb{R}^n$, inherits a norm from $\mathbb{R}^n$, which we will denote $\|\cdot\| : \mathcal{X} \to \mathbb{R}$. For a constant $\alpha > 0$, a point $x \in \mathcal{X}$, and a set $S \subset \mathcal{X}$, define the open sets $B_\alpha(x) = \{y \in \mathcal{X} : \|y - x\| < \alpha\}$ and $B_\alpha(S) = \{y \in \mathcal{X} : \|y\|_S < \alpha\}$. We denote by $\mathrm{cl}(S)$ the closure of the set S, and by $\mathcal{N}(S)$ a generic open neighbourhood of S, that is, an open subset of $\mathcal{X}$ containing S. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f on $\mathcal{X}$, and $dV(x)$ to denote the differential map of V. We denote by $\mathrm{ad}_f\, g$ the Lie bracket of two vector fields f and g on $\mathcal{X}$, and by $\mathrm{ad}_f^k\, g$ its k-th iteration.

2.2 Passivity

Throughout this paper it is assumed that (1) is passive with smooth nonnegative storage function $V : \mathcal{X} \to \mathbb{R}$, i.e., V is a C^r ($r \geq 1$) nonnegative function such that, for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of

$\phi_u(\cdot, x_0)$,

$$V(\phi_u(t, x_0)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau,$$

where $y(t) = h(\phi_u(t, x_0))$. It is well-known (see [14]) that the passivity property above is equivalent to the two conditions

$$(\forall x \in \mathcal{X}) \ L_f V(x) \leq 0 \text{ and } \ L_g V(x) = h(x)^\top, \tag{2}$$

where $L_g V$ denotes the row vector $[L_{g_1} V \ \cdots \ L_{g_m} V]$. Our main objective is to investigate the stabilization of closed sets by means of *passivity-based feedbacks* of the form

$$u = -\varphi(x), \text{ with } \varphi(\cdot)\Big|_{h(x)=0} = 0, \ h(x)^\top \varphi(x)\Big|_{h(x)\neq 0} > 0, \tag{3}$$

where $\varphi : \mathcal{X} \to \mathcal{U}$ is a smooth function. The class of passivity-based feedbacks in (3) includes that of output feedback controllers $u = -\varphi(h(x))$ commonly used in the literature on passive systems.

2.3 Set stability and attractivity

We now introduce the basic notions of set stability and attractivity used in this paper. All definitions below, except that of a uniform semi-attractor, are standard and can be found in [31]. Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant for a dynamical system

$$\Sigma : \dot{x} = f(x), \quad x \in \mathcal{X}. \tag{4}$$

Definition 2.1 (Set stability and attractivity)**.** (i) Γ is *stable* for Σ if for all $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(ii) Γ is a *semi-attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$.

(iii) Γ is a *global attractor* for Σ if it is a semi-attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.

(iv) Γ is a *uniform semi-attractor* for Σ if for all $x \in \Gamma$, there exists $\lambda > 0$ such that, for all $\varepsilon > 0$, there exists $T > 0$ yielding $\phi([T, +\infty), B_\lambda(x)) \subset B_\varepsilon(\Gamma)$.

(v) Γ is a *[globally] semi-asymptotically stable* for Σ if it is stable and semi-attractive [globally attractive] for Σ.

Remark. If Γ is a compact positively invariant set, then the concepts of stability, semi-attractivity, and semi-asymptotic stability are equivalent to the familar ϵ-δ notions of uniform stability, attractivity, and asymptotic stability found, e.g., in [32, Definition 8.1]. Moreover, in the compact case, the notion of uniform semi-attractivity is equivalent to that of uniform attractivity used, e.g., in [33]. In the non-compact case, however, the notions of asymptotic stability and uniform attractivity are much stronger than semi-asymptotic stability and uniform semi-attractivity. For instance, the domain of attraction of an asymptotically stable set must contain a neighbourhood of Γ of the form $B_\delta(\Gamma)$ (a "tube" of constant radius), while the domain of attraction of a semi-attractor does not have to contain such a neighbourhood, as its "width" may shrink to zero at infinity.

Definition 2.2 (Relative set stability and attractivity)**.** Let $\mathcal{O} \subset \mathcal{X}$ be such that $\mathcal{O} \cap \Gamma \neq \emptyset$. We say that Γ is *stable relative to* $\mathcal{O}$ for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Similarly, one modifies all other notions in Definition 2.1 by restricting initial conditions to lie in $\mathcal{O}$.

Definition 2.3 (Local stability and attractivity near a set)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be positively invariant sets. The set $\mathcal{O}$ is *locally stable near* Γ if for all $x \in \Gamma$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma)$ and all $t > 0$, whenever $\phi([0,t], x_0) \subset B_c(x)$ one has that $\phi([0,t], x_0) \subset B_\varepsilon(\mathcal{O})$. The set $\mathcal{O}$ is *locally (semi-) attractive near* Γ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\phi(t, x_0) \to \mathcal{O}$ at $t \to +\infty$.

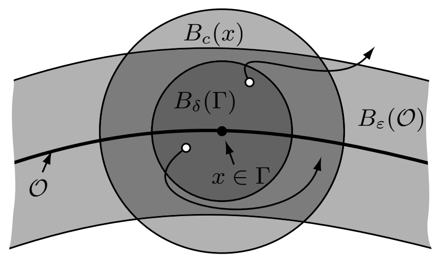

Figure 1: An illustration of the notion of local stability near Γ

The property of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centred at a point x in Γ, trajectories originating in $B_c(x)$ sufficiently close to Γ cannot travel far away from $\mathcal{O}$ before first exiting $B_c(x)$; see Figure 1. It is immediate to see that if Γ is stable, then $\mathcal{O}$ is locally stable near Γ.

Definition 2.4 (Local uniform boundedness)**.** The system Σ is *locally uniformly bounded near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

Remark. If Γ is a stable compact set, then Σ is locally uniformly bounded near Γ. For, the stability of Γ implies the existence of a compact neighbourhood S of Γ which is positively invariant for Σ. Let $\lambda > 0$ and $m > 0$ be such that, for all $x \in \Gamma$, $B_\lambda(x) \subset S \subset B_m(x)$ (λ and m exist by compactness). Then, for any $x \in \Gamma$, the ball $B_\lambda(x)$ is contained in S, and thus by positive invariance, $\phi(\mathbb{R}^+, B_\lambda(x)) \subset S \subset B_m(x)$.

The next lemma, proved in Appendix A, clarifies the relationship between uniform semi-attractivity and semi-asymptotic stability. It is used in Section 4.2 and Appendix D.

Lemma 2.5. Let Γ be a closed set which is positively invariant for Σ in (4), and let $U \supset \Gamma$ be a closed set. If Γ is a uniform semi-attractor [relative to U], then it is semi-asymptotically stable [relative to U]. Furthermore, if Σ is locally uniformly bounded near Γ, then Γ is semi-asymptotically stable [relative to U] if, and only if, it is a uniform semi-attractor [relative to U].

2.4 Limit Sets

In order to characterize the asymptotic properties of bounded solutions, we will use the well-known notion of limit set, due to G. D. Birkhoff (see [34]), and that of prolongational limit set, due to T. Ura (see [35]). Given a smooth feedback $u(x)$ and a point $x_0 \in \mathcal{X}$, the *positive limit set* (or ω-limit set) of the closed-loop solution $\phi_u(t, x_0)$ is defined as

$$L_u^+(x_0) := \{p \in \mathcal{X} : (\exists \{t_n\} \subset \mathbb{R}^+)\, t_n \to +\infty,\, \phi_u(x_0, t_n) \to p\}.$$

The positive limit set of the open-loop solution $\phi(t, x_0)$, defined in an analogous way, is denoted $L^+(x_0)$. The *negative limit sets* (or α-limit sets) $L_u^-(x_0)$ and $L^-(x_0)$ of $\phi_u(t, x_0)$ and $\phi(t, x_0)$, respectively, are defined using time sequences diverging to $-\infty$. We let $L_u^+(S) := \bigcup_{x_0 \in S} L_u^+(x_0)$ and $L^+(S) := \bigcup_{x_0 \in S} L^+(x_0)$.

The *prolongational limit set* $J_u^+(x_0)$ of a closed-loop solution $\phi_u(t, x_0)$ is defined as

$$J_u^+(x_0) := \{p \in \mathcal{X} : (\exists \{(x_n, t_n)\} \subset \mathcal{X} \times \mathbb{R}^+),\, x_n \to x_0,\, t_n \to +\infty,\, \phi_u(x_n, t_n) \to p\}.$$

If $U \subset \mathcal{X}$ and $x_0 \in \mathrm{cl}(U)$, the *prolongational limit set of* $\phi_u(t, x_0)$ *relative to* U is defined as

$$J_u^+(x_0, U) := \{p \in \mathcal{X} : (\exists \{(x_n, t_n)\} \subset U \times \mathbb{R}^+),\, x_n \to x_0,\, t_n \to +\infty,\, \phi_u(x_n, t_n) \to p\}.$$

The corresponding prolongational limit sets of an open-loop solution $\phi(t, x_0)$ are denoted by $J^+(x_0)$ and $J^+(x_0, U)$. We let $J_u^+(S) := \bigcup_{x_0 \in S} J_u^+(x_0)$, and in an analogous manner we define $J^+(S)$, $J_u^+(S, U)$, and $J^+(S, U)$.

Obviously, $L_u^+(x_0) \subset J_u^+(x_0)$ and $L^+(x_0) \subset J^+(x_0)$. Moreover, if $x_0 \in \mathrm{cl}(U)$, then

$$L_u^+(x_0) \subset J_u^+(x_0, U) \subset J_u^+(x_0), \quad L^+(x_0) \subset J^+(x_0, U) \subset J^+(x_0).$$

Proposition 2.6 (Theorem II.4.3 and Lemma V.1.10 in [36])**.** Consider the dynamical system Σ in (4). For any $x \in \mathcal{X}$, $J^+(x)$ is closed and invariant. Moreover, for any $\omega \in L^+(x)$, $J^+(x) \subset J^+(\omega)$.

