String Instability in Classes of Linear Time Invariant Formation Control With Limited Communication Range

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***Abstract*—This paper gives sufficient conditions for string instability in an array of linear time-invariant autonomous vehicles with communication constraints. The vehicles are controlled autonomously and are subject to a rigid or semi-rigid formation policy. The individual controllers are assumed to have a limited range of forward and backward communication with other vehicles. Sufficient conditions are given that imply a lower bound on the peak of the frequency response magnitude of the transfer function mapping a disturbance to the leading vehicle to a vehicle in the chain. This lower bound quantifies the effect of spacing separation policy, intervehicle communication policy, and vehicle settling response performance. These results extend earlier works to give a unified treatment of heterogeneous, non-nearest neighbor communication and semi-rigid one-dimensional formation control.**

***Index Terms*—Bullwhip effect, distributed systems, formation control, performance limitations, string stability.**

I. Introduction

THERE has recently been extensive interest in a range of cooperative control problems, including those of controlling the formation of a large number of autonomous vehicles; see for example [1]–[3]. The simplest case of such formations is 1-D systems, as for example the control of intelligent vehicle highway systems [4].

As early as [5], a difficulty known as 'string instability' has been observed in tight formation control of long strings of vehicles based on local information. Here we use the term string instability to describe the amplification along the string of the response to a disturbance to the lead vehicle. Different measures of disturbance amplification have been proposed in the literature. For example, [5] uses a frequency domain (H_∞) definition, whilst a more complete discussion in [6] gives more formal definitions and uses the norm induced by the L_∞ signal norm to characterize string stability. Discussions in an H_∞ setting are presented in [7], where the authors use the terminology of *ill-conditioning* and *non-scalability*, used in more general settings of networked dynamic systems, to describe phenomena similar to string instability. In this paper, we use the term 'string stability' to denote the situation where an appropriately defined H_∞ norm is bounded independently of the string length.

Although the problem setting above has been described for formation control of autonomous vehicles, very similar network structures and dynamics have been described in other application domains. For example, in the area of irrigation flow control (see for example [8]), a series of "pools" connected by gates with local control laws is studied with the same phenomena being present. Another example of closely related dynamics occurs in supply chain, or production and inventory control systems [9]. These are often modeled in discrete time (see for example [10, Fig. 1]) and use feedforward to achieve the equivalent of type II servo response (zero steady state error to ramp references), in some cases.[1] In this context, concepts similar to string instability are sometimes known as the "bullwhip effect," or the "Forrester effect" [11].

String instability is clearly undesirable[2] and has lead to a number of analyses of the difficulties and proposed solutions. Some of the main solutions proposed to combat string instability and non-scalability issues include:

Extending Information Flow: The most obvious examples of string instability occur when each vehicle only has access to its relative position error to the preceding vehicle. In [5], [12] and other related references, control laws are designed so that both separation from the preceding and succeeding vehicles (sometimes called "bidirectional" control [7]) in the platoon are used in computing a vehicle's actions. One extension of this idea is "multi-look ahead" control, see for example [13], [14]. In other cases, transmission of some global information is used in individual control calculations. Such schemes include "leader following" control [12] where each vehicle has access to information from the lead vehicle. Analysis and discussions in [15] also point to the need for some global information in the problem formulation and control.

Relaxing Formation Rigidity: It is also known that maintaining a strict formation position separation exacerbates

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[1]In terms of [10], type II servo response occurs when the average estimate of the production lead time is correct, which is required to ensure inventory levels "lock-on" to their target values.

[2]However, in some applications, where it is known a-priori that the length of the string is bounded, it may be possible to tolerate string instability for sufficiently small rates of amplification.

string stability problems. For example, in [6], weak coupling (which relaxes formation rigidity) is shown to give string stability. Other approaches that relax formation rigidity include both a position headway, and a time headway in the tracking error definition (see for example [16]–[18]).

Heterogeneous Controller Tuning: The concatenation of identical transfer functions, implicit in some homogeneous strings, implies that any magnitude peak above 0 dB in the transfer function will result in unbounded amplification as the string length grows. This suggests that by having nonuniform controller tuning in different vehicles, it may be possible to avoid string stability issues. This approach is pursued in, for example, [16, §3.E], [17, Remark 4], [19], and [20].

Nonlinear Controllers: A range of different controller nonlinearities (for example [16], [17] including some cases of switched or hybrid elements [21]) have been proposed to improve stability properties in strings of vehicles.

A key question therefore is to analyze general underlying causes and remedies for string instability problems. We wish to extend the work of [12] where some specific classes of Linear Time Invariant (LTI) feedback control systems are analyzed. In particular, an analysis of the implications of the complementary sensitivity integral [22] can be used to establish conditions under which $\|T(\cdot)\|_{H_\infty} > 1$, where $T(\cdot)$ denotes the closed loop transmission from one vehicle in the string to the next. Certain types of control strategies (namely, homogeneous controller tuning with nearest-neighbor-only communication) give rise to these conditions on the transfer function $T(s)$ and thereby dictate that sequential disturbance amplification must occur. The authors in [23] analyze a generalization of the Bode Sensitivity integral to asymptotically infinite dimensional circulant systems that satisfy a spatial invariance assumption. However, the authors of [23] do not analyze the specific impact on stability and scalability issues discussed in the present paper.

A vast number of studies exist on decentralized control with other information structures, including, for example, the study of string stability in platoons with ring couplings [24], and the application of graph theory concepts to the control of general formations with complex information paths (e.g., [25], [26]). However, to the best of the authors' knowledge, these studies only apply to structures with homogeneous individual feedback loop dynamics.

The present paper extends the work in [12] by providing a unified analysis including:

1) Heterogeneous individual feedback loop dynamics, that is, non-uniform vehicle or controller dynamics. Here, we consider the consequences of a uniform bound on the high frequency behavior (see Assumption 7) of the feedback loops, and a bound on the settling response behavior of the closed loop system (see Assumption 8).
2) More general information structures, with the only restriction being that communications are restricted to be between vehicles within a limited range of each other.
3) A slightly broader class of spacing policies. In particular, we show that relaxing the constant spacing policy to allow a sufficiently small time headway (semi-rigid formation) does not qualitatively alter the results.

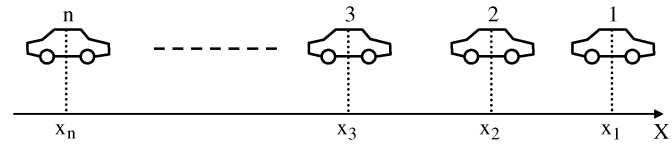

Fig. 1. Diagram depicting a vehicle platoon.

The paper is organized as follows. Section II defines the system considered and presents some preliminary results. Section III presents the main result of the paper: a lower bound on the peak gain from a disturbance at the leading vehicle to the last vehicle in the platoon. Section IV provides interpretations and implications of the main result, which are illustrated by examples in Section V. Conclusions and final remarks are given in Section VI.

An earlier version of these results was presented in [27].

Notation: Most of the notation used is fairly standard in the systems and control literature. the Laplace transform and inverse Laplace transform operators are denoted by $\mathcal{L}$ and $\mathcal{L}^{-1}$. The Laplace transform complex variable is $s \in \mathbb{C}$, and Laplace transforms will typically be denoted by an upper case letter, that is: $\mathcal{L}\{u(t)\} = U(s)$, and $\mathcal{L}^{-1}\{U(s)\} = u(t)$. The notation $P(s) \star u(t)$ is used to denote the time response (with zero initial conditions) of a linear time invariant system with transfer function $P(s)$ and input $u(t)$. The relative degree r of a rational transfer function is the difference between the degrees of its denominator and numerator polynomials. A transfer function is proper if $r \geq 0$, and strictly proper if $r > 0$. A real scalar-valued function of time, $x : \mathbb{R} \to \mathbb{R}$, is denoted $x(t) \in \mathbb{R}$. Similarly, a complex scalar-valued function of s, $X : \mathbb{C} \to \mathbb{C}$, is denoted $X(s) \in \mathbb{C}$. Vector and matrix-valued functions are denoted $\underline{x}(t) \in \mathbb{R}^n$ and $C(s) \in \mathbb{C}^{n\times n}$. Given a number $x \in \mathbb{R}$, the notation $\lceil x \rceil$ represents the smallest integer no smaller than x. We extend the standard product notation $\prod$ to include matrices as follows: $\prod_{i=1}^{n} M_i \triangleq M_n M_{n-1} \ldots M_2 M_1$. The imaginary unit is j, that is, $j^2 = -1$. The string length will be denoted by n.

