H-infinity filter for bilinear systems using LPV approach

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$\mathcal{H}_\infty$ filter for bilinear systems using LPV approach

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Abstract

The aim of this paper is to present an LPV (Linear Paramater Varying) approach for the design of a functional filter for bilinear systems with a disturbance attenuation specification. The unbiasedness requirement on the observation error is guaranteed by the solution of Sylvester equations obtained by parametrizing the filter gain matrix. This parametrization leads to a non convex optimization problem which is overcome by introducing a constraint on the gain matrix. To take into account the whole set of the inputs in the filter design, an LPV approach is used to obtain the gain matrix guaranteeing the quadratic stability of the observation error and a given level of disturbance attenuation. This approach is then applied to the high gain observer design in order to consider the level of disturbance attenuation.

Keywords : Bilinear systems, observability, functional filter, high gain filter, $\mathcal{H}_\infty$ performance, LPV approach.

1 Introduction

As many physical processes may be appropriately modeled as bilinear systems, such systems have attracted an increasing attention of many researchers as in [21, 16]. The main reason is that some important actual processes, as nuclear kinetics, cannot be modeled realistically by the classical linear systems, while bilinear systems fit these processes with more accuracy. Furthermore, bilinear systems offer considerable intrinsic theoretical problems since they form an intermediary class between the linear and the general nonlinear systems.

For the study of observability and the design of observers for bilinear systems, the influence of control inputs is crucial [27]. Some observers can have a linear estimation error [14], [13] and [28]. It has been shown in [13] and [28] that these observers are equivalent to an unknown input observer [6] for a specific linear system, then these observers have strong existence conditions. When bilinear systems are uniformly observable, Williamson [27] has proposed observers requiring differentiators of control inputs, while in [2] the observers need on-line integration of a differential Riccati matrix equation. Gauthier *et al.* [11] suggested to use the canonical form of an observable bilinear system to design a straightforward high gain observer. In [3], the authors propose to modified the high gain observer methodology by writing the observer gain as a function of the state estimate. In [20], a high gain observer is computed using LMIs (Linear Matrix Inequalities) techniques. When the bilinear system is not uniformly observable, the observers proposed in the literature include explicitly the values of the bounds of the control inputs (see [7], [24], [26], [17], [18], [23]). These observers

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are appropriated since the bilinear system is detectable for all admissible inputs, i.e. the unobservable states with respect to some non universal inputs (see [15]) is stable.

In this work, the control inputs and their derivatives are considered to be bounded, this assumption is generally not too conservative for the state estimation of physical systems. With the hypothesis of boundedness, an LPV approach can be used as in [12]. Indeed, considering the control inputs as a varying parameter allows to look for a bilinear filter adapted to the whole domain of reachability associated to the system dynamics. The LPV approach enables to introduce performance specifications like disturbance attenuation in the filter design. In this paper, a new approach is presented to solve the functional filter problem, which is less conservative than the one given in [23] where the input was considered as uncertainties. Even if LPV approach is classical, parametrization of observers for bilinear systems with this approach hasn't been done yet according to our knowledge. Moreover, it is shown that the LPV approach provides a methodology enabling to choose the tuning parameter to optimize the disturbance attenuation in the high gain observer design.

This paper is organized as follows. First the formulation of the problem is presented in Section 2. In Section 3, for the functional filter, we present two parametrizations of the filter matrices and the LPV approach is used to determine the gain matrix ensuring the quadratic stability of the observation error and optimizing the disturbance attenuation. Then in Section 4, a "high gain"-like observer dedicated to uniformly observable system is studied with the LPV approach which enables to ensure the stability of the filtering error and to optimize the disturbance attenuation contrary to classical "high gain" approach. Simulation results are shown in Section 5 and conclusions are drawn in Section 6.

Notations. Throughout this paper, $\|x\| = \sqrt{x^T x}$ is the Euclidean vector norm. $A^\dagger$ is a generalized inverse of matrix A satisfying $A = AA^\dagger A$ [19] and $(*)$ represents a term induced by symmetry.