The results in Proposition 2.6 still hold if one replaces $J^+(x)$ by $J^+(x, U)$, with $U \subset \mathcal{X}$. While $L^+(x_0)$ is used to characterize the asymptotic convergence properties of $\phi(t, x_0)$, $J^+(x_0)$ is used to characterize *uniform* convergence, as shown in the next result, which is used in Section 4.2 to determine sufficient conditions for Γ-detectability.

Proposition 2.7. Suppose that Σ in (4) is locally uniformly bounded near a closed and positively invariant set Γ. Let $U \subset \mathcal{X}$ be a closed set, $\Gamma \subset U$. Then, for each x in some neighbourhood of Γ, $J^+(x) \neq \emptyset$ [$J^+(x, U) \neq \emptyset$]. Moreover, Γ is a uniform semi-attractor [relative to U] for Σ if, and only if, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$ [$J^+(\mathcal{N}(\Gamma), U) \subset \Gamma$].

The proof of sufficiency can be found in Appendix B, while that of necessity is omitted because it is not used in the sequel. An analogous result holds for compact sets without the local uniform boundedness assumption, see Proposition V.1.2 in [36].

2.5 Stabilization and reduction problems

The main objective of this paper is the stabilization of a closed set Γ using passivity-based feedback.

Problem 1 (Set Stabilization)**.** Given a closed set $\Gamma \subset V^{-1}(0) = \{x \in \mathcal{X} : V(x) = 0\}$ which is positively invariant for the open-loop system in (1), and given a passivity-based feedback of the form (3), find conditions guaranteeing that Γ is [globally] semi-asymptotically stable for the closed-loop system.

The rationale behind passivity-based feedback is the following. Using (2) and the properties of the passivity-based feedback (3), the time derivative of the storage function V along trajectories of the closed-loop system formed by (1) with feedback (3) is given by

$$\begin{aligned}\frac{dV(\phi_u(t, x_0))}{dt} &= L_f V(\phi_u(t, x_0)) - L_g V(\phi_u(t, x_0)) \varphi(\phi_u(t, x_0)) \\ &\leq -h(\phi_u(t, x_0))^\top \varphi(\phi_u(t, x_0)) \leq 0.\end{aligned} \tag{5}$$

Thus, a passivity-based feedback renders the storage function V nonincreasing along solutions of the closed-loop system. One expects that if the system enjoys suitable properties, then the storage function should decrease asymptotically to zero and the solutions should approach a subset of $V^{-1}(0)$, hopefully the set Γ.

Our point of departure in understanding what system properties yield the required result is the well-known property, found in the proof of Theorem 3.2 in [18], that, for all $x_0 \in \mathcal{X}$, the positive limit set $L_u^+(x_0)$ of the closed-loop system is invariant for the open-loop system and such that $L_u^+(x_0) \subset h^{-1}(0)$. Let $\mathcal{O}$ denote the *maximal* set contained in $h^{-1}(0)$ which is invariant for the open-loop system. In light of the property above, if $L_u^+(x_0)$ is non-empty, then it must be contained in $\mathcal{O}$. Therefore, all bounded trajectories of the closed-loop system asymptotically approach $\mathcal{O}$. Since $L_f V \leq 0$, V is nonincreasing along solutions of the open-loop system, and so $V^{-1}(0)$ is an invariant set for the open-loop system. Moreover, since V is nonnegative, any point $x \in V^{-1}(0)$ is a local minimum of V and hence $dV(x) = 0$. Therefore, $L_g V(x) = h(x)^\top = 0$ on $V^{-1}(0)$, and so $\Gamma \subset V^{-1}(0) \subset h^{-1}(0)$. Since $V^{-1}(0)$ is invariant and contained in $h^{-1}(0)$, it is necessarily a subset of $\mathcal{O}$ (this implies that $\mathcal{O}$ is not empty). Putting everything together, we conclude that

$$\Gamma \subset V^{-1}(0) \subset \mathcal{O} \subset h^{-1}(0). \tag{6}$$

It is then clear that if the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, the least a passivity-based feedback can guarantee is the semi-attractivity of $\mathcal{O}$; but this is not sufficient for our purposes. Notice that, on $\mathcal{O}$, $\varphi(\cdot) = 0$ and so the closed-loop dynamics on $\mathcal{O}$ coincide with the open-loop dynamics. In particular, then, $\mathcal{O}$ is an invariant set for the closed-loop system. In order to ensure the property of semi-asymptotic stability of Γ, the open-loop system *must* enjoy the same property *relative to* $\mathcal{O}$. Therefore, a necessary condition for Γ to be semi-asymptotically stable for the closed-loop system is that Γ be semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system. *Is this condition also sufficient or are extra-properties needed?* As discussed in the introduction, the question we have just raised is fundamental in control theory, as it often arises whenever one wants to infer stability properties of a control system based on its properties on a subset of the state space. It is then worth formalizing this problem and investigating it in its generality.

Problem 2 (Reduction Problem, [25], [26])**.** Consider the dynamical system Σ in (4). Let Γ and $\mathcal{O}$ be two closed positively invariant sets such that $\Gamma \subset \mathcal{O} \subset \mathcal{X}$. Assume that Γ is, respectively, stable, semi-attractive, and semi-asymptotically stable relative to $\mathcal{O}$. Find what additional conditions are needed to guarantee that Γ is, respectively, stable, semi-attractive, and semi-asymptotically stable for Σ. We also seek to solve the global version of each of the problems above.

Problem 2 was originally formulated by P. Seibert and J.S. Florio in 1969-1970. Seibert and Florio developed reduction principles for stability and asymptotic stability (but not attractivity) for dynamical systems on metric spaces assuming that Γ is compact. Their conditions first appeared in [25] and [26], while the proofs are found in [30] (see also the work in [37] for related results). In the next section, we present novel reduction principles for semi-attractivity and, for the case of unbounded Γ, semi-asymptotic stability (a reduction principle for stability is found in Appendix D). Using these reduction principles, in Section 4 we solve Problem 1.

3 Reduction principles

In this section we focus on the reduction problem, Problem 2, and we put aside until Section 4 the investigation of the original set stabilization problem, Problem 1. Consider the dynamical system

$$\Sigma: \ \dot{x} = f(x), \ x \in \mathcal{X}, \tag{7}$$

with f locally Lipschitz on $\mathcal{X}$, and let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be closed sets which are positively invariant for system Σ. As pointed out earlier, Seibert and Florio developed reduction principles for stability (see Theorem 3.4 in [30]) and asymptotic stability (see Theorem 4.13 and Corollary 4.11 in [30]). Here, we present a novel reduction principle for semi-attractivity (Theorem 3.1) and a reduction principle for semi-asymptotic stability (Theorem 3.2) which extends Seibert and Florio's result to the non-compact case. A consequence of our reduction principles is a novel stability result for cascade-connected systems, presented in Corollary 3.3.

Theorem 3.1 (Reduction principle for semi-attractivity)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ is semi-attractive if the following conditions hold:

(i) Γ is semi-asymptotically stable relative to $\mathcal{O}$

(ii) $\mathcal{O}$ is locally semi-attractive near Γ,

(iii) there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all initial conditions in $\mathcal{N}(\Gamma)$, the associated solutions are bounded and such that the set $\mathrm{cl}(\phi(\mathbb{R}^+, \mathcal{N}(\Gamma))) \cap \mathcal{O}$ is contained in the domain of attraction of Γ relative to $\mathcal{O}$.

The set Γ is globally attractive if:

(i)' Γ is globally semi-asymptotically stable relative to $\mathcal{O}$,

(ii)' $\mathcal{O}$ is a global attractor,

(iii)' all trajectories in $\mathcal{X}$ are bounded.

Conditions (ii) and (ii') are also necessary. The proof of this theorem is found in Appendix C. Part of the proof was inspired by the stability results using positive semidefinite Lyapunov functions presented in [38] and by the proof of Lemma 1 in [39].

Remark. Being of a rather technical nature, Assumption (iii) is difficult to check and of limited practical use. It has, however, theoretical significance because it is used to prove the reduction principle for semi-asymptotic stability stated in the sequel. A similar, but slightly stronger, assumption is found in Theorem 10.3.1 in [29] concerning the attractivity of equilibria of cascade-connected systems. In fact, the result in [29] is a corollary of Theorem 3.1. If condition (i) is replaced by the stronger (i)', then one can replace (iii) by the simpler requirement that trajectories in some neighbourhood of Γ be bounded.

It is interesting to note that it is not enough to assume, in place of condition (i), that Γ is a semi-attractor relative to $\mathcal{O}$ (or, in place of condition (i)', that Γ is a global attractor relative to $\mathcal{O}$), as the next example shows.

Example. Consider the following system

$$\begin{aligned}\dot{x}_1 &= (x_2^2 + x_3^2)(-x_2)\\ \dot{x}_2 &= (x_2^2 + x_3^2)(x_1)\\ \dot{x}_3 &= -x_3^3.\end{aligned}$$

Let $\Gamma = \{(x_1, x_2, x_3) : x_2 = x_3 = 0\}$ and $\mathcal{O} = \{(x_1, x_2, x_3) : x_3 = 0\}$, both invariant sets. Obviously, $\mathcal{O}$ is global attractor (in fact, it is globally asymptotically stable). The system dynamics on $\mathcal{O}$ take the form

$$\begin{aligned}\dot{x}_1 &= -x_2(x_2^2)\\ \dot{x}_2 &= x_1(x_2^2).\end{aligned}$$

On $\Gamma \subset \mathcal{O}$, every point is an equilibrium. Phase curves on $\mathcal{O}$ off of Γ are concentric semicircles $\{x_1^2+x_2^2 = c\}$, and therefore Γ is a global, but unstable, attractor relative to $\mathcal{O}$. For initial conditions x_0 not in $\mathcal{O}$, the trajectories are bounded and their positive limit set $L^+(x_0)$ is a circle inside $\mathcal{O}$ which intersects Γ at equilibrium points. Thus, $\phi(t, x_0) \not\to \Gamma$ because $L^+(x_0) \not\subset \Gamma$. This is illustrated in Figure 2. This example shows that if all assumptions in Theorem 3.1 are satisfied except the relative stability requirement in condition (i), then the attractivity of Γ cannot be guaranteed.