II. Preliminaries

A. System Definition

We consider a one-dimensional array of vehicles as depicted in Fig. 1. In this diagram, each vehicle, $1, 2, \ldots n$ is traveling in the positive X direction, and the i vehicle has x-coordinate denoted by $x_i(t) \in \mathbb{R}$.

The dynamics for the i-vehicle are assumed to be linear time invariant with a scalar transfer function $P_i(s) \in \mathbb{C}$, and scalar input $u_i(t) \in \mathbb{R}$. The vehicle dynamics are then given by

$$x_i(t) = P_i(s) \star u_i(t); \quad i = 1, 2, \ldots, n. \tag{1}$$

In vector form, let $\underline{x}(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)]^T$, and similarly define the vector control variable $\underline{u}(t) = [u_1(t)\ u_2(t)\ \ldots\ u_n(t)]^T$. We further define the multivariable plant transfer function, $P(s) = \mathrm{diag}\{P_i(s)\} \in \mathbb{C}^{n\times n}$, and therefore rewrite (1) as

$$\underline{x}(t) = P(s) \star \underline{u}(t). \tag{2}$$

The vehicle dynamics are typically modeled as a second order system including damping (see for example [5], [15]), in which case $P_i(s) = g_i/s(s+\mu_i)$. Other references such as [17] use a double integrator model, sometimes augmented with first order actuator dynamics [12]. Here we shall not be concerned directly with the details of the vehicle dynamics and make the following initial assumption on the plant whilst later assumptions deal with the overall dynamics of the system.

Assumption 1 (Plant): Each of the n individual vehicle transfer functions, $P_i(s)$, for $i = 1,\dots n$, is strictly proper, has no unstable hidden modes, and has no zeros at $s = 0$. □

Key aspects of the performance of the platoon are regulation of the vehicles' relative positions whilst maintaining a target velocity generated by the first vehicle. Therefore, we introduce as performance variables the vehicle separations $e_i(t)$ defined for $i = 1,\dots,n$, by

$$e_i(t) \triangleq \begin{cases} d_1(t) - x_1(t), & \text{for } i = 1, \\ \delta_i(t) + x_{i-1}(t) - x_i(t), & \text{for } i = 2,3,\dots,n \end{cases} \tag{3}$$

where $d_1(t) \in \mathbb{R}$ denotes the desired position for the string lead vehicle, and $\delta_i(t) \in \mathbb{R}$ for $i > 1$ denotes the *target separation* (negative) for the i-vehicle. In vector form, using the notation $\underline{e}(t) = [e_1(t)\ e_2(t)\ \dots\ e_n(t)]^T$ and $\underline{\delta}(t) = [0\ \delta_2(t)\ \dots\ \delta_n(t)]^T$, we rewrite (3) as

$$\underline{e}(t) = \underline{\delta}(t) - M\underline{x}(t) + V_1^n d_1(t) \tag{4}$$

where $V_1^n = [1\ 0\ \dots\ 0]^T$, and $M \in \mathbb{R}^{n\times n}$ denotes the coupling matrix

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -1 & 1 \end{bmatrix}. \tag{5}$$

Note that in the most general case the vehicles' separation $\underline{e}(t)$ could be permitted to be a general function of both position in the platoon, and also time (which permits constant time headway policies). Here, we restrict attention to the following class of separation policies.

Assumption 2 (Vehicle Separation Policy): The target vehicle separations $\delta_i(t)$, with $i = 2,3,\dots,n$, are either constant or increase linearly with the vehicle's own velocity. That is

$$\underline{\delta}(t) = \underline{\delta}_0 - H\frac{d\underline{x}(t)}{dt} \tag{6}$$

where $H \in \mathbb{R}^{n\times n}$, defined as $H = \mathrm{diag}\{h_i\} \geq 0$, is the matrix of *time headways*, and $\underline{\delta}_0 \in \mathbb{R}^n$ is a vector of constant reference spacings. □

We shall see that, as might be expected, using negative time headways (equivalently, deliberately introducing 'negative damping') aggravates the frequency domain constraints described later. Thus, negative headways would seem to offer no benefit in the control design, and we therefore consider only the case of positive time headways.

Also, as a consequence of the above definitions, under normal circumstances $\underline{\delta}_0$ will consist of negative elements, and $h_i \in \mathbb{R}$ will be positive, indicating that at higher speeds increased spacing is desired. If $H = 0$, then the vehicle separation policy is termed a *constant spacing* policy; otherwise, (6) is referred to as a *constant time headway* policy.

Subject to Assumption 2, the vehicle separation vector $\underline{e}(t)$ from (4) can be expressed as

$$\begin{aligned} \underline{e}(t) &= \underline{\delta}(t) - M\underline{x}(t) + V_1^n d_1(t) \\ &= -(M + sH) \star \underline{x}(t) + V_1^n d_1(t) + \underline{\delta}_0. \end{aligned} \tag{7}$$

Assumption 3 (Control Policy): We assume that the control is linear time invariant, possibly multivariable (depending on the communications range to be defined in Assumption 4), and based on error measurements, $\underline{e}(t)$ as defined by (7). That is

$$\underline{u}(t) = C(s) \star \underline{e}(t) \tag{8}$$

where $C(s) \in \mathbb{C}^{n\times n}$. □

If the controller of each vehicle uses exclusively information about the separation from the vehicle immediately ahead, we say that the control communication range of the string is limited to one vehicle forward. This situation is represented by a diagonal matrix $C(s)$.

To model communication range more generally, we shall use *banded* matrices (see for example [28]): Given integers k,ℓ, a matrix $A \in \mathbb{R}^{n\times n}$ is called (k,ℓ)-*banded* if $a_{ij} = 0$ for all $i > j + k$ and $a_{ij} = 0$ for all $j > i + \ell$. For example, the bidiagonal coupling matrix M in (5) is (1, 0)-banded.

We make the following structural assumption on the controller $C(s)$.

Assumption 4 (Communications Range): There are fixed natural numbers, $c_r, c_f \in \mathbb{N}$ (independent of the string length, n), with $c_f \geq 1$, which we term the *reverse* and *forward communication ranges*, such that the control transfer function matrix $C(s)$ is $(c_f - 1, c_r)$-banded. We will refer to the integer $\ell_r = \lceil c_r/c_f \rceil$ as the *communication range ratio* and, for simplicity, assume the number of vehicles n to be divisible by the forward communication range c_f, that is, $n = Nc_f$. □

The forward communication range c_f in Assumption 4 specifies the number of vehicles in front of the i-vehicle that are permitted to communicate with the i-vehicle. Conversely, the reverse communication range c_r specifies the number of vehicles behind the i-vehicle that are permitted to communicate with the i-vehicle. Other common communication strategies, such us 'leader broadcast' [18], allow string stability at the expense of a communication range that increases without bound as a function of the length of the platoon.