2 Problem statement

Let us consider bilinear systems of the following form

$$\begin{cases} \dot{x} = A^0 x + \sum_{i=1}^{m} A^i u^i x + Ru + Bw \\ y = Cx + Dw \\ z = Lx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u = \left[\begin{array}{ccc} u^1 & \dots & u^m \end{array}\right]^T \in \mathbb{R}^m$ is the vector of control inputs, y$\in \mathbb{R}^p$ is the output. The vector $w \in \mathbb{R}^q$ represents the unknown disturbance vector with bounded energy. $z \in \mathbb{R}^r$ is the vector to be estimated where $r \leqslant n$. As usual, let us consider $\widehat{z} \in \mathbb{R}^r$ a given estimate of z (given in sections 3 and 4) and $e \in \mathbb{R}^r$ is the filtering error with $e = z - \widehat{z}$. Without loss of generality, the control inputs are assumed to be bounded, *i.e.* $u \in \Omega$ where

$$\Omega = \left\{ u : t \to \mathbb{R}^m \mid \forall t \in \mathbb{R}^+,\ u^i_{\min} \leqslant u^i \leqslant u^i_{\max},\ \mu^i_{\min} \leqslant \dot{u}^i \leqslant \mu^i_{\max},\ i = 1, \dots, m \right\}.$$

To characterize the disturbance attenuation, we introduce the following definition that can be seen as a generalization of the $\mathcal{H}_\infty$ norm for linear systems to nonlinear ones (see [25]).

Definition 1. *Let $\gamma > 0$, if $\forall\, w \in \mathcal{L}_2[0,\infty)$, with zero initial conditions,*

$$J_{ew} = \int_0^\infty \left(\|e\|^2 - \gamma^2 \|w\|^2 \right) dt \leqslant 0, \tag{2}$$

then the mapping from w to e is said to have $\mathcal{L}_2$ gain less than or equal to γ. ■

The design of the filter for bilinear system (1) is stated as follows.

Problem 1. *In this paper, the problem is to design a filter such that*

1. *the filtering error e is quadratically stable for $u : t \mapsto \mathbb{R}^m \in \Omega$ and $w = 0$,*
2. *the mapping from the disturbance input w to the filtering error e has $\mathcal{L}_2$ gain less than a given scalar γ for $u \in \Omega$ (see definition 1).* ■

Before designing the filter, we recall the following bounded real lemma for LPV system.

Lemma 1. *[10, 4] The LPV system*

$$\begin{cases} \dot{e} = \mathbb{A}(u)e + \mathbb{B}(u)w \\ s = \mathbb{C}e + \mathbb{D}w \end{cases} \tag{3}$$

with $u \in \Omega$ is quadratically stable and has a disturbance attenuation inferior to a given scalar γ if there exists a matrix $P(u) = P(u)^T > 0$ and a matrix F such that the following inequality

$$\begin{bmatrix} (1,1) & P(u) - F + \mathbb{A}^T(u)F^T & F\mathbb{B}(u) & \mathbb{C}^T \\ * & -F - F^T & F\mathbb{B}(u) & 0 \\ * & * & -\gamma^2 I & \mathbb{D}^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{4}$$

is satisfied $\forall\ u \in \Omega$, with $(1,1) = \dot{P}(u) + F\mathbb{A}(u) + \mathbb{A}^T(u)F^T$. ▽