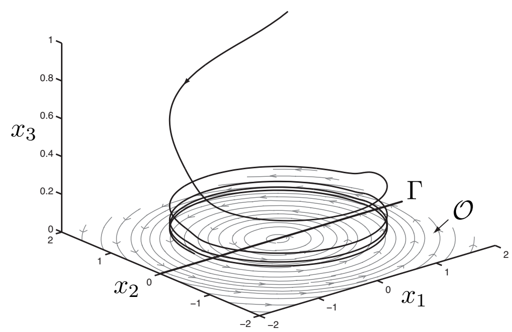

Figure 2: Γ is globally semi-attractive rel. to $\mathcal{O}$, $\mathcal{O}$ is globally asymptotically stable, and yet Γ is not semi-attractive.

The next semi-asymptotic stability result relies, in part, on the reduction principle in Theorem 3.1.

Theorem 3.2 (Reduction principle for semi-asymptotic stability)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ is [globally] semi-asymptotically stable if the following conditions hold:

(i) Γ is [globally] semi-asymptotically stable relative to $\mathcal{O}$,

(ii) $\mathcal{O}$ is locally stable near Γ,

(iii) $\mathcal{O}$ is locally semi-attractive near Γ [$\mathcal{O}$ is globally attractive],

(iv) if Γ is unbounded, then Σ is locally uniformly bounded near Γ,

(v) [all trajectories of Σ are bounded.]

Conditions (i), (ii), and (iii) above are necessary. If Γ is a compact set, then the above theorem is equivalent to the results presented in Theorem 4.13 and Corollary 4.11 in [30].

Proof. We prove the theorem assuming that Γ is unbounded, since the compact case is already covered by Theorem 4.13 and Corollary 4.11 in [30]. In Appendix D, we show that assumptions (i), (ii), (iv) imply that Γ is stable. By Theorem 3.1, the global version (i.e., including statements in square brackets) of assumptions (i), (iii), and (v) imply global attractivity, and hence global semi-asymptotic stability. To prove that the local version of the assumptions imply semi-asymptotic stability of Γ, we need to show that assumption (iii) in Theorem 3.1 is satisfied.

By assumption (i), Γ is semi-attractive relative to $\mathcal{O}$. Let $N \subset \mathcal{O}$ denote the domain of attraction of Γ relative to $\mathcal{O}$. By assumption (iv), for each $x \in \Gamma$ there exist two positive numbers $\lambda(x)$ and $m(x)$ such that $\phi(\mathbb{R}^+, B_{\lambda(x)}(x)) \subset B_{m(x)}(x)$. Fix $x \in \Gamma$, and let $\varepsilon(x) > 0$ be small enough that

$$\mathrm{cl}\left(B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x)\right) \cap \mathcal{O} \subset N.$$

The constant ε is guaranteed to exist because the set on left-hand side of the inclusion is compact and can be made arbitrarily small. Since Γ is stable, there exists a neighbourhood $\mathcal{N}_x(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}_x(\Gamma)) \subset B_{\varepsilon(x)}(\Gamma)$. Now define

$$U = \bigcup_{x \in \Gamma} B_{\lambda(x)}(x) \cap \mathcal{N}_x(\Gamma).$$

Clearly, U is a neighbourhood of Γ. By definition, for each $y \in U$, there exists $x \in \Gamma$ such that $y \in B_{\lambda(x)}(x) \cap \mathcal{N}_x(\Gamma)$, so that the solution originating in y is bounded and

$$\phi(\mathbb{R}^+, y) \subset B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x).$$

Therefore, $\mathrm{cl}(\phi(\mathbb{R}^+, y)) \cap \mathcal{O} \subset \mathrm{cl}\left(B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x)\right) \cap \mathcal{O} \subset N$. □

As remarked earlier, the usefulness of reduction principles is not limited to the stabilization of closed sets for passive systems. As a matter of fact, stability theorems for cascade-connected systems of the form

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(y), \end{aligned} \tag{8}$$

well-known in the control literature (see [27, Theorem 3.1], [28, Corollary 5.2], [29, Corollaries 10.3.2, 10.3.3]), are consequences of Seibert and Florio's reduction theory, specialized to the case when Γ is the origin and $\mathcal{O} = \{(x, y) : y = 0\}$. Motivated by this observation, we present a straightforward application of Theorem 3.2 which has independent interest.

Corollary 3.3. Consider system (8), with $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and let $\Gamma \subset \mathbb{R}^{n_1}$ be a positively invariant set for system $\dot{x} = f(x, 0)$. Suppose that $g(0) = 0$. Then, $\tilde{\Gamma} := \{(x, y) : x \in \Gamma,\ y = 0\}$ is [globally] semi-asymptotically stable for (8) if the following conditions hold:

(i) Γ is [globally] semi-asymptotically stable for $\dot{x} = f(x, 0)$,

(ii) $y = 0$ is a [globally] asymptotically stable equilibrium of $\dot{y} = g(y)$,

(iii) if Γ is unbounded, then (8) is locally uniformly bounded near $\tilde{\Gamma}$,

(iv) [all trajectories of (8) are bounded.]

4 Passivity-based set stabilization

We now return to the passive control system (1) and apply the reduction principles presented in Section 3 to solve Problem 1. First, we review the main stabilization results for passive systems available in the literature.

4.1 Previous results

When the storage function V is positive definite, and $\Gamma = V^{-1}(0) = \{0\}$ is an equilibrium, the most general stabilization result is that by Byrnes, Isidori, and Willems in [18]; it relies on the following notion of detectability[1].

[1] The zero-state detectability definition in [18] uses the slightly *stronger* condition $h(\phi(t, x_0)) = 0$ for all $t \geq 0 \implies \phi(t, x_0) \to 0$. Our relaxation of the definition has no effects on any of the results in [18].

Definition 4.1 (Zero-state detectability)**.** System (1) is *locally zero-state detectable* if there exists a neighbourhood U of 0 such that, for all $x_0 \in U$,

$$h(\phi(t, x_0)) = 0 \text{ for all } t \in \mathbb{R} \implies \phi(t, x_0) \to 0 \text{ as } t \to +\infty.$$

If $U = \mathcal{X}$, the system is *zero-state detectable.*

Note that the definition above involves open-loop trajectories. The work in [18] provides sufficient conditions for detectability. Assuming that V is C^r, $r \geq 1$, define the distribution

$$\mathcal{D} = \operatorname{span}\{\operatorname{ad}_f^k g_i : 0 \leq k \leq n-1, 1 \leq i \leq m\}, \tag{9}$$

and the set

$$S = \{x \in \mathcal{X} : L_f^j L_\tau V(x) = 0, \text{ for all } \tau \in \mathcal{D}, \text{ and all } 0 \leq j < r\}. \tag{10}$$

Proposition 4.2 (Proposition 3.4 in [18])**.** If $S \cap L^+(\mathcal{X}) = \{0\}$ and V is proper and positive definite, then system (1) is zero-state detectable.

Theorem 4.3 (Theorem 3.2 in [18])**.** Suppose that the storage function V is positive definite and (1) is locally zero-state detectable. Then any passivity-based feedback of the form (3) asymptotically stabilizes the equilibrium $x = 0$. Moreover, if V is proper (i.e., all its sublevel sets are compact) and (1) is zero-state detectable, then the passivity-based feedback globally asymptotically stabilizes $x = 0$.

In a series of papers, [19, 20, 21], Shiriaev and Fradkov extended Theorem 4.3 to the case when Γ is compact and $\Gamma = V^{-1}(0)$, relying on the following notion of detectability.

Definition 4.4 (V-detectability)**.** System (1) is *locally V-detectable* if there exists a constant $c > 0$ such that for all $x_0 \in V^{-1}([0, c])$,

$$h(\phi(t, x_0) = 0 \text{ for all } t \in \mathbb{R} \implies V(\phi(t, x_0)) \to 0 \text{ as } t \to +\infty.$$

If $c = \infty$, the system is *V-detectable.*

Proposition 4.5 (Theorem 10 in [20])**.** If $S \cap L^+(\mathcal{X}) \subset V^{-1}(0)$ and V is proper and positive semi-definite, then system (1) is V-detectable.

Theorem 4.6 (Theorem 2.3 in [21])**.** Suppose that $V^{-1}(0)$ is a compact set, and (1) is locally V-detectable. Then, any passivity-based feedback of the form (3) asymptotically stabilizes $V^{-1}(0)$. Moreover, if V is proper and (1) is V-detectable, then the passivity-based feedback globally asymptotically stabilizes $V^{-1}(0)$.

4.2 Γ-detectability and its characterization

We now turn our attention to Problem 1. For convenience, we repeat the definition of the set $\mathcal{O}$ given in Section 2.5.

Definition 4.7 (Set $\mathcal{O}$)**.** Given the control system (1), we denote by $\mathcal{O}$ the maximal set contained in $h^{-1}(0)$ which is invariant for the open-loop system $\dot{x} = f(x)$.