Using (1), (4), and (8), the state variables $\underline{x}$ may be related to the target separation variables $\underline{\delta}_0$ and the lead vehicle target position $d_1(t)$ by the expression

$$\underline{x}(t) = (I + L(s))^{-1} P(s)C(s) \star (\underline{\delta}_0 + V_1^n d_1(t)) \tag{9}$$

where $L(s) \in \mathbb{C}^{n\times n}$ is the multivariable *loop transfer function*

$$L(s) = P(s)C(s)(M + sH). \tag{10}$$

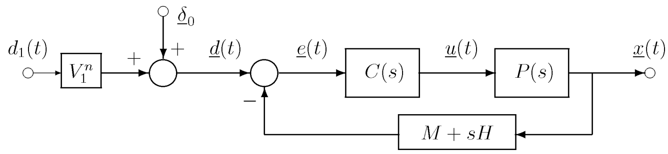

Fig. 2. Multivariable feedback loop representation of the vehicle string.

From Assumption 4 and (10), the loop transfer function $L(s)$ is a (c_f, c_r)-banded matrix.

Remark 1 (On More General Control Structures): More general linear time invariant control structures could be considered. For example, we could permit the local control actions, $u_i(t)$ to depend on the local state variable, $x_i(t)$ as well as the error variables in the form

$$\underline{u}(t) = C(s) \star (\underline{e}(t) - C_X(s) \star \underline{x}(t)) \tag{11}$$

where $C_X(s)$ is a diagonal transfer function matrix.

In the case of (11), the derivations below follow except that the loop transfer function becomes $L(s) = P(s)C(s)(M + sH + C_X(s))$. In this case, unless $C_X(0) = 0$, the closed loop system will have an unbounded steady state error in response to a ramp input. It would then seem reasonable to restrict attention to the case where we can factor $C_X(s) = sH_X(s)$, in which case the analysis that follows is identical except: (i) H is replaced by $H + H_X(s)$; (ii) the effective time headway for the ith vehicle is $h_i + H_{Xi}(0)$; and, (iii) some alterations to later analysis would be needed if $\|(1 + s(H + H_X(s)))^{-1}\|_{\mathcal{H}_\infty} > 1$. Therefore, for simplicity of exposition, we analyze control laws of the form (8). □

With the above notation, the vehicle string is represented by the multivariable feedback loop illustrated in Fig. 2. For further reference, we also introduce from (9) the closed-loop multivariable transfer function matrix $H_{xd}(s) \in \mathbb{C}^{n\times n}$

$$\begin{aligned} H_{xd}(s) &= (I + L(s))^{-1} P(s)C(s) \\ &= \left(I - (I + L(s))^{-1}\right)(M + sH)^{-1} \end{aligned} \tag{12}$$

which represents the frequency response from inputs $\underline{d}(t)$ to vehicle positions $\underline{x}(t)$ in Fig. 2.

In broad terms, we shall be interested in examining conditions on the feedback loop dynamics, and the communications structure, such that the closed loop behavior captured in H_{xd}, including that represented by (9), cannot be made 'well behaved' for arbitrarily large n.

To ensure that we can achieve asymptotically zero tracking error for any constant speed reference signal $\delta_1(t) = \delta_x + \delta_v t$ we require a multivariable type-II servomechanism controller. This, along with other standing assumptions on the loop dynamics, are described next.

Assumption 5 (Feedback Loop): The loop transfer function $L(s)$ in (10) satisfies:

(a) $L(s)$ is strictly proper. In other words, every element of $L(s)$ has relative degree $r \geq 1$.
(b) $L(s)$ is free of unstable hidden modes.
(c) $P(s)C(s)$ can be factored as $P(s)C(s) = s^{-2}\bar{L}(s)$, where $\bar{L}(0)$ is non-singular. □

B. Basic Loop Properties

Assumption 5(c) allows us to establish some initial properties of the low frequency portion of the closed loop response matrix, H_{xd}.

Lemma 1 (Values of H_{xd} at $s = 0$): Consider H_{xd} as defined in (12). Then subject to Assumption 5 we have

$$H_{xd}(0) = M^{-1} \tag{13}$$
$$H'_{xd}(0) = -M^{-1}HM^{-1}. \tag{14}$$

Proof: From the definition of H_{xd} in (12) we have

$$\begin{aligned} H_{xd}(s) &= (I + P(s)C(s)(M + sH))^{-1} P(s)C(s) \\ &= \left(M^{-1}(P(s)C(s))^{-1} + I + sM^{-1}H\right)^{-1} M^{-1} \\ &= \left(I + sM^{-1}H + s^2M^{-1}\bar{L}^{-1}(s)\right)^{-1} M^{-1} \end{aligned} \tag{15}$$

where the last line in (15) follows from Assumption 5(c). Evaluating (15) at $s = 0$ gives (13) since $\bar{L}(0)$ is assumed to be invertible. Similarly, differentiating (15) at $s = 0$ gives (14). ■

The analysis that follows makes use of the Bode Complementary Sensitivity integral (see for example [29, Theorem 3.1.5]) in a similar fashion to that in [12]. We restate this theorem here for completeness.

Lemma 2 (Bode Complementary Sensitivity Integral): Let $T(s)$ be a real rational scalar function of the complex variable s. Suppose that $T(0) = 1$ and also that $T(s)$ is stable (analytic in the closed right half complex plane). Then

$$\int_0^\infty \log_e |T(j\omega)| \frac{d\omega}{\omega^2} \geq \frac{\pi}{2} T'(0). \tag{16}$$

Proof: This result follows immediately from [29, Theorem 3.1.5], where we have equality if $T(s)$ has no zeros in the closed right half complex plane. ■

The lower left element of H_{xd} describes the response of the state of the last vehicle to a disturbance at the first vehicle. String instability has been observed in this response, and we are therefore interested in analyzing this particular component[3] of the overall closed loop response. We shall be particularly interested in applying Lemma 2 to the lower left element of H_{xd}, namely the scalar transfer function

$$H_{x_n d_1}(s) = (V_n^n)^T H_{xd} V_1^n \tag{17}$$

where $V_1^n, V_n^n \in \mathbb{R}^n$ are the 1st and nth canonical basis vectors respectively. We now apply Lemma 2 to the transfer function $H_{x_n d_1}(s)$.

[3]Note that since we are considering only one component of the closed loop response, we can only make precise statements about conditions under which the overall response is not well behaved. The reverse implication, determining conditions under which the complete response is well behaved would require more extensive analysis.

Lemma 3 (Bode Integral for H_{xd}): Consider $H_{x_n d_1}$ as defined in (17). Then subject to Assumption 5 we have

$$\int_0^\infty \log_e |H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\frac{n\pi}{2}\bar{h} \tag{18}$$

where $\bar{h}$ is the *average time headway*

$$\bar{h} = \frac{1}{n}\sum_{i=1}^{n} h_i. \tag{19}$$

Proof: Note from the definition of M in (5) that

$$M^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{20}$$

Using (20) in Lemma 1 yields

$$\begin{aligned} H_{x_n d_1}(0) &= 1 \\ H'_{x_n d_1}(0) &= -n\bar{h}. \end{aligned} \tag{21}$$

The result then follows by using (21) in Lemma 2. ■

We now turn to the analysis of feedback systems of the form described above. In particular, we derive conditions for broad classes of systems under which it is not possible to retain certain 'well behaved' closed-loop properties for large platoon sizes. These are described in terms of a set of seemingly reasonable specifications and objectives that dictate a lower bound on the achievable performance. In addition, we will be able to show that some combinations of performance specifications and assumptions are infeasible.

III. Lower Bounds on Achievable Performance

This section contains the main technical result of the paper in Theorem 7. This theorem presents a lower bound on the worst case disturbance amplification along the string in terms of communication constraints, high frequency behavior, and transient performance. In order to state this result, we will first present some preliminary results and refined assumptions on the system structural properties induced by the communications range (Section III-A), high frequency behavior (Section III-B), and time domain performance specifications (Section III-C). Theorem 7 is stated in Section III-D, followed by interpretations and discussion, given in Section IV, and a number of illustrative numerical examples in Section V.