3 Functional LPV filter design with gain parametrization

In this section, for bilinear systems in the general form (1), we propose the following functional filter

$$\begin{cases} \dot{\eta} = H^0\eta + \sum_{i=1}^m H^i u^i \eta + J^0 y + \sum_{i=1}^m J^i u^i y + Gu \\ \widehat{z} = \eta + Ey \end{cases} \tag{5}$$

where $\eta \in \mathbb{R}^r$ is the state vector of the filter and $\widehat{z} \in \mathbb{R}^r$ is the estimate of z (with $r \leqslant n$). The filtering error is defined as follows

$$e = z - \widehat{z} = Lx - \widehat{z} = \overline{e} - EDw \tag{6}$$

where $\overline{e} = \Psi x - \eta$ and

$$\Psi = L - EC. \tag{7}$$

From (6), note that the time derivative of the error e is a function of the time derivative of the disturbances w. To avoid the use of $\dot{w}$, we introduce the operator from w to e with the following state space realization

$$\begin{cases} \dot{\overline{e}} = (H^0 + \sum_{i=1}^m u^i H^i)\overline{e} + (\Psi R - G)u + (\Psi A^0 - H^0\Psi - \Upsilon^0 C - H^0 EC)x \\ \qquad\qquad + \sum_{i=1}^m (\Psi A^i - H^i\Psi - \Upsilon^i C - H^i EC)u^i x \\ \qquad\qquad + (LB - ECB - \Upsilon^0 D - H^0 ED - \sum_{i=1}^m (\Upsilon^i D + H^i ED)u^i)w \\ e = \overline{e} - EDw \end{cases} \tag{8}$$

with

$$\Upsilon^i = J^i - H^i E \text{ for } i = 0, \ldots, m. \tag{9}$$

To satisfy the stability condition required in problem 1, the observer must be unbiased *i.e.* the dynamic of the filtering error must be independant of the state x and u (see [22], p. 176). Then the following Sylvester equations

$$\Psi A^i - H^i \Psi - \Upsilon^i C - H^i E C = 0 \qquad i = 0, \ldots, m \tag{10}$$

must hold and matrix G must be chosen as

$$G = \Psi R. \tag{11}$$

These Sylvester equations are related to unbiased observer. Using (7), equation (10) becomes [5]

$$LA^i - ECA^i - H^i L - \Upsilon^i C = 0 \qquad i = 0, \ldots, m. \tag{12}$$

Notice that equation (12) can be rewritten in the following compact form

$$\mathcal{N}\mathcal{F} = A_L \tag{13}$$

where $\mathcal{N}$, $\mathcal{F}$ and A_L are given by

$$\mathcal{N} = \begin{bmatrix} E & H^0 & \ldots & H^m & \Upsilon^0 & \ldots & \Upsilon^m \end{bmatrix}, \quad \mathcal{F}^T = \begin{bmatrix} A_C^T & \widetilde{L}^T & \widetilde{C}^T \end{bmatrix}, \quad A_L = L\widetilde{A}, \tag{14}$$

with

$$\widetilde{A} = \begin{bmatrix} A^0 & \ldots & A^m \end{bmatrix}, \; A_C = C\widetilde{A}, \; \widetilde{L} = \begin{bmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{bmatrix}, \; \widetilde{C} = \begin{bmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{bmatrix}.$$

Equation (13) admits a solution $\mathcal{N}$ if and only if the following condition is satisfied [19]

$$\operatorname{rank} \begin{bmatrix} A_L^T & \mathcal{F}^T \end{bmatrix}^T = \operatorname{rank} \mathcal{F} \tag{15}$$

and the general solution of (13) is given by [19]

$$\mathcal{N} = A_L \mathcal{F}^\dagger + Z(I_\alpha - \mathcal{F}\mathcal{F}^\dagger) \tag{16}$$

where Z is an arbitrary matrix parameter with appropriate dimension and $\alpha = p + (m+1)(r+p)$. From equation (16), matrices E, H^i and Υ^i are given by

$$E = E_1 + ZE_2, \; H^i = H_1^i + ZH_2^i, \; \Upsilon^i = \Upsilon_1^i + Z\Upsilon_2^i, \tag{17}$$