When system (1) is linear time-invariant (LTI), the set $\mathcal{O}$ is the unobservable subspace. As discussed in Section 2.5, as long as the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, a passivity-based feedback renders the set $\mathcal{O}$ semi-attractive. In order to guarantee semi-asymptotic stability of $\Gamma \subset \mathcal{O}$, the reduction principle in Theorem 3.2 suggests that Γ should be semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system. We call this property Γ-detectability.

Definition 4.8 (Γ-detectability)**.** System (1) is *locally* Γ*-detectable* if Γ is semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system. The system is Γ*-detectable* if Γ is globally semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system.

Our notion of detectability is parameterized by Γ, and not by $\mathcal{O}$, although the set $\mathcal{O}$ figures in its definition. This is due to the fact that $\mathcal{O}$ is entirely determined by the open-loop vector field f and the output function h and, therefore, $\mathcal{O}$ is intrinsically related to the open-loop system. In the case of LTI systems, when $\Gamma = \{0\}$, the above definition requires that all tracjectories on the unobservable subspace $\mathcal{O}$ converge to 0. Therefore, in the LTI setting, Γ-detectability coincides with the classical notion of detectability. Further, the notion of Γ-detectability generalizes that of zero-state detectability. As a matter of fact, when V is positive definite, and thus $\Gamma = \{0\}$, the two detectability notions coincide.

Lemma 4.9. If V is positive definite and $\Gamma = V^{-1}(0) = \{0\}$, then the following three conditions are equivalent:

(a) System (1) is locally zero-state detectable [zero-state detectable],

(b) the equilibrium $x = 0$ is [globally] attractive relative to $\mathcal{O}$ for the open-loop system,

(c) system (1) is locally Γ-detectable [Γ-detectable].

Proof. The set of points $x_0 \in \mathcal{X}$ such that the open-loop solution satisfies $h(\phi(t, x_0)) \equiv 0$ is precisely the maximal open-loop invariant subset of $h^{-1}(0)$, i.e., the set $\mathcal{O}$. Thus, conditions (a) and (b) are equivalent. Since (1) is passive, by (2) we have $L_f V \leq 0$. By the assumption that V is positive definite, it follows that $x = 0$ is a stable equilibrium of the open-loop system. Thus, $x = 0$ is [globally] asymptotically stable relative to $\mathcal{O}$ for the open-loop system if and only if $x = 0$ is [globally] attractive relative to $\mathcal{O}$ for the open-loop system, proving that conditions (b) and (c) are equivalent. □

The next lemma shows that Γ-detectability also encompasses the notion of V-detectability.

Lemma 4.10. If $\Gamma = V^{-1}(0)$ is a compact set, then the following three conditions are equivalent:

(a) System (1) is locally V-detectable,

(b) the set Γ is attractive relative to $\mathcal{O}$ for the open-loop system,

(c) system (1) is locally Γ-detectable.

Moreover, if V is proper, then the global versions of conditions (a)-(c) are equivalent.

Proof. Suppose that (1) is locally V-detectable. Then, for all $x_0 \in V^{-1}([0, c]) \cap \mathcal{O}$, $V(x(t)) \to 0$. Since $V^{-1}(0)$ is compact, in a sufficiently small neighbourhood of Γ, $V^{-1}(\phi(t, x_0)) \to 0$ implies $\phi(t, x_0) \to V^{-1}(0)$, and thus $\Gamma = V^{-1}(0)$ is attractive relative to $\mathcal{O}$ for the open-loop system, showing that condition (a) implies (b). Since $L_f V \leq 0$, Γ is also stable for the open-loop system. Thus, condition (b) implies (c). Now suppose that (1) is locally Γ-detectable. Then, there exists a neighbourhood S of Γ such that, for all $x_0 \in S \cap \mathcal{O}$, $\phi(t, x_0) \to \Gamma$. Since $\Gamma = V^{-1}(0)$ is compact and V is continuous, there exists $c > 0$ such that $V^{-1}([0, c]) \subset S$. Hence, for all $x_0 \in V^{-1}([0, c]) \cap \mathcal{O}$ or, equivalently for all $x_0 \in V^{-1}([0, c])$ such that $h(\phi(t, x_0)) \equiv 0$, we have $\phi(t, x_0) \to V^{-1}(0)$. By the continuity of V and the compactness of $V^{-1}(0)$ the latter fact implies that $V(\phi(t, x_0)) \to 0$. This proves that condition (c) implies (a).

The proof of equivalence of the global notions of detectability follows directly from the fact that if V is proper, then $V(\phi(t, x_0)) \to 0$ if and only if $\phi(t, x_0) \to V^{-1}(0)$. □

Despite their equivalence when $\Gamma = V^{-1}(0)$ is compact, the two notions of Γ- and V-detectability have a different flavor, in that the latter notion utilizes the storage function $V(x)$ to define a property of the open-loop system, detectability, which is independent of V. On the other hand, the definition of Γ-detectability, being independent of V, is closer in spirit to the original definition of zero-state detectability. Finally, the notion of V-detectability cannot be generalized to the case when Γ is unbounded, even if $\Gamma = V^{-1}(0)$, because in this case $V(\phi(t, x_0)) \to 0$ no longer implies $\phi(t, x_0) \to V^{-1}(0)$.

We now give sufficient conditions for (1) to be Γ-detectable that extend the results in Propositions 4.2 and 4.5. Recall the definition of the set S in (9)-(10) and let

$$S' = \{x \in \mathcal{X} : L_f^m h(x) = 0, 0 \leq m \leq r + n - 2\}.$$

Notice that the definition of S', unlike that of S, does not directly involve the storage function (but recall that $h^\top = L_g V$, so it does indirectly depend on V). The next result clarifies the relationship between S and S'.

Lemma 4.11. Given any subset $X \subset \mathcal{X}$, $S' \cap L^+(X) = S \cap L^+(X)$.

This result is interesting in its own right because it implies that the conditions in Propositions 4.2 and 4.5 can be equivalently stated as $S' \cap L^+(\mathcal{X}) \subset \Gamma$. This condition can be checked without directly knowing the storage function.

Proof. We show that $(S' \cap L^+(X)) \subset (S \cap L^+(X))$. Let $\bar{x}$ be an arbitrary point in $S' \cap L^+(X)$. Since $\bar{x}$ is a positive limit point of an open-loop trajectory of (1), and since $L_f V \leq 0$, then $V(\phi(t, \bar{x}))$ is constant and hence

$$\frac{dV(\phi(t, \bar{x}))}{dt} = L_f V(\phi(t, \bar{x})) \equiv 0.$$

The identity $L_f V(\phi(t, \bar{x})) \equiv 0$ implies that $L_f V(\phi(t, \bar{x}))$ is maximal. Therefore, $dL_f V(\phi(t, \bar{x})) \equiv 0$, yielding $L_{g_i} L_f V(\phi(t, \bar{x})) \equiv 0$. This and the fact that $\bar{x} \in S'$ give

$$\begin{aligned} L_{[f,g_i]} V(\bar{x}) &= L_f L_{g_i} V(\bar{x}) - L_{g_i} L_f V(\bar{x}) \\ &= L_f L_{g_i} V(\bar{x}) = L_f h(\bar{x}) = 0. \end{aligned}$$

Next, notice that since $L_{g_i} L_f V(\phi(t, \bar{x})) \equiv 0$, we have

$$0 \equiv \frac{d^m}{dt^m} L_{g_i} L_f V(\phi(t, \bar{x})) = L_f^m L_{g_i} L_f V(\phi(t, \bar{x})), \ 0 \leq m < r.$$

Thus, for $0 \leq m < r$,

$$\begin{aligned} L_f^m L_{[f,g_i]} V(\bar{x}) &= L_f^{m+1} L_{g_i} V(\bar{x}) - L_f^m L_{g_i} L_f V(\bar{x}) \\ &= L_f^{m+1} h_i(\bar{x}) = 0. \end{aligned}$$

A simple extension of this argument leads to

$$L_f^m L_\tau V(\bar{x}) = 0, \text{ for all } \tau \in D, \ 0 \leq m < r,$$

and thus $\bar{x} \in S \cap L^+(X)$. The proof that $S \cap L^+(X) \subset S' \cap L^+(X)$ is almost identical and is therefore omitted. $\square$

Proposition 4.12. Suppose that all open-loop trajectories that originate and remain on S' are bounded and that the open-loop system in (1) is locally uniformly bounded near Γ. If

$$S' \cap J^+(S', S') \subset \Gamma, \tag{11}$$

then system (1) is Γ-detectable. Moreover, if $\Gamma = V^{-1}(0)$, then condition (11) may be replaced by the following one:

$$S' \cap L^+(S') \subset V^{-1}(0). \tag{12}$$

Proof. In order to show that Γ is globally semi-asymptotically stable relative to $\mathcal{O}$, it is sufficient to show that $J^+(\mathcal{O}, \mathcal{O}) \subset \Gamma$. For, $L^+(\mathcal{O}) \subset J^+(\mathcal{O}, \mathcal{O}) \subset \Gamma$ implies that Γ is a global attractor. Moreover, $J^+(\mathcal{O}, \mathcal{O}) \subset \Gamma$ implies, by Proposition 2.7, that Γ is a uniform semi-attractor relative to $\mathcal{O}$ and so, by Lemma 2.5, it is stable.

Since $\mathcal{O} \subset h^{-1}(0)$ is open-loop invariant, we have $h(\phi(t, x)) \equiv 0$ for all $x \in \mathcal{O}$, and thus also $L_f^m h(\phi(t, x)) \equiv 0$, for $m = 0, 1, \ldots$, showing that $\mathcal{O} \subset S'$.