A. Structural Properties

As pointed out above, $L(s)$ is a (c_f, c_r)-banded transfer matrix. Since from Assumption 4 the number of vehicles n is divisible by the forward communication range c_f, then $L(s)$ can be written as an $N \times N$ block matrix, with $N = n/c_f$

$$L(s) = \begin{bmatrix} L_{1,1}(s) & L_{1,2}(s) & L_{1,3}(s) & \cdots & 0 \\ L_{2,1}(s) & L_{2,2}(s) & L_{2,3}(s) & \cdots & 0 \\ 0 & L_{3,2}(s) & L_{3,3}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & L_{N-1,N}(s) \\ 0 & \cdots & 0 & L_{N,N-1}(s) & L_{N,N}(s) \end{bmatrix} \tag{22}$$

where each block element $L_{i,j}(s)$ is a $c_f \times c_f$ dimensional transfer function matrix, and $L_{i,j}(s) = 0$ for $j > i + \ell_r$, where $\ell_r = \lceil c_r/c_f \rceil$ is the communication range ratio introduced in Assumption 4.

It follows that $I + L(s)$ can be conveniently factorized in a block LU form [28] described in the following lemma, which is not difficult to prove by direct algebraic substitution.

Lemma 4 (Block LU Factorisation of $L(s)$): Under Assumption 4, let $L(s)$ be the (c_f, c_r)-banded transfer function matrix defined in (22). Then

$$(I + L(s)) \triangleq M_L(s) M_U(s) \tag{23}$$

where $M_U(s)$ is given as

$$M_U(s) = \begin{bmatrix} I & U_{1,2}(s) & U_{1,3}(s) & \cdots & 0 \\ 0 & I & U_{2,3}(s) & \cdots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{24}$$

and $M_L(s)$ as

$$M_L(s) = \begin{bmatrix} \tilde{S}_{1,1}^{-1}(s) & 0 & 0 & \cdots & 0 \\ L_{2,1}(s) & \tilde{S}_{2,2}^{-1}(s) & 0 & \cdots & 0 \\ 0 & L_{3,2}(s) & \tilde{S}_{3,3}^{-1}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{N,N-1}(s) & \tilde{S}_{N,N}^{-1}(s) \end{bmatrix} \tag{25}$$

and $\tilde{S}_{k,k}(s), U_{k,j}$ are defined recursively by

$$\begin{aligned} \tilde{S}_{1,1}(s) &= (I + L_{1,1}(s))^{-1} \\ U_{1,j}(s) &= \tilde{S}_{1,1}(s) L_{1,j}(s) : j = 2, 3 \ldots, N \\ \tilde{S}_{k,k}(s) &= (I + L_{k,k}(s) - L_{k,k-1}(s) U_{k-1,k}(s))^{-1} : \\ &\qquad k = 2, 3, \ldots, N \\ U_{k,j}(s) &= \tilde{S}_{k,k}(s) \left(L_{k,j}(s) - L_{k,k-1}(s) U_{k-1,j}(s) \right) : \\ &\qquad 1 < k < j \leq N. \end{aligned} \tag{26}$$

□

Remark 2 (Forward Communications Case): Note that in the case where we permit only forward communications, the loop transfer function is lower triangular and therefore also lower block triangular, in which case, $M_U(s) = I$. This case is therefore a simpler special case of the general situation discussed below. □

From (22)–(26) it follows that the multivariable sensitivity function $S(s) = (I + L(s))^{-1}$ can be written as a product of upper and lower block triangular matrices

$$\begin{aligned} S(s) =& M_U^{-1}(s) M_L^{-1}(s) \\ =& \begin{bmatrix} I & * & \dots & * \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & I \end{bmatrix} \\ &\times \begin{bmatrix} \tilde{S}_{1,1}(s) & 0 & \dots & 0 \\ \tilde{S}_{2,1}(s) & \tilde{S}_{2,2}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{S}_{N,1}(s) & \dots & \tilde{S}_{N,N-1}(s) & \tilde{S}_{N,N}(s) \end{bmatrix} \end{aligned} \tag{27}$$

where '$*$' denotes possibly non-zero transfer function blocks within the matrix $M_U^{-1}(s)$ and from (25) we have that for all $j > i$

$$\tilde{S}_{j,i}(s) = -\tilde{S}_{j,j}(s) L_{j,j-1}(s) \tilde{S}_{j-1,i}(s). \tag{28}$$

Then, selecting $j = N$ in (28), and recursively using this equation gives

$$\tilde{S}_{N,i}(s) = \tilde{S}_{N,N}(s) \prod_{k=i}^{N-1} \left(-L_{k+1,k}(s) \tilde{S}_{k,k}(s) \right). \tag{29}$$

From (27) and (29), the bottom block row of $S(s)$ satisfies

$$S_{N,i}(s) = \tilde{S}_{N,i}(s) = \tilde{S}_{N,N}(s) \prod_{k=i}^{N-1} \tilde{T}_{k+1,k} \tag{30}$$

where

$$\tilde{T}_{k+1,k} \triangleq -L_{k+1,k} \tilde{S}_{k,k} \quad \text{for } k = 1, 2, \dots, N-1. \tag{31}$$

Note from (30) that $\tilde{S}_{k,k}(s)$ is precisely the lower right hand block of the multivariable sensitivity function that results from a string of k vehicles. Therefore, if (for example) the individual controllers are not permitted to reconfigure themselves based on information about their position in the string, it is reasonable to restrict $\tilde{S}_{k,k}(s)$ as in the following assumption.

Assumption 6 (Uniform Bounds on $\tilde{S}_{k,k}$): There exists a finite number $\sigma > 0$ such that

$$\left\| \tilde{S}_{k,k}(s) \right\|_{\mathcal{H}_\infty} \leq \sigma \quad \text{for } k = 1, 2, \dots, N. \tag{32}$$

□

Note in (32) that necessarily $\sigma \geq 1$, which follows from the definition of $\tilde{S}_{k,k}(s)$ in Lemma 4 and the fact that $L(s)$ is strictly proper (Assumption 5a).

We now analyze the high frequency response of the system.

B. High Frequency Bounds on $H_{x_n d_1}$

Using block notation, and letting $(H_{xd})_{N,1}$ denote the lower leftmost $c_f \times c_f$ block of H_{xd}, we can write the transfer function from lead disturbance to the nth vehicle as

$$H_{x_n d_1}(s) = \left(V_{c_f}^{c_f} \right)^T (H_{xd})_{N,1} V_1^{c_f} \tag{33}$$

where $V_1^{c_f}, V_{c_f}^{c_f} \in \mathbb{R}^{c_f}$ are the 1st and c_fth elementary basis vectors. We then have the following result giving a bound on the frequency response of the state of the last vehicle in the platoon to a disturbance at the leading vehicle.

We make the following assumption on the behavior of the loop transfer functions.

Assumption 7 (Loop High Frequency Bound): The loop transfer functions $\tilde{T}_{k+1,k}(s)$ with $k \in \{1, 2, \dots, N-1\}$, defined in (31), obey the uniform high

$$\left\| \tilde{T}_{k+1,k}(j\omega) \right\| \leq \left(\frac{\omega_H}{\omega} \right)^r, \quad \text{for all } \omega > \omega_H \tag{34}$$

for some $\omega_H > 0$ independent of N and (relative degree) $r \geq 1$ and all $k \in \{1, 2, \dots, N-1\}$. In addition, we assume that $\forall \omega \geq \omega_H$

$$\left\| (PC)_{1,1}(j\omega) V_1^{c_f} \right\| \leq \bar{p} \tag{35}$$

for some $\bar{p} < \infty$. □

Note that uniform high frequency roll off in $L_{k+1,k}(s)$ together with (32) is sufficient to ensure the existence of constants ω_H and r such that (34) is satisfied. Also, provided any imaginary axis poles of the plant and controller have imaginary part less than ω_H, then the existence of a $\bar{p}$ that satisfies (35) is assured.