with

$$\begin{array}{lll} E_1 = A_L \mathcal{F}^\dagger M_E, & H_1^i = A_L \mathcal{F}^\dagger M_{H^i}, & \Upsilon_1^i = A_L \mathcal{F}^\dagger M_{\Upsilon^i}, \\ E_2 = (I_\alpha - \mathcal{F}\mathcal{F}^\dagger) M_E, & H_2^i = (I_\alpha - \mathcal{F}\mathcal{F}^\dagger) M_{H^i}, & \Upsilon_2^i = (I_\alpha - \mathcal{F}\mathcal{F}^\dagger) M_{\Upsilon^i} \end{array}$$

and

$$M_{H^i} = \begin{bmatrix} 0_{p\times r}^T & 0_{(i-1)r\times r}^T & I_r & 0_{(m+1-i)r\times r}^T & 0_{(m+1)p\times r}^T \end{bmatrix}^T,$$
$$M_{\Upsilon^i} = \begin{bmatrix} 0_{p\times p}^T & 0_{(m+1)r\times p}^T & 0_{(i-1)p\times p}^T & I_p & 0_{(m+1-i)p\times p}^T \end{bmatrix}^T,$$

$$M_E = \begin{bmatrix} I_p & 0^T_{(m+1)r\times p} & 0^T_{(m+1)p\times p} \end{bmatrix}^T.$$

In the sequel of this paper, it is assumed that the condition of unbiasedness (15) is fulfilled. Using (11) and (17), system (8) can be rewritten as follows

$$\begin{cases} \dot{\overline{e}} = (H_1^0 + ZH_2^0 + \sum_{i=1}^{m} u^i(H_1^i + ZH_2^i))\overline{e} \\ \qquad +(LB - (E_1 + ZE_2)CB - (\Upsilon_1^0 + Z\Upsilon_2^0)D - \underbrace{(H_1^0 + ZH_2^0)(E_1 + ZE_2)D}_{\Gamma^0} \\ \qquad -\sum_{i=1}^{m}((\Upsilon_1^i + Z\Upsilon_2^i)D + \underbrace{(H_1^i + ZH_2^i)(E_1 + ZE_2)D}_{\Gamma^i})u^i)w \\ e = \overline{e} - (E_1 + ZE_2)Dw. \end{cases} \tag{18}$$

Notice that Γ^i is quadratic in the gain Z, *i.e.* we have quadratic terms such ZH_2^iZ. To use an LMI-LPV approach based on lemma 1 in the filter design, the expressions of Γ^i in (18) must be linear in the gain matrix Z. This linearization can be made by introducing one of the two following constraints which lead to two different parametrizations of the gain matrix Z : $ZE_2D = 0$ (section 3.1) and $ED = 0$ (section 3.2).

3.1 First case : filter design with the constraint $ZE_2D = 0$

The matrix parameter Z is chosen such that

$$ZE_2D = 0. \tag{19}$$

The matrix parameter Z is then given by $Z = Z_1\left(I_\alpha - E_2D(E_2D)^\dagger\right)$, where Z_1 is an arbitrary matrix of appropriate dimensions, so Γ^i becomes (see (18))

$$\Gamma^i = (H_1^i + Z_1\left(I_\alpha - E_2D(E_2D)^\dagger\right)H_2^i)E_1)D \tag{20}$$

and equations (17) can be written as

$$E = E_{11} + Z_1\,E_{21},\ H^i = H_{11}^i + Z_1H_{21}^i,\ \Upsilon^i = \Upsilon_{11}^i + Z_1\,\Upsilon_{21}^i \tag{21}$$

where $E_{11} = E_1$, $H_{11}^i = H_1^i$, $\Upsilon_{11}^i = \Upsilon_1^i$, $E_{21} = \Phi M_E$, $H_{21}^i = \Phi M_{H^i}$ and $\Upsilon_{21}^i = \Phi M_{\Upsilon^i}$, with

$$\Phi = \left(I_\alpha - E_2D(E_2D)^\dagger\right)(I_\alpha - \mathcal{F}\,\mathcal{F}^\dagger). \tag{22}$$