It can be shown (see the proof of Proposition 3.4 in [18], which is Proposition 4.2 in this paper) that $L^+(\mathcal{O}) \subset S$, and so $L^+(\mathcal{O}) \subset S \cap L^+(S')$. By Lemma 4.11, $L^+(\mathcal{O}) \subset S \cap L^+(S') = S' \cap L^+(S')$. Using condition (11) or, when $\Gamma = V^{-1}(0)$, condition (12), we obtain

$$L^+(\mathcal{O}) \subset S' \cap L^+(S') \subset \Gamma.$$

Since all open-loop trajectories contained in S', and hence in $\mathcal{O}$, are bounded, the above inclusion implies that Γ is a global attractor relative to $\mathcal{O}$. Let $p \in \mathcal{O}$ be arbitrary. We next show that $J^+(p, \mathcal{O})$ is compact. Let $\omega \in L^+(p) \subset \Gamma$. By local uniform boundedness of the open-loop system near Γ, there exist two positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(\omega)) \subset B_m(\omega)$. By definition of prolongational limit set, for any $\delta > 0$, $J^+(\omega, \mathcal{O}) \subset \mathrm{cl}\,(\phi(\mathbb{R}^+, B_\delta(\omega)))$. Taking $\delta = \lambda$, we have that $J^+(\omega, \mathcal{O}) \subset \mathrm{cl}(B_m(\omega))$. Thus, $J^+(\omega, \mathcal{O})$ is a compact set. By Proposition 2.6, $J^+(p, \mathcal{O}) \subset J^+(\omega, \mathcal{O})$, and so $J^+(p, \mathcal{O})$ is a compact set as well.

We claim that, for all $p \in \mathcal{O}$, $J^+(p, \mathcal{O}) \subset V^{-1}(0)$. Suppose that the claim is false. Then, by the compactness of $J^+(p, \mathcal{O})$, there exists $y \in J^+(p, \mathcal{O})$ such that $V(y) > 0$. Put $\mu = V(y)$. Since $y \in J^+(p, \mathcal{O})$, there exist two sequences $\{x_k\} \subset \mathcal{O}$ and $\{t_k\}$ such that $x_k \to p$, $t_k \to +\infty$, and $\phi(t_k, x_k) \to y$. By the continuity of V, one can find $K > 0$ such that, for all $k > K$, $V(\phi(t_k, x_k)) > 3\mu/4$. Since $p \in \mathcal{O}$ and Γ is a global attractor relative to $\mathcal{O}$, $\phi(t, p) \to \Gamma \subset V^{-1}(0)$. Since all solutions on $\mathcal{O}$ are bounded and V is continuous, $V(\phi(t, p)) \to 0$ and hence there exists $T > 0$ such that, for all $t \geq T$, $V(\phi(t, p)) < \mu/4$. Using again the continuity of V, there exists $\varepsilon > 0$ such that, for all $x \in B_\varepsilon(\phi(T, p))$, $V(x) < \mu/2$. Now, by continuous dependence on initial conditions, there exists $\delta > 0$ such that, for all $x \in B_\delta(p)$, $\|\phi(t, x) - \phi(t, p)\| < \varepsilon$ for all $t \in [0, T]$. Since, for sufficiently large $k > K$, $x_k \in B_\delta(p)$ and $t_k > T$, we have $V(\phi(t_k, x_k)) > 3\mu/4 > \mu/2 > V(\phi(T, x_k))$ which contradicts the fact that $L_f V \leq 0$, proving the claim.

So far we have established that $J^+(\mathcal{O}, \mathcal{O}) \subset V^{-1}(0)$. If $\Gamma = V^{-1}(0)$, we are done. If $\Gamma \subsetneq V^{-1}(0)$, we reach the desired conclusion by means of condition (11) as follows. Note that $J^+(\mathcal{O}, \mathcal{O}) \subset V^{-1}(0) \subset \mathcal{O} \subset S'$, and, further, $J^+(\mathcal{O}, \mathcal{O}) \subset J^+(S', S')$. In conclusion, $J^+(\mathcal{O}, \mathcal{O}) \subset S' \cap J^+(S', S') \subset \Gamma$, as required. ☐

Remark. The natural way to check Γ-detectability is to compute the set $\mathcal{O}$ in Definition 4.7, and then assess the semi-asymptotic stability of Γ relative to $\mathcal{O}$. Should the computation of the set $\mathcal{O}$ be too difficult, Proposition 4.12 above provides an alternative, but conservative, criterion for Γ-detectability that may prove useful in some cases. The example below illustrates this result. It is important to notice that condition (11) may be hard to check in practice because it involves the computation of the prolongational limit set $J^+(S', S')$. The conditions used in Propositions 4.2 and 4.5 suffer from the same limitation because they too involve the computation of limit sets.

Propositions 4.2 and 4.5 are corollaries of Proposition 4.12 above. As a matter of fact, Proposition 4.12 relaxes the sufficient conditions for detectability found in [18] and [21]. To see this fact, note that, when V is proper and $\Gamma = V^{-1}(0)$, *all* trajectories of the open-loop system are bounded and system (1) is locally uniformly bounded with respect to Γ. Therefore, in this setting Proposition 4.12 states that a sufficient condition for Γ-detectability is the inclusion $S' \cap L^+(S') \subset V^{-1}(0)$. Since $S' \cap L^+(S') = S \cap L^+(S') \subset S \cap L^+(\mathcal{X})$, this condition is weaker than the condition $S \cap L^+(\mathcal{X}) \subset V^{-1}(0)$ used in Propositions 4.2 and 4.5.

Example. Consider the control system on $\mathcal{X} = \mathbb{R}^5$

$$
\begin{aligned}
\dot{x}_1 &= -x_1 - x_1 x_4 \\
\dot{x}_2 &= -x_2 + x_1 - x_4^2 \\
\dot{x}_3 &= x_5^2 + u_1 \\
\dot{x}_4 &= x_1^2 + e^{-1/x_4^2} u_2 \\
\dot{x}_5 &= -x_3 x_5
\end{aligned}
$$

(we set $e^{-1/x_4^2}|_{x_4=0} := 0$) with output $y = \mathrm{col}\left(x_3, x_4 e^{-1/x_4^2}\right)$. This system is passive with storage $V(x) = 1/2(x_1^2 + x_3^2 + x_4^2 + x_5^2)$. The goal set is $\Gamma = \{0\}$. It is not hard to see that $S' = \{x : x_3 = x_4 = x_5 = 0\}$. Let $(x_1(t), x_2(t), 0, 0, 0)$ be any solution of the open-loop system lying in S' for all time. Since $\dot{x}_1(t) = -x_1(t)$ and $\dot{x}_2(t) = -x_2(t) + x_1(t)$, any such solution is bounded. Next, we check condition (11). Pick any $x_0 \in S'$, i.e., $x_0 = (x_{10}, x_{20}, 0, 0, 0)$, and consider the corresponding open-loop solution $x(t)$. If $x_{10} \neq 0$, then $x_4(t) \to \infty$, and so $J^+(x_0, S') = \emptyset$. On the other hand, if $x_{10} = 0$, then we claim that $J^+(x_0, S') = \{0\}$. For, the equilibrium $x = 0$ is globally asymptotically stable relative to the set $\{x_1 = x_3 = x_4 = x_5 = 0\}$, and hence a uniform attractor relative to the same set. By Proposition 2.7, then, $J^+(x_0, S') = \{0\}$. In conclusion, $S' \cap J^+(S', S') = \{0\}$, and the system is Γ-detectable.

In this example, Γ-detectability can be checked without using Proposition 4.12, since it is easily seen that the maximal open-loop invariant subset of $h^{-1}(0)$ is $\mathcal{O} = \{x_1 = x_3 = x_4 = x_5 = 0\}$. As noted above, $\{0\}$ is globally asymptotically stable relative to this set.

4.3 Solution to Problem 1

In this section we solve the set stabilization problem, Problem 1, by presenting conditions that guarantee that a passivity-based controller of the form (3) makes Γ stable, attractive, or semi-asymptotically stable for the closed-loop system. All results are straightforward consequences of the reduction principles presented in Section 3, and they rely on the next fundamental observation.

Proposition 4.13. Consider the passive system (1) with a passivity-based feedback of the form (3), and the set $\mathcal{O}$ in Definition 4.7. Then, the set $\mathcal{O}$ is locally stable near Γ for the closed-loop system.

Proof. Given arbitrary x in Γ and $c > 0$, we need to show that

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ s.t. } (\forall x_0 \in B_\delta(\Gamma))(\forall t \geq 0)\ \phi_u([0,t], x_0) \subset B_c(x) \implies \phi_u([0,t], x_0) \subset B_\varepsilon(\mathcal{O}).$$

Let $U = \mathrm{cl}(B_c(x))$ and pick any $\varepsilon > 0$. Define

$$v := \min\{V(x) : x \in U \cap \{x : \|x\|_{V^{-1}(0)} = \varepsilon\},$$

and notice that $v > 0$ because $U \cap \{x : \|x\|_{V^{-1}(0)} = \varepsilon\}$ is compact and disjoint from $V^{-1}(0)$. Using v, we define

$$\delta := \min\{\|x\|_{V^{-1}(0)} : x \in U \cap V^{-1}(v)\}.$$

Since $U \cap V^{-1}(v)$ is compact and disjoint from $V^{-1}(0)$, then $\delta > 0$. Note that $\delta \leq \varepsilon$ for, if not, then we would have that

$$(\forall x \in U \cap V^{-1}(v))\ \|x\|_{V^{-1}(0)} > \varepsilon,$$

and this would contradict the definition of v. By the definitions of v and δ it follows that

$$U \cap B_\delta(V^{-1}(0)) \subset U \cap V^{-1}([0, v]) \subset U \cap B_\varepsilon(V^{-1}(0)).$$