We then have the following bound on the high frequency tail of the integral of $H_{x_n d_1}$.

Corollary 5 (High Frequency Bound on $H_{x_n d_1}$): Under Assumptions 4, 5, 6 and 7

$$\int_{\omega_H}^{\infty} \log |H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \leq k_0 - k_1 n \tag{36}$$

where $k_0 = (1/\omega_H)(r + \log(\sigma \bar{p}))$ and $k_1 = (r / c_f \omega_H)$.

Proof: Note from (33) that $H_{x_n d_1}$ is one element of the lower left block of the transfer function, H_{xd}. Then from (12), (23) and Assumption 5(c), $H_{x_n d_1}$ is one element of $(H_{xd}(s))_{N,1} V_1^{c_f} = (S(s)(PC)(s))_{N,1} V_1^{c_f}$, where $(PC)(s) \triangleq P(s)C(s)$. Also, because of the banded structure of $(PC)(s)$ it follows that:

$$\begin{aligned} &(S(s)(PC)(s))_{N,1} V_1^{c_f} \\ &\quad = (S_{N,1}(s)(PC)_{1,1}(s) + S_{N,2}(s)(PC)_{2,1}(s)) V_1^{c_f} \\ &\quad = S_{N,1}(s)(PC)_{1,1}(s) V_1^{c_f} \\ &\quad = \tilde{S}_{N,N}(s) \left(\prod_{k=1}^{N-1} \tilde{T}_{k+1,k}(s) \right) \left((PC)_{1,1}(s) V_1^{c_f} \right) \end{aligned} \tag{37}$$

where the second equality in (37) follows since $(PC)_{2,1}$ is strictly upper triangular, and the third equality follows from (30).

Note first that by using Assumptions 7 and 6 in (37) it follows that for all $\omega \geq \omega_H$

$$|H_{x_n d_1}(j\omega)| \leq \sigma \bar{p} \left(\frac{\omega_H}{\omega} \right)^{r(N-1)}. \tag{38}$$

Then using (38) we have

$$\begin{aligned}\int_{\omega_H}^{\infty} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} &\leq \int_{\omega_H}^{\infty} \left(\log(\sigma\bar{p}) + (N-1)r\log\left(\frac{\omega_H}{\omega}\right)\right)\frac{d\omega}{\omega^2} \\ &= \frac{\log(\sigma\bar{p})}{\omega_H} + \left[\frac{(N-1)r}{\omega}\left(1 - \log\frac{\omega_H}{\omega}\right)\right]_{\omega_H}^{\infty} \\ &= \frac{r + \log(\sigma\bar{p})}{\omega_H} - \frac{Nr}{\omega_H}\end{aligned}$$

and (36) follows since $N = n/c_f$. ■

C. Time Domain Performance Specifications

We now consider specifications on the response at the last vehicle to a ramp input as the target trajectory for the first vehicle. This is governed by the transfer function $H_{x_n d_1}(s)$ as defined in (17). We impose an integral absolute error (IAE) specification on the response of the vehicle separations to a ramp disturbance to the first vehicle.

Assumption 8 (IAE Specification on Transient Response): For $k = 1, 2, \ldots, n$, let $e_k^{ramp}(t)$ be the error response of the kth vehicle to a unit ramp $d_1(t) = t$. We assume that for all n the integral of the absolute value of $e_k^{ramp}(t)$ is bounded as

$$\sum_{k=1}^{n} \int_0^{\infty} |e_k^{ramp}(t)|\, dt \leq n\bar{\alpha}(n) \triangleq \alpha(n) \tag{39}$$

for some positive function $\bar{\alpha}(n)$. □

If possible, a uniform bound in Assumption 8 ($\bar{\alpha}$ constant) would be preferable. However, this may be incompatible with other performance as we shall see later in Section IV.

One immediate consequence of Assumption 8 follows.

Lemma 6 (Low Frequency Bound on $H_{x_n d_1}$): Let Assumption 8 hold. Then, for all $\omega \in \mathbb{R}$

$$|H_{x_n d_1}(j\omega)| \leq 1 + \alpha(n)\omega^2. \tag{40}$$

Furthermore, for any $\omega_L > 0$

$$\begin{aligned}\int_0^{\omega_L} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \leq &-\frac{1}{\omega_L}\log\left(1 + \alpha(n)\omega_L^2\right) \\ &+ 2\sqrt{\alpha(n)}\tan^{-1}\left(\sqrt{\alpha(n)}\omega_L\right) \\ &\triangleq \eta\left(\alpha(n), \omega_L\right).\end{aligned} \tag{41}$$

Proof: From the definition of vehicle separations (7) and the expression (12) for $H_{x_n d_1}(s)$, we have

$$\underline{e}(t) = \left(I - (M + sH)H_{xd}\right)V_1^n \star d_1(t). \tag{42}$$

By letting $d_1(t) = t$ (that is, $\mathcal{L}\{d_1(t)\} = 1/s^2$), and taking the kth row of (42) we obtain

$$\begin{aligned}\int_0^{\infty} e^{-st} e_k^{ramp}(t)dt &= V_k^{nT}\left(I - (M + sH)H_{xd}\right)V_1^n \frac{1}{s^2} \\ &= \left((1 + h_k s)H_{x_k d_1}(s) - H_{x_{k-1} d_1}(s)\right) \\ &\quad \times \frac{1}{s^2}.\end{aligned} \tag{43}$$

Following the recursion in k in (43) yields

$$\begin{aligned}H_{x_n d_1}(s) = \prod_{\ell=1}^{n}(1 + h_\ell s)^{-1} &+ \sum_{k=1}^{n}\left(\prod_{\ell=k}^{n}(1 + h_\ell s)^{-1}\right) \\ &\times \left(s^2 \int_0^{\infty} e^{-st} e_k^{ramp}(t)dt\right).\end{aligned} \tag{44}$$

By evaluating (44) at $s = j\omega$ we obtain the bound

$$|H_{x_n d_1}(j\omega)| \leq 1 + \omega^2 \sum_{k=1}^{n} \int_0^{\infty} |e_k^{ramp}(t)|\, dt$$

from which (40) follows. Finally, using (40) we have

$$\begin{aligned}\int_0^{\omega_L} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} &\leq \int_0^{\omega_L} \log\left(1 + \alpha(n)\omega^2\right)\frac{d\omega}{\omega^2} \\ &= \left[-\frac{1}{\omega}\log\left(1 + \alpha(n)\omega^2\right)\right. \\ &\quad \left. + 2\sqrt{\alpha(n)}\tan^{-1}\left(\sqrt{\alpha(n)}\omega\right)\right]_0^{\omega_L}\end{aligned} \tag{45}$$

and (41) follows immediately. ■

We now state our main result: a lower bound on the worst-case disturbance amplification in terms of the communications range, high frequency behavior and transient behavior.