Taking the constraint (19) into account, the system (18) is parametrized as follows

$$\begin{cases} \dot{\overline{e}} = (A_{u1}(u) + Z_1A_{z1}(u))\overline{e} + (B_{u1}(u) + Z_1B_{z1}(u))w \\ e = \overline{C}\overline{e} + \overline{D}_1w \end{cases} \tag{23}$$

where

$$A_{u1}(u) = H_{11}^0 + \sum_{i=1}^{m} u^iH_{11}^i, \overline{C} = I_r, \tag{24a}$$

$$A_{z1}(u) = H_{21}^0 + \sum_{i=1}^{m} u^iH_{21}^i, \overline{D}_1 = -E_{11}D \tag{24b}$$

$$B_{u1}(u) = LB - E_{11}CB - \left(\Upsilon_{11}^0 + \sum_{i=1}^{m} u^i \Upsilon_{11}^i + \left(H_{11}^0 + \sum_{i=1}^{m} u^i H_{11}^i\right) E_{11}\right) D, \quad (24c)$$

$$B_{z1}(u) = -E_{21}CB - \left(\Upsilon_{21}^0 + \sum_{i=1}^{m} u^i \Upsilon_{21}^i + \left(H_{21}^0 + \sum_{i=1}^{m} u^i H_{21}^i\right) E_{11}\right) D. \quad (24d)$$

To reduce the conservatism in the choice of the Lyapunov matrix $P(u)$ in lemma 1, we consider that matrix $P(u)$ has the following structure similar to the structure of the bilinear system (1), where P_i are constant matrices

$$P(u) = P^0 + \sum_{i=1}^{m} u^i P^i, \dot{P}(u) = \sum_{i=1}^{m} \dot{u}^i P^i \quad (25)$$

and we define the following vector ρ

$$\rho^T = \begin{bmatrix} \rho^1 & \dots & \rho^m & \rho^{m+1} & \dots & \rho^{2m} \end{bmatrix}^T = \begin{bmatrix} u^1 & \dots & u^m & \dot{u}^1 & \dots & \dot{u}^m \end{bmatrix}^T. \quad (26)$$

Using (25) and (26), we can define new parameter dependent matrices $\widetilde{P}$, $\overline{P}$, $\widehat{A}_{\rho 1}$, $\widehat{A}_{z1}$, $\widehat{B}_{\rho 1}$ and $\widehat{B}_{z1}$ as follows

$$\widetilde{P}(\rho) = P^0 + \sum_{i=1}^{m} \rho^i P^i = P(u), \quad (27a)$$

$$\overline{P}(\rho) = \sum_{i=1}^{m} \rho^{m+i} P^i = \dot{P}(u), \quad (27b)$$

$$\widehat{A}_{\rho 1}(\rho) = H_{11}^0 + \sum_{i=1}^{m} \rho^i H_{11}^i = A_{u1}(u), \quad (27c)$$

$$\widehat{A}_{z1}(\rho) = H_{21}^0 + \sum_{i=1}^{m} \rho^i H_{21}^i = A_{z1}(u), \quad (27d)$$

$$\widehat{B}_{\rho 1}(\rho) = LB - E_{11}CB - \left(\Upsilon_{11}^0 + \sum_{i=1}^{m} \rho^i \Upsilon_{11}^i + \left(H_{11}^0 + \sum_{i=1}^{m} \rho^i H_{11}^i\right) E_{11}\right) D = B_{u1}(u), \quad (27e)$$

$$\widehat{B}_{z1}(\rho) = -E_{21}CB - \left(\Upsilon_{21}^0 + \sum_{i=1}^{m} \rho^i \Upsilon_{21}^i + \left(H_{21}^0 + \sum_{i=1}^{m} \rho^i H_{21}^i\right) E_{11}\right) D = B_{z2}(u). \quad (27f)$$