Since $\Gamma \subset V^{-1}(0) \subset \mathcal{O}$, for any $x \in \mathcal{X}$ we have $\|x\|_{\mathcal{O}} \leq \|x\|_{V^{-1}(0)} \leq \|x\|_\Gamma$, and so $B_\delta(\Gamma) \subset B_\delta(V^{-1}(0))$ and $B_\varepsilon(V^{-1}(0)) \subset B_\varepsilon(\mathcal{O})$. By inequality (5) we have that all level sets of V are positively invariant for the closed-loop system. Putting everything together we have

$$\begin{aligned} x_0 \in U \cap B_\delta(\Gamma) &\implies x_0 \in U \cap B_\delta(V^{-1}(0)) \implies x_0 \in U \cap V^{-1}([0, v]) \\ &\implies \phi(\mathbb{R}^+, x_0) \subset V^{-1}([0, v]). \end{aligned}$$

From the above, for any $t \geq 0$, the condition $\phi([0, t], x_0) \subset U$ implies

$$\phi([0, t], x_0) \subset U \cap V^{-1}([0, v]) \subset B_\varepsilon(V^{-1}(0)) \subset B_\varepsilon(\mathcal{O}),$$

and thus $\mathcal{O}$ is locally stable near Γ for the closed-loop system. □

Theorem 4.14 (Semi-asymptotic stability of Γ)**.** Consider system (1) with a passivity-based feedback of the form (3). If Γ is compact, then

- Γ is asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable,
- if all trajectories of the closed-loop system are bounded, then Γ is globally asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

If Γ is unbounded and the closed-loop system is locally uniformly bounded near Γ, then

- Γ is semi-asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable.
- if all trajectories of the closed-loop system are bounded, then Γ is globally semi-asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

Proof. The sufficiency part of the theorem follows from the following considerations. By Proposition 4.13, $\mathcal{O}$ is locally stable near Γ. If Γ is compact, by Theorem D.1 local Γ-detectability stability of Γ. The stability of Γ and its compactness in turn imply that all closed-loop trajectories in some neighbourhood of Γ are bounded. Since all bounded trajectories asymptotically approach $\mathcal{O}$, $\mathcal{O}$ is locally semi-attractive near Γ. If all trajectories of the closed-loop system are bounded, then $\mathcal{O}$ is globally attractive. Theorem 3.2 yields the required result.

Now suppose that Γ is unbounded. By local uniform boundedness near Γ we have that all closed-loop solutions in some neighbourhood of Γ are bounded and hence $\mathcal{O}$ is locally semi-attractive near Γ. Once again, if all closed-loop trajectories are bounded, then $\mathcal{O}$ is globally attractive. The required result now follows from Theorem 3.2.

The various necessity statements follow from the following basic observation. Any passivity-based feedback of the form (3) makes $\mathcal{O}$ an invariant set for the closed-loop system (see Section 2.5). Therefore, if Γ is [globally] semi-asymptotically stable for the closed-loop system, necessarily Γ is [globally] semi-asymptotically stable relative to $\mathcal{O}$ for the closed-loop system. In other words, (1) is necessarily locally Γ-detectable [Γ-detectable]. $\square$

We conclude this section with the following result, which gives conditions that are alternatives to the Γ-detectability assumption.

Proposition 4.15. Theorem 4.14 still holds if the local Γ-detectability [Γ-detectability] assumption is replaced by the following condition:

(i') Γ is stable relative to $V^{-1}(0)$ and Γ is [globally] semi-attractive relative to $\mathcal{O}$.

We omit the proof of this proposition because it relies on essentially identical arguments as those used to prove the reduction principles in Theorems 3.1 and 3.2. If the sufficient conditions for Γ-detectability in Proposition 4.12 fail, rather than checking for Γ-detectability one may find it easier to check condition (i') in Proposition 4.15. This is because verifying whether Γ is stable relative to $V^{-1}(0)$ does not require finding the maximal open-loop invariant subset $\mathcal{O}$ of $h^{-1}(0)$; moreover, checking that Γ is semi-attractive relative to $\mathcal{O}$ amounts to checking the familiar condition

$$h(\phi(t, x_0)) \equiv 0 \implies \phi(t, x_0) \to \Gamma \text{ as } t \to +\infty.$$

Note that, in the framework of [18] and [21], the requirement that Γ be stable relative to $V^{-1}(0)$ is trivially satisfied because in these references it is assumed that $\Gamma = V^{-1}(0)$.

4.4 Discussion

Theorems 4.3 and 4.6, dealing with the special case when $\Gamma = V^{-1}(0)$ $(= \{0\})$ and Γ is compact, become corollaries of our main result, Theorem 4.14. We have already shown (see Lemmas 4.9 and 4.10) that in this special case the properties of zero-state detectability (when $\Gamma = \{0\}$), and V-detectability coincide with our notion of Γ-detectability. Therefore, Theorems 4.3 and 4.6 state that local Γ-detectability is a sufficient condition for the asymptotic stabilization of the origin using a passivity-based feedback. We have shown that actually this condition is also *necessary.* When the storage function is proper, Theorems 4.3 and 4.6 assert that Γ-detectability is a sufficient condition for the global stabilization of Γ by means of a passivity-based feedback of the form (3). If V is proper, then all trajectories of the closed-loop system are bounded, and so Theorem 4.14 gives the same result. Moreover, once again, the theorem states that Γ-detectability is necessary for the stabilizability of Γ by means of a passivity-based feedback.

The theory in [18] and [21] does not handle the special case when Γ is compact and $\Gamma \subsetneq V^{-1}(0)$, while our theory does. This case includes the important situation when one wants to stabilize an equilibrium ($\Gamma = \{0\}$) but the storage is only positive semi-definite. Based on the results in [18] and [21], it may be tempting to conjecture that Theorems 4.3 and 4.6 still hold if one employs the following notion of detectability:

$$(\forall x_0 \in \mathcal{N}(\Gamma)) \; h(\phi(t, x_0)) = 0 \text{ for all } t \in \mathbb{R} \implies \phi(t, x_0) \to \Gamma, \tag{13}$$

which corresponds to requiring that $\mathcal{O}$ in Definition 4.7 is a semi-attractor for the open-loop system. This conjecture is false: we have shown that (local) Γ-detectability (i.e., the semi-asymptotic stability of Γ relative to $\mathcal{O}$ for the open-loop system) is a necessary condition for the stabilization of Γ. Even if one relaxes the asymptotic stability requirement and just asks for semi-attractivity of Γ relative to $\mathcal{O}$, the above conjecture is still false. As a matter of fact, Theorem 3.1 suggests that even in this case (local) Γ-detectability is a key property. A counter-example illustrating this loss of semi-attractivity is the pendulum. The upright equilibrium is globally attractive, but unstable, relative to the homoclinic orbit of the pendulum. Despite the fact that a passivity-based feedback can be used to asymptotically stabilize the homoclinic orbit (see, e.g., [40], [41], and the related work in [42]), the upright equilibrium is unstable for the closed-loop system. This well-known phenomenon finds explanation in the theory developed in this paper: the cause of the problem is the instability of the upright equilibrium relative to the homoclinic orbit. We next present another explicit counter-example illustrating our point.

Example. Consider the control system with state (x_1, x_2, x_3),

$$\begin{aligned}\dot{r} &= -r(r-1)\\ \dot{\theta} &= \sin^2(\theta/2) + x_3\\ \dot{x}_3 &= u\\ y &= x_3^3,\end{aligned} \tag{14}$$

where $(r, \theta) \in (0, +\infty) \times S^1$ represent polar coordinates for (x_1, x_2). The control system is passive with storage $V(x) = x_3^4/4$. Let Γ be the equilibrium point $\{(x_1, x_2, x_3) : x_1 = 1, x_2 = x_3 = 0\}$ and note that $\mathcal{O} = \{(x_1, x_2, x_3) : x_3 = 0\}$. On $\mathcal{O}$, the open-loop dynamics read as

$$\begin{aligned}\dot{r} &= -r(r-1)\\ \dot{\theta} &= \sin^2(\theta/2),\end{aligned} \tag{15}$$

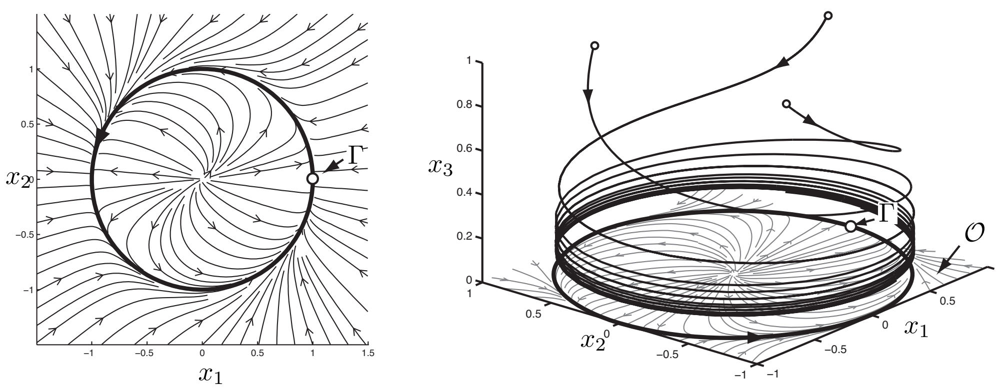

Figure 3: On the left-hand side, phase portrait on $\mathcal{O}$ for the open-loop system (15). On the right-hand side, closed-loop system (14) with feedback $u = -y$. Note that Γ is not attractive.

and it is easily seen that the equilibrium Γ attracts every point in $\mathcal{O}$ except the origin. Hence, Γ is attractive relative to $\mathcal{O}$, but unstable (indeed, the unit circle is a homoclinic orbit of the equilibrium); see Figure 3. Therefore, condition (13) holds but the system is not locally Γ-detectable. Consider the passivity-based feedback $u = -y$, which renders $\mathcal{O}$ globally asymptotically stable. Now for any initial condition off of $\mathcal{O}$ such that $(x_1(0), x_2(0)) \neq (0, 0)$, $x_3(0) > 0$, the corresponding trajectory is bounded, but its positive limit set is the unit circle on $\mathcal{O}$, and therefore it is not a subset of Γ; see Figure 3. In conclusion, Γ is not attractive for the closed-loop system (and neither is it stable). This example illustrates the fact that, when $\Gamma \subsetneq V^{-1}(0)$ is compact, simply requiring condition (13) in place of Γ-detectability may not be enough for attractivity of Γ.