D. Main Theorem—Lower Bound on Disturbance Amplification

Theorem 7: Consider a system subject to Assumptions 1 to 8. Then for any $\omega_L \in (0, \omega_H)$

$$\begin{aligned}\max_{\omega \in [\omega_L, \omega_H]} &\log|H_{x_n d_1}(j\omega)| \\ &\geq \left(\frac{\omega_L \omega_H}{\omega_H - \omega_L}\right) \times \left(\beta(n) - \eta\left(\alpha(n), \omega_L\right)\right)\end{aligned} \tag{46}$$

with $\alpha(n)$ as in Assumption 8, and

$$\beta(n) = \frac{n\pi}{2}(h_c - \bar{h}) - k_0 \tag{47}$$

where h_c is the *critical average time headway* defined as

$$h_c \triangleq \frac{2r}{\pi \omega_H c_f}. \tag{48}$$

Proof: We establish this result by splitting the interval of integration in Lemma 3. In particular, from (18)

$$\int_{\omega_L}^{\omega_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\int_{0}^{\omega_L} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} - \int_{\omega_H}^{\infty} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} - \frac{n\pi}{2}\bar{h}. \quad (49)$$

Then, using (49) together with Corollary 5 and Lemma 6 we obtain

$$\int_{\omega_L}^{\omega_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\eta(\alpha(n), \omega_L) + \frac{1}{\omega_H}\left(\frac{nr}{c_f}\right) - k_0 - \frac{n\pi}{2}\bar{h}. \quad (50)$$

Also, we can derive for the left hand side of (50) the inequality

$$\begin{aligned}\int_{\omega_L}^{\omega_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} &\leq \max_{\omega\in[\omega_L,\omega_H]} \{\log|H_{x_n d_1}(j\omega)|\} \int_{\omega_L}^{\omega_H} \frac{d\omega}{\omega^2} \\ &= \left(\frac{\omega_H - \omega_L}{\omega_L\omega_H}\right) \max_{\omega\in[\omega_L,\omega_H]} \log|H_{x_n d_1}(j\omega)|. \quad (51)\end{aligned}$$

The result then follows by combining (50) and (51). ∎

We now turn to consider various consequences and interpretations of Theorem 7.

IV. Consequences of Theorem 7

A. Sufficient Conditions for Exponential Growth in Disturbance Amplification

Exponential growth in disturbance amplification for some classes of distributed control problems has been observed by a number of authors (e.g. [5], [12]), though this has generally been restricted to systems that involve one or more of homogeneous platoons, nearest neighbor communications, and no time headway. Here we extend these results by obtaining sufficient conditions for exponential growth that include heterogeneous strings, with limited range of communications and with sufficiently small time headway.

Corollary 8 (Sufficient Condition for Exponential Growth): Suppose in Theorem 7 that $\omega_H\alpha(n) > \beta(n) > 0$. Then

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| \geq \exp\left(\frac{\beta^2(n)}{2\alpha(n)}\right). \quad (52)$$

Furthermore, if $\bar{\alpha} \leq \alpha_1$, with $\alpha_1 > (\pi/2)(h_c - \bar{h})$ then

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| \geq \exp\left(n\frac{\pi^2}{8\alpha_1}(h_c - \bar{h})^2\right). \quad (53)$$

Proof: Firstly, we note from the definition of η [see (41)] that

$$\eta(\alpha(n), \omega_L) \leq \alpha(n)\omega_L. \quad (54)$$

Using (54) and (47) in Theorem 7 gives

$$\sup_{\omega} \log|H_{x_n d_1}(j\omega)| \geq \left(\frac{\omega_L\omega_H}{\omega_H - \omega_L}\right)(\beta(n) - \alpha(n)\omega_L) \quad (55)$$

for all $\omega_L \in (0, \omega_H)$. Under the conditions of Corollary 8, we can substitute $\omega_L = (\beta/2\alpha) < (\omega_H/2)$ in (55) and with some simple algebra obtain (52). ∎

The case where the condition $\omega_H\alpha(n) > \beta(n) > 0$ is not satisfied is covered in Corollary 9.

B. Infeasible Specifications

It turns out that in some cases, demands for certain types of high frequency and transient performance may be incompatible with communication and time headway constraints. This incompatibility can be demonstrated by proving that in such cases, the lower bound on the frequency response peak is infinite. Performance specifications in Assumptions 1, 4, 7 and 8 that are sufficient to guarantee an unbounded peak in the closed loop transfer function $H_{x_n d_1}(s)$ will be referred to as *infeasible*. The following corollary examines this situation.

Corollary 9 (Infeasible Specifications Test): Suppose that

$$\beta(n) > \eta(\alpha(n), \omega_H). \quad (56)$$

Then the specifications for communication ranges, uniform high frequency bounds and transient performance in Assumptions 4, 7 and 8 are infeasible in the sense that any closed loop stable system subject to the conditions of Theorem 7 satisfies

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| = +\infty. \quad (57)$$

Proof: Since (46) applies for all $\omega_L \in (0, \omega_H)$, under the condition (56), (57) follows directly from:

$$\lim_{\omega_L\to\omega_H^-} \left\{\left(\frac{\omega_L\omega_H}{\omega_H - \omega_L}\right) \times (\beta(n) - \eta(\alpha(n), \omega_L))\right\} = +\infty.$$

∎

We can add further interpretations on this result in special cases, as indicated in the following corollary.

Corollary 10 (Tradeoffs in Time Domain Performance): Suppose that for all n the average IAE transient specification (39) satisfies

$$\bar{\alpha}(n) \leq \alpha_2 n + \alpha_1 \quad (58)$$

for some non-negative constants α_1 and α_2, where

$$\alpha_2 < (h_c - \bar{h})^2/4. \quad (59)$$

Then, for sufficiently large n, the closed loop specifications are infeasible in the sense of Corollary 9.

Proof: Note that from the definition of $\eta(\alpha(n),\omega_L)$ in (41) that for any ω_L

$$\eta\left(\alpha(n),\omega_L\right) < \pi\sqrt{\alpha(n)}. \tag{60}$$

Then using (47), (60) and (58) we obtain

$$\beta(n)-\eta\left(\alpha(n),\omega_H\right) > n\pi\left(\frac{h_c-\bar{h}}{2}-\sqrt{\alpha_2}\right)-k_0-\pi\sqrt{n\alpha_1}. \tag{61}$$

Under the condition (59), the RHS of (61) is positive for sufficiently large n, and by Corollary 9 the specifications are infeasible. ■

Corollary 10 shows that to avoid infeasibility for sufficiently large n in a system with average time headway below h_c, the transient specification $(\alpha(n))$ of Assumption 8 must grow at least quadratically with n $(\alpha_2 > 0)$, with a minimum rate determined by the high frequency limit ω_H and the communications range c_f. That is, in rigid or semi-rigid formations, string instability can be avoided only at the expense of transient performance in the IAE sense.

C. Interpretations and Discussion

Several interpretations may be drawn from the analysis above.

1) Sufficient Time Headway May Avert String Instability: Other authors (e.g. [18]), in slightly less general settings, have shown that a sufficiently large time headway may avert string instability. From the previous analysis, if the average time headway satisfies

$$\bar{h} > h_c \tag{62}$$

then none of the results above demand growth in the disturbance response (or infeasibility of the specifications) with large n, regardless of the transient specification. In particular, (62) guarantees that the condition $\beta(n) > 0$ in Corollary 8 is false. Similarly, the condition (56) in Corollary 9 is false regardless of $\eta > 0$. Furthermore, the condition (59) in Corollary 10 is never satisfied.

Therefore, (62) is an important benchmark for time headway allowance in the design of distributed controllers for strings of dynamics systems.

2) Without Sufficient Time Headway (That is, $\bar{h} < h_c$), the Average Transient Performance Specification, $\bar{\alpha}(n)$ May Need to Grow at Least Linearly With n: Clearly from Corollary 10, under the other assumptions, to avoid string stability problems, we will require that for large n and for $\bar{h} < h_c$

$$\bar{\alpha}(n) \geq \left(\frac{h_c-\bar{h}}{2}\right)^2 n + O(1). \tag{63}$$

One of the implications of this result is that requiring a uniform bound on the average IAE in the response to a ramp may not be feasible for large strings, under the conditions of Theorem 7.

3) Factors That May Improve String Stability Properties: Apart from increasing the time headway as noted above, we can also identify a range of factors that the analysis indicates may allow improved string stability properties. These include:

a) *Improved Forward Communications Range*: Note that larger forward communications, c_f, directly reduces the rate of exponential growth (see for example Corollary 8) and indeed, may avert conditions that guarantee string instability. Reverse communications, on the other hand do not have any direct effect on the results. Note however that in some cases (see Example V-E) reverse communications have been observed to give rise to very long transient responses and in such cases, would demand relaxed time domain specifications, and therefore may indirectly avoid string stability problems.

b) *Increased Loop High Frequency Response*: Improvements in both the high frequency roll-off, ω_H, and reductions in the loop relative degree, r, are both seen to be beneficial in reducing the lower bound on performance.

c) *Relaxing the requirements on the low frequency transient performance*: Increasing the permissible IAE specification on transient performance, that is, increasing α, reduces the demands imposed by the 'waterbed effect' and thereby may permit improved string stability properties.

d) *System Nonlinearities*: The analysis above requires the formation of a single closed loop transfer function, and this is clearly not possible for many systems incorporating nonlinear control elements. Several schemes proposed in practice for distributed vehicle control incorporate a number of non-linear elements (see for example [16]) which may circumvent some of the difficulties described above. Of course, if the nonlinearities are sufficiently smooth, then local approximation by linear behavior may predict small-signal string instability using the above analysis.

Note however, that there is no indication in any of the results presented above that heterogeneous system design is advantageous. This is in contrast to some earlier results, where string stability is assured, but only by using a heterogeneous design that demands one or both of: (i) non-uniform high frequency bounds (for example, controller gains that increase with position within the string [19]), (ii) IAE performance that deteriorates polynomially with string length.

V. Examples

We present several simple examples illustrating the results above. Consider a string of N identical vehicles defined by the plant transfer functions

$$P_k(s) = \frac{1}{s}, \quad \text{for } k = 1,\ldots,N. \tag{64}$$

A. Homogeneous Predecessor Following, Constant Spacing

We first consider a simple control policy illustrating string instability. Suppose we use a constant spacing policy ($H = 0$ in Assumption 2) and homogeneous, fully decentralized (no communication between vehicles) control. In this case, it is well known that the system is string unstable (see for example [12]), and we briefly repeat some analysis of this string instability

$$C_k(s) = \frac{5s+1}{4s} \triangleq c(s), \quad \text{for } k = 1,\ldots,N. \tag{65}$$

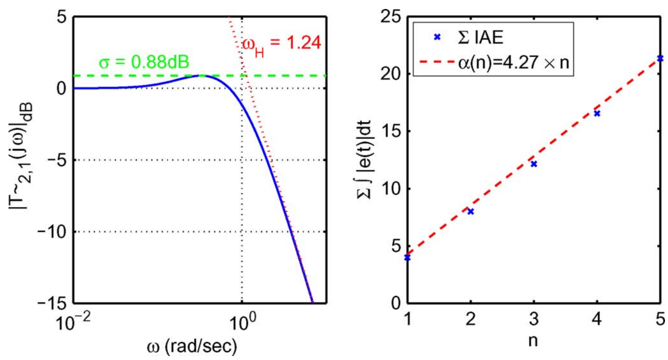

Fig. 3. Example Section V-A: string transfer function, $\tilde{T}(s)$ (left) and numerical evaluation of integral absolute error (right).

The control policy for each vehicle is fully decentralized, that is $u_k(t) = c(s) \star e_k(t)$, for $k = 2, \ldots, N$ that is, $C(s) = c(s)I$.

In this case, we can show that $\tilde{T}_{k+1,k}(s) = (5s+1/4s^2+5s+1)$ for $k = 1, \ldots, N$, and that this transfer function has relative degree $r = 1$, with $\omega_H = 1.24$, as shown in Fig. 3 (left).

If we examine the transient response for a small range of platoon sizes (e.g., $n = 1 \ldots 5$), we obtain integral absolute error values as illustrated in Fig. 3 (right).

If we temporarily[4] take the bound on the IAE response predicted from Fig. 3, that is, assume $\alpha(n) = 4.27n$, then from Corollary 8, we predict that

$$|H_{x_n d_1}(s)|_{\mathcal{H}_\infty} \geq \exp\left(\frac{\left(n\omega_H^{-1} - k_0\right)^2}{(2 \times 4.27n)}\right). \tag{66}$$

From (66), for large n, we predict string instability with a growth in the peak of the frequency response of at least a factor of $\exp(1/2 \times 4.27\omega_H^2)$ per vehicle, or equivalently, 0.66 dB per vehicle.

Note that in this particular case, this analysis could have more easily been performed, simply by evaluating $\|\tilde{T}_{k+1,k}\|_{\mathcal{H}_\infty}$, which is 0.88 dB, and it is clear that the peak in the disturbance response grows at this rate. The advantage of the analysis presented here, is that the lower bound on the rate of growth applies to any heterogeneous linear control scheme that is predecessor only, and that satisfies both the high frequency bound of Assumption 7 (with $r = 1$, $\omega_H = 1.24$), and the integral absolute error bound, Assumption 8 with $\alpha(n) = 4.27n$.

B. Homogeneous Predecessor Following, Time Headway

Since in Section V-A we clearly have string instability, we note from (62) in Section IV-C-1 that string stability may be achievable if we retain the high frequency bound, but introduce a time headway policy with $h > h_c = (2/\pi\omega_H) = 0.5134$.

Motivated by this, we find that we can achieve string stability, with a time headway policy and controller as follows. We take a time headway $h = 0.8$ sec and include additional dynamics (see Remark 1) of $H_X(s) = (h/(1+5s))$. We also modify

[4]Note that this system is in fact string unstable, and in this case, this exhibits itself in an exponential growth (for large n) in $\alpha(n)$.

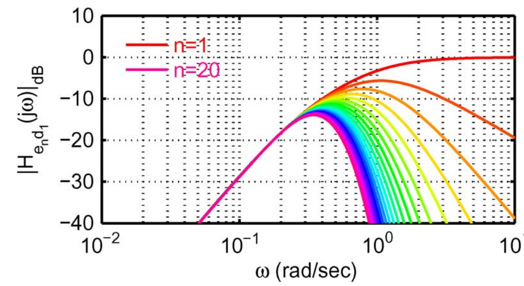

Fig. 4. Magnitude plot of $H_{e_n d_1}(j\omega)$ for example from Section V-B.

the controller design so that the individual closed loop poles are identical to those in Section V-A and take

$$C_k(s) = \frac{(5s+1)}{4s}\frac{(5s+1)}{(5.8s+1)}, \quad \text{for } k = 1, \ldots, N. \tag{67}$$

The control policy for each vehicle is fully decentralized, that is $u_k(t) = C_k(s) \star e_k(t) + sH_X(s) \star x_k(t)$, for $i = 2, \ldots, N$. In this case, we can show that

$$\tilde{T}_{k+1,k}(s) = \frac{(5s+1)}{(4s^2+5s+1)}\frac{(5s+1)}{((5+h)s+1)} \tag{68}$$

and $\|\tilde{T}_{k+1,k}\|_{\mathcal{H}_\infty} = 1$. This indicates, as shown in Fig. 4, string stability for this case.

C. Heterogeneous Predecessor Following, Constant Spacing

As an alternative to introducing a time headway policy, consider the use of a heterogeneous control policy, where the controller gains are permitted to depend on vehicle position within the string as follows:

$$C_k(s) = \frac{5s + g_I(k)}{4s}, \quad \text{for } k = 1, \ldots, N \tag{69}$$

where $g_I(k) = (1/k^2)$ is the integral gain of the controller for the kth vehicle in the string.[5]

The control policy for each vehicle is fully decentralized, that is $u_k(t) = C_k(s) \star e_k(t)$, for $k = 2, \ldots, N$, that is, $C(s) = C_k(s)I$. It can then be shown that

$$\tilde{T}_{k+1,k}(s) = \frac{5s + g_I(k+1)}{4s^2 + 5s + g_I(k)}, \quad \text{for } k = 1, \ldots, N-1 \ldots \tag{70}$$

The magnitude plot of the transfer function $\tilde{T}_{k+1,k}$ is shown in Fig. 5 (left). Note that it can be shown that $\|\tilde{T}_{k+1,k}(s)\|_{\mathcal{H}_\infty}^2 - 1 \sim O(k^{-2})$, and therefore, in view of (30) it follows that we have string stability. This is consistent with simulation studies for this case, where the error responses to a ramp set point are shown in Fig. 5 (right).