In the light of Theorem 4.14 and the example above, it is clear that the addition of the stability requirement on Γ, relative to $\mathcal{O}$, is a crucial enhancement to the notions of detectability in [18] and [21].

5 Conclusions

We have investigated the problem of stabilizing a closed set Γ which is open-loop invariant for a passive system, and contained in the zero level set of the storage function. Allowing Γ to be a subset of, and not necessarily coincide with the zero level set of the storage function adds great flexibility in control design. This point is illustrated through several examples in a companion paper [43]. In this paper we gave conditions for semi-asymptotic stability of Γ relying on the new notion of Γ-detectability. When Γ is compact, among other things we showed that Γ-detectability is both necessary and sufficient for the asymptotic stabilization of Γ by means of a passivity-based feedback. When Γ is unbounded, our results rely on the assumption that, near Γ, trajectories of the closed-loop system enjoy a type of boundedness property. Our theory crucially relies on three novel reduction principles for stability, semi-attractivity, and semi-asymptotic stability. We envision that these principles will be relevant to other problems in control theory. The investigation of semi-asymptotic stability in the presence of unbounded trajectories must necessarily rely on different tools than the ones used in this paper. Birkhoff's notion of limit set and Ura's notion of prolongational limit set are not applicable in this setting, and the same holds for the various reduction principles presented in this paper.

A Proof of Lemma 2.5

We first show that if Γ is a uniform semi-attractor, then it is semi-asymptotically stable. Suppose, by way of contradiction, that Γ is unstable. This implies that there exists $\varepsilon > 0$ and sequences $\{x_i\} \subset \mathcal{X}$ and $\{t_i\} \subset \mathbb{R}^+$, with $\|x_i\|_\Gamma \to 0$ such that $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$. By Lemma D.2, we can assume, without loss of generality, that $\{x_i\}$ is bounded and has a limit $\bar{x} \in \Gamma$. Using $\bar{x}$ and ε in the definition of uniform semi-attractivity, we get $\lambda > 0$ and $T > 0$ such that $\phi([T, +\infty), B_\lambda(\bar{x})) \subset B_\varepsilon(\Gamma)$. For sufficiently large i, $x_i \in B_\lambda(\bar{x})$ and therefore, necessarily, $0 < t_i < T$. Having established that $\{t_i\}$ is a bounded sequence, we can assume that t_i has a limit $\tau < \infty$. Since Γ is positively invariant, $\phi(\tau, \bar{x}) \in \Gamma$. This gives a contradiction since $\phi(t_i, x_i) \to \phi(\tau, \bar{x})$ and, for all i, $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$.

Next we show that if Σ is locally uniformly bounded near Γ and Γ is semi-asymptotically stable, then Γ is a uniform semi-attractor for Σ. By Proposition 2.7, we need to show that there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$. By local uniform boundedness, for all x in a neighbourhood of Γ, $J^+(x) \neq \emptyset$. Moreover, since Γ is a semi-attractor, by Proposition 2.6 we have $J^+(x) \subset J^+(L^+(x)) \subset J^+(\Gamma)$. Therefore, to prove uniform semi-attractivity it is enough to show that $J^+(\Gamma) \subset \Gamma$. Consider an arbitrary point $x \in \Gamma$, and let $p \in J^+(x)$. By local uniform boundedness, there exist positive constants λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$. By the definition of prolongational limit set, there exist sequences $\{x_n\} \subset \mathcal{X}$ and $\{t_n\} \subset \mathbb{R}^+$, with $x_n \to x$ and $t_n \to +\infty$, such that $\phi(t_n, x_n) \to p$. Without loss of generality, we can assume that $\{x_n\} \subset B_\lambda(x)$. Take a decreasing sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$, with $\varepsilon_n \to 0$. By the stability of Γ, there exists a nested sequence of neighborhoods $\mathcal{N}_{n+1}(\Gamma) \subset \mathcal{N}_n(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}_n(\Gamma)) \subset B_{\varepsilon_n}(\Gamma)$. Since $\mathcal{N}_n(\Gamma) \cap B_\lambda(x)$ is a bounded set, for each n there exists $\delta_n > 0$ such that $B_{\delta_n}(\Gamma) \cap B_\lambda(x) \subset \mathcal{N}_n(\Gamma) \cap B_\lambda(x)$. We thus obtain a decreasing sequence $\{\delta_n\}$, $\delta_n \to 0$, such that $\phi(\mathbb{R}^+, B_{\delta_n}(x)) \subset B_m(x) \cap B_{\varepsilon_n}(\Gamma)$. Take subsequences $\{x_{n_k}\}$ and $\{B_{\delta_{n_k}}(x)\}$ such that, for each k, $x_{n_k} \in B_{\delta_{n_k}}(x)$. Since $x_n \to x \in \Gamma$, for each n there are infinitely many x_n's in $B_{\delta_n}(x)$, and therefore the subsequences just defined have infinite elements. We have that $\phi(t_{n_k}, x_{n_k}) \to p$ and, by construction, $\phi(t_{n_k}, x_{n_k}) \in B_{\varepsilon_{n_k}}(\Gamma)$. This implies that $p \in \Gamma$, and so $J^+(x) \subset \Gamma$.

The proofs of the statements involving relative stability concepts are identical. □

B Proof of Proposition 2.7

We only prove sufficiency. Assume that there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$. By local uniform boundedness, we can assume that all trajectories on $\mathcal{N}(\Gamma)$ are bounded, and hence for each $x \in \mathcal{N}(\Gamma)$, $L^+(x) \neq \emptyset$. Since $L^+(\mathcal{N}(\Gamma)) \subset J^+(\mathcal{N}(\Gamma), U) \subset J^+(\mathcal{N}(\Gamma))$, we have that for each $x \in \mathcal{N}(\Gamma)$, $J^+(x)$ and $J^+(x, U)$ are not empty. To prove that Γ is a uniform semi-attractor, we need to show that, for all $x \in \Gamma$,

$$(\exists \delta > 0)(\forall \varepsilon > 0)(\exists T > 0) \text{ s.t. } \phi([T, +\infty), B_\delta(x)) \subset B_\varepsilon(\Gamma).$$

Suppose, by way of contradiction, that there exists $x \in \Gamma$ such that

$$(\forall \delta > 0)(\exists \varepsilon > 0) \text{ s.t. } (\forall T > 0)(\exists \bar{x} \in B_\delta(x), \exists \bar{t} \geq T) \text{ s.t. } \|\phi(\bar{t}, \bar{x})\|_\Gamma \geq \varepsilon. \tag{16}$$

By the local uniform boundedness assumption, there exist positive λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$. We can take small enough δ that $\delta \leq \lambda$ and $\mathrm{cl}(B_\delta(x)) \subset \mathcal{N}(\Gamma)$. Let $\varepsilon > 0$ be as in (16). Take a sequence $\{T_i\} \subset \mathbb{R}^+$, with $T_i \to \infty$. By (16), there exist sequences $\{\bar{x}_i\} \subset B_\delta(x)$ and $\{\bar{t}_i\} \subset \mathbb{R}^+$, with $\bar{t}_i \to \infty$, such that $\|\phi(\bar{t}_i, \bar{x}_i)\|_\Gamma \geq \varepsilon$. Since $\bar{x}_i \in B_\delta(x) \subset B_\lambda(x)$, then $\phi(\bar{x}_i, \bar{t}_i) \in B_m(x)$. By boundedness of $\{\bar{x}_i\}$ and $\{\phi(\bar{t}_i, \bar{x}_i)\}$, we can assume that $\bar{x}_i \to x^\star \in \mathrm{cl}(B_\delta(x))$, and $\phi(\bar{t}_i, \bar{x}_i) \to p$, with $\|p\|_\Gamma \geq \varepsilon$. We have thus obtained that there exists $x^\star \in \mathrm{cl}(B_\delta(x))$ such that $J^+(x^\star) \not\subset \Gamma$. However, $\mathrm{cl}(B_\delta(x)) \subset \mathcal{N}(\Gamma)$, and so $J^+(\mathrm{cl}(B_\delta(x))) \subset \Gamma$, a contradiction.

The proof that Γ is a uniform semi-attractor relative to U if and only if there exists $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma), U) \subset \Gamma$ is identical.