Note that the time response exhibits very slow settling as string length grows, and the sum of the integral absolute errors (as defined in Assumption 8) can be bounded by $\alpha(n) = 4n^2 + 1.2n^3$ or equivalently $\bar{\alpha}(n) = 4n + 1.2n^2$. Therefore

[5]Note that we chose not to alter the proportional gain in this case since: (i) increasing the proportional gain will increase the high frequency roll off, ω_H, with consequent reduction in high frequency stability robustness; and (ii) reduction of the proportional gain in this case results in poorer performance.

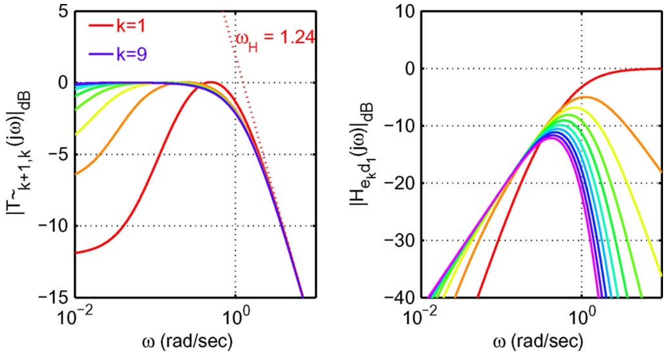

Fig. 5. Example Section V-C: magnitude of string transfer functions, $\tilde{T}_{k+1,k}(s)$ (left), and error response to a ramp setpoint (right).

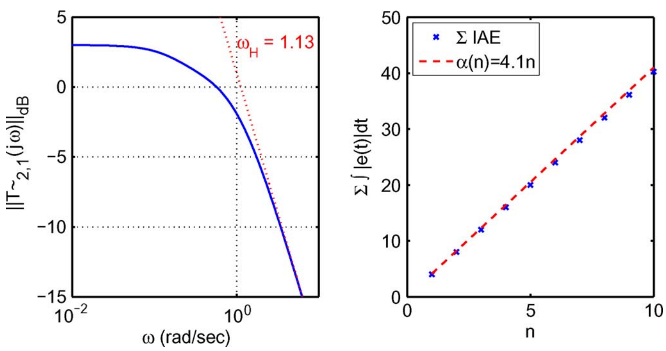

Fig. 6. Example Section V-D: $\|\tilde{T}_{k+1,k}(j\omega)\|$ (left), and integral absolute error (right).

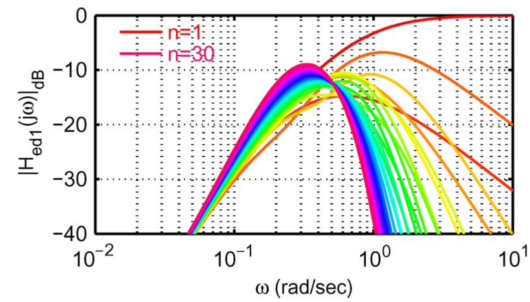

Fig. 7. $|H_{e_n d_1}(j\omega)|$ for example Section V-D.

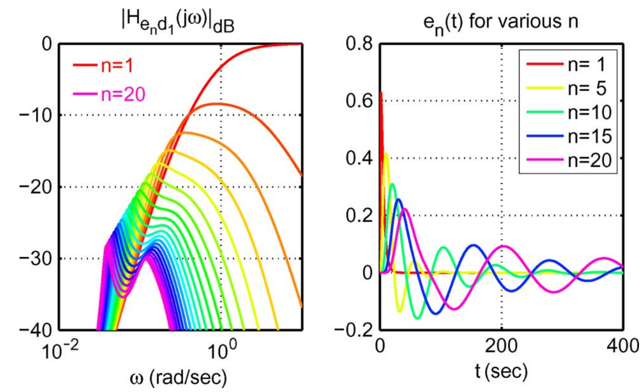

Fig. 8. Example error transfer functions (left) and transient error response (right).

we note that the average IAE grows quadratically with n, and therefore, as discussed in Section IV-C-2, this large growth in IAE may permit string stability. Note however, that to achieve this, some level of global coordination is required, since each controller gain depends on the position, k, within the string.

D. Homogeneous, Constant Spacing, Extended Forward Communications

We now turn to the effects of increased forward communications range. We return to the case of homogeneous control, but now with a forward communications range of 2. The particular control we consider is specified as

$$u_k(t) = \left(\frac{5s+1}{4s}\right) \star e_k(t) + e_{k-1}(t), \quad \text{for all } k = 2, \ldots, N. \tag{71}$$

This gives rise to a (2×2) multivariable transfer function $\tilde{T}_{k+1,k}(s)$ with magnitude illustrated in Fig. 6 (left).

Also, if we compute numerically the integral absolute error performance for this situation, we obtain (for small values of n) the bound $\alpha(n) = 4.1n$ as illustrated in Fig. 6 (right).

From (66), for large n, we predict string instability with a growth in the peak of the frequency response of at least a factor of $\exp(1/2 \times 4.1\omega_H^2 2^2)$ per vehicle, or equivalently, 0.21 dB per vehicle. Numerical evaluation of the frequency response gives the results shown in Fig. 7 which gives a disturbance amplification of 0.312 dB per vehicle.

E. Bidirectional Control

Bi-directional control has been discussed in references such as [7], [20] for improving string stability. Here we simulate the case where the control is symmetric, that is, includes equally weighted forward and reverse errors

$$u_k(t) = \left(\frac{5s+1}{4s}\right) \star \left(e_k(t) + e_{k+1}(t)\right), \quad \text{for all } k = 1, \ldots, n-1. \tag{72}$$

The symmetric bidirectional control scheme of (72) does give rise to limited disturbance amplification as illustrated in Fig. 8 (left). It can also be shown that this control strategy does satisfy a uniform high frequency roll-off bound (as in Assumption 7), however, since there is no time headway used, in view of the analysis of Section IV-C-2, clearly it must have an average IAE performance, $\bar{\alpha}$ that grows at least linearly with n. The increase in settling time with string length is illustrated in in Fig. 8 (right), where we see that even for relatively modest string lengths, a slow very lightly damped oscillatory mode is present in the transient response. Other simulations (not shown here) confirm the results of [20], that 'mistuning' (or asymmetric) bidirectional control improves the settling response of the string, but simultaneously give unbounded disturbance amplification.

VI. Conclusion

This paper reexamines and expands the string instability analysis presented in [12]. The analysis in the present paper includes heterogeneous, non-zero time headway, limited communication range systems, and shows that:

1) System heterogeneity, within reasonable confines of bounded high frequency response and IAE performance restrictions, does not circumvent string instability.
2) Extra, though limited, forward communication range does not avoid string stability problems in a qualitative sense. It can, however, significantly reduce the rate of disturbance amplification.
3) Relaxing a rigid formation control policy, to allowing a small time headway, does not qualitatively alter the string instability results, though it does reduce the rate of disturbance amplification. A sufficiently large time headway may permit string stability.
4) Bidirectional control, or reverse communication, appears to offer an advantage in terms of string stability. However, this improvement is necessarily at the cost of very long transients as string length grows. More specifically, all else being equal, the average integral absolute value of the error in response to a ramp grows at least linearly with string length, if bidirectional control is to be used to avoid string instability.

Topics for future research include studies of the potential advantages of nonlinear and/or time-varying control schemes. In addition, there is a need to extend the analysis to more general graph structures, with high order dynamics at each node.

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