□

C Proof of Theorem 3.1

By assumption (ii), there exists a neighbourhood $\mathcal{N}_1(\Gamma)$ of Γ such that all trajectories originating there asymptotically approach $\mathcal{O}$ in positive time. Let $\mathcal{N}_2(\Gamma)$ be the neighbourhood in assumption (iii), and define $\mathcal{N}_3(\Gamma) = \mathcal{N}_1(\Gamma) \cap \mathcal{N}_2(\Gamma)$. Clearly, $\mathcal{N}_3(\Gamma)$ is a neighbourhood of Γ. By construction, for all $x_0 \in \mathcal{N}_3(\Gamma)$, the solution is bounded and approaches $\mathcal{O}$. Therefore, the positive limit set $L^+(x_0)$ is non-empty, compact, invariant, and $L^+(x_0) \subset \mathcal{O}$. Moreover, by definition of positive limit set, and by assumption (iii) we have the following inclusion,

$$L^+(x_0) \subset \mathrm{cl}(\phi(\mathbb{R}^+, x_0)) \cap \mathcal{O} \subset \{\text{domain of attraction of } \Gamma \text{ rel. to } \mathcal{O}\}. \tag{17}$$

We need to show that $L^+(x_0) \subset \Gamma$. Assume, by way of contradiction, that there exists $\omega \in L^+(x_0)$ and $\omega \notin \Gamma$. By the invariance of $L^+(x_0)$, $\phi(\mathbb{R}, \omega) \subset L^+(x_0)$, and therefore $L^-(\omega) \subset L^+(x_0)$. By the inclusion in (17), all trajectories in $L^-(\omega)$ asymptotically approach Γ in positive time, and so since $L^-(\omega)$ is closed, $L^-(\omega) \cap \Gamma \neq \emptyset$. Let $p \in L^-(\omega) \cap \Gamma$. Pick $\varepsilon > 0$ such that $\|\omega\|_\Gamma > \varepsilon$. By the stability of Γ relative to $\mathcal{O}$, there exists a neighbourhood $\mathcal{N}_4(\Gamma)$ of Γ such that $\phi(\mathbb{R}^+, \mathcal{N}_4(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Since $p \in L^-(\omega)$, there exists a sequence $\{t_k\} \subset \mathbb{R}^+$, with $t_k \to +\infty$, such that $\phi(-t_k, \omega) \to p$ at $k \to +\infty$. Since $p \in \Gamma$, we can pick $k^\star$ large enough that $\phi(-t_{k^\star}, \omega) \in \mathcal{N}_4(\Gamma)$. Let $T = t_{k^\star}$ and $z = \phi(-t_{k^\star}, \omega)$. We have thus obtained that $z \in \mathcal{N}_4(\Gamma)$, but $\phi(T, z) = \omega$ is not in $B_\varepsilon(\Gamma)$. This contradicts the stability of Γ, and therefore, for all $x_0 \in \mathcal{N}_3(\Gamma)$, $L^+(x_0) \subset \Gamma$, proving that Γ is a semi-attractor for Σ.

To prove global attractivity of Γ it is sufficient to notice that by assumptions (ii)' and (iii)', for all $x_0 \in \mathcal{X}$, $L^+(x_0)$ is non-empty and $L^+(x_0) \subset \mathcal{O}$. On $\mathcal{O}$, by assumption (i)' all trajectories approach Γ, so by the contradiction argument above we conclude that $L^+(x_0) \subset \Gamma$. □

D Reduction principle for stability

Theorem D.1. Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. If assumptions (i), (ii), and (iv) of Theorem 3.2 hold, then Γ is stable.

To prove the theorem, we need the next lemma.

Lemma D.2. Let $\Gamma \subset \mathcal{X}$ be a closed set which is positively invariant set for Σ in (7). If Γ is unstable, then there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $x_i \to \bar{x} \in \Gamma$, and $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$.

Proof. The instability of Γ implies that there exists $\varepsilon > 0$, a sequence $\{x_i\} \subset \mathcal{X}$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $\|x_i\|_\Gamma \to 0$, and $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$. If we show that we there exists a bounded $\{x_i\}$ as above, then we are done. Let S be defined as follows

$$S = \{x \in B_\varepsilon(\Gamma) : (\exists t > 0)\ \|\phi(x, t)\| = \varepsilon\}.$$

The instability of Γ implies that S is not empty. Moreover, since Γ is positively invariant, $S \cap \Gamma = \emptyset$. Suppose that the lemma is not true. Then, for any bounded sequence $\{x_i\} \subset S$, we have $x_i \not\to \Gamma$. This implies that, for any $x \in \Gamma$, there exists $\delta(x) > 0$ such that $B_{\delta(x)}(x) \cap S = \emptyset$. For, if this were not true, then there would exist a bounded sequence $\{x_i\} \subset S$, with $x_i \to \Gamma$ contradicting the assumption we have made. Let $U = \bigcup_{x\in\Gamma} B_{\delta(x)}(x)$. By construction, U is a neighbourhood of Γ such that $U \cap S = \emptyset$. In other words, for all $x \in U$, there does *not* exist $t > 0$ such that $\|\phi(t, x)\|_\Gamma = \varepsilon$, contradicting the assumption that Γ is unstable. □

Proof of Theorem D.1: By way of contradiction, suppose that Γ is unstable. Then, by Lemma D.2, there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, with $x_i \to \bar{x} \in \Gamma$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that

$$\|\phi(t_i, x_i)\|_\Gamma = \varepsilon, \text{ and } \phi([0, t_i), x_i) \in B_\varepsilon(\Gamma).$$

By local uniform boundedness of Σ near Γ, there exist two positive numbers λ and m such that $\phi(\mathbb{R}^+, B_\lambda(\bar{x})) \subset B_m(\bar{x})$. We can assume $\{x_i\} \subset B_\lambda(\bar{x})$. Take a decreasing sequence $\{\varepsilon_i\} \subset \mathbb{R}^+$, $\varepsilon_i \to 0$. By assumption (ii), $\mathcal{O}$ is locally stable near Γ. Using the definition of local stability with $c = m$ and $\varepsilon = \varepsilon_i$, there exists $\delta_i > 0$ such that for all $x_0 \in B_{\delta_i}(\bar{x})$ and all $t > 0$, if $\phi([0, t], x_0) \subset B_m(\bar{x})$, then $\phi([0, t], x_0) \subset B_{\varepsilon_i}(\mathcal{O})$. By taking $\delta_i \leq \lambda$ we have

$$(\forall x_0 \in B_{\delta_i}(\bar{x}))\ \phi(\mathbb{R}^+, x_0) \subset B_{\varepsilon_i}(\mathcal{O}).$$

By passing, if needed, to a subsequence we can assume without loss of generality that, for all i, $x_i \in B_{\delta_i}(\bar{x})$ so that

$$\limsup_{i\to\infty} d(\phi([0, t_i], x_k), \mathcal{O}) = 0.$$

Using assumptions (i) and (iii) (if Γ is unbounded), by Lemma 2.5 it follows that Γ is a uniform semi-attractor relative to $\mathcal{O}$. Therefore,

$$(\forall x \in \Gamma)(\exists \mu > 0)(\forall \varepsilon' > 0)(\exists T > 0) \text{ s.t. } \phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{18}$$

Consider the set $\Gamma' = \Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))$. Since Γ' is compact, using (18) we infer the existence of $\mu > 0$ such that

$$(\forall x \in \Gamma')(\forall \varepsilon' > 0)(\exists T > 0)\ \phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{19}$$

By reducing, if necessary, ε in the instability definition, we may assume that $\varepsilon < \mu$. Now choose $\varepsilon' < \varepsilon/2$. Using again a compactness argument, by (19) one infers the following condition

$$(\exists T > 0)(\forall x \in \Gamma')\phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{20}$$

We claim that $B_\mu(\Gamma) \cap B_m(\bar{x}) \subset B_\mu(\Gamma')$. For, if $\mu \geq m$, then

$$B_\mu(\Gamma) \cap B_m(\bar{x}) = B_m(\bar{x}) \subset B_\mu(\bar{x}) \subset B_\mu\Big(\Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))\Big).$$

If $\mu < m$, then $x \in B_\mu(\Gamma) \cap B_m(\bar{x})$ if and only if $\|x\|_\Gamma < \mu$ and $\|x - \bar{x}\| < m$; in particular, there exists $y \in \Gamma$ such that $\|x - y\| < \mu$. Since $\|y - \bar{x}\| \leq \|x - y\| + \|x - \bar{x}\| \leq \mu + m < 2m$, we have that $y \in \Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))$, and thus $x \in B_\mu(\Gamma \cap \mathrm{cl}(B_{2m}(\bar{x})))$.

Using (20) and the claim we've just proved we obtain

$$(\forall x \in B_\mu(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O})\ \phi([T, +\infty), x) \subset B_{\varepsilon'}(\Gamma). \tag{21}$$

Now, since $\{t_k\}$ is unbounded there exists $K_1 > 0$ such that $t_k > T$ for all $k \geq K_1$. Since $\phi([0, t_k), x_k) \subset B_\varepsilon(\Gamma)$ we have $\phi(t_k - T, x_k) \in B_\varepsilon(\Gamma)$ for all $k \geq K_1$. Let

$$y_k = \phi(t_k, x_k), \text{ and } z_k = \phi(t_k - T, x_k).$$

Thus, $y_k = \phi(T, z_k)$, $\|y_k\|_\Gamma = \varepsilon$ and $z_k \in B_\varepsilon(\Gamma)$. By local uniform boundedness, it also holds that $z_k \in B_m(\bar{x})$. Pick $\delta \in (0, \mu - \varepsilon)$. Since $z_k \in \phi([0, t_k), x_k) \subset B_m(\bar{x})$, and since

$$\limsup_{k \to \infty} d(\phi([0, t_k], x_k), \mathcal{O}) = 0,$$

then there exists $K_2 \geq K_1$ such that, for all $k \geq K_2$, there exists $z'_k \in B_m(\bar{x}) \cap \mathcal{O}$ such that $\|z_k - z'_k\| < \delta$. Since $z_k \in B_\varepsilon(\Gamma)$, then

$$z'_k \in B_{\varepsilon+\delta}(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O} \subset B_\mu(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O}$$

and, by (21), $\phi([T, +\infty), z'_k) \subset B_{\varepsilon'}(\Gamma)$. By continuous dependence on initial conditions, δ can be chosen small enough that

$$(\forall x \in B_m(\bar{x}))(\forall x_0 \in B_\delta(x))\ \|\phi(T, x) - \phi(T, x_0)\| < \varepsilon/2.$$

We have $z_k \in B_m(\bar{x})$ and $\|z_k - z'_k\| < \delta$, hence $\|\phi(T, z_k) - \phi(T, z'_k)\| < \varepsilon/2$, which implies

$$y_k \in B_{\varepsilon/2}(\phi(T, z'_k)) \subset B_{\varepsilon/2+\varepsilon'}(\Gamma) \subset B_\varepsilon(\Gamma),$$

contradicting $\|y_k\|_\Gamma = \varepsilon$. $\square$

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