Thus, one can see that the parameter ρ belongs to the following convex polytope $\mathcal{P}$

$$\mathcal{P} = [u_{\min}^1, u_{\max}^1] \times \ldots \times [u_{\min}^m, u_{\max}^m] \times \left[\mu_{\min}^1, \mu_{\max}^1\right] \times \ldots \times [\mu_{\min}^m, \mu_{\max}^m]. \quad (28)$$

Let $\mathcal{S}$ be the set of $\nu = 2^{2m}$ vertices of polytope $\mathcal{P}$ given by

$$\mathcal{S} = \Big\{ \beta = \left[\beta^1 \ldots \beta^i \ldots \beta^{2m}\right]^T \in \mathbb{R}^{2m} \mid \forall\, i \in [1, m], \\ \beta^i \in \left\{u_{\min}^i, u_{\max}^i\right\} \text{ and } \forall\, i \in [m+1, 2m], \beta^i \in \left\{\mu_{\min}^i, \mu_{\max}^i\right\} \Big\}. \quad (29)$$

Using notation and definition of (27), the following theorem gives the gain matrix Z_1 used in (23) through LMIs by using the information on the control inputs, their derivatives and the structure of the system.

Theorem 1. *Assume that condition* (15) *holds. If there exist matrices* $P^i \in \mathbb{R}^{r\times r}$ *(for* $i = 0, \ldots, m$*),* $F \in \mathbb{R}^{r\times r}$ *and* $Y \in \mathbb{R}^{r\times\alpha}$ *such that, for* $j = 1, \ldots, \nu$*,* $\widetilde{P}(\overline{\beta}^j) = \widetilde{P}(\overline{\beta}^j)^T > 0$ *and*

$$\begin{bmatrix} (1,1)^j & (1,2)^j & F\widehat{B}_{\rho 1}(\overline{\beta}^j) + Y\widehat{B}_{z1}(\overline{\beta}^j) & I_r \\ * & -F - F^T & F\widehat{B}_{\rho 1}(\overline{\beta}^j) + Y\widehat{B}_{z1}(\overline{\beta}^j) & 0 \\ * & * & -\gamma^2 I_q & -\overline{D}_1^T \\ * & * & * & -\overline{C} \end{bmatrix} < 0, \tag{30}$$

where

$$\begin{aligned} (1,1)^j &= \overline{P}(\overline{\beta}^j) + F\widehat{A}_{\rho 1}(\overline{\beta}^j) + Y\widehat{A}_{z1}(\overline{\beta}^j) + \widehat{A}_{\rho 1}^T(\overline{\beta}^j)F^T + \widehat{A}_{z1}^T(\overline{\beta}^j)Y^T, \\ (1,2)^j &= \widetilde{P}(\overline{\beta}^j) - F + \widehat{A}_{\rho 1}^T(\overline{\beta}^j)F^T + \widehat{A}_{z1}(\overline{\beta}^j)^T Y^T \end{aligned}$$

and $\overline{\beta}^j \in \mathcal{S}$ *(see* (29)*), then the filter* (5) *for the bilinear system* (1) *has a filtering error* e *which is quadratically stable and a* $\mathcal{L}_2$ *gain attenuation from* w *to* e *less than* γ*, with the gain matrix* Z_1 *given by* $Z_1 = F^{-1}Y$*.* •

Proof. Under the condition (15), the system (23) represents the filtering error of the filter (5). If the LMIs (30) has a solution for each element $\overline{\beta}^j$ of $\mathcal{S}$ given by the equation (29) (see [1]), then $F + F^T > 0$ and F is invertible, thus Z_1 can be determined.

Using $Y = FZ_1$, if LMIs (30) are satisfied on the ν vertices of polytope $\mathcal{S}$ then the inequality (4) in lemma 1 holds with the system (3) replaced by the system (23).

Using lemma 1, the filtering error e of filter (5) is quadratically stable and a $\mathcal{L}_2$ gain attenuation from w to e less than γ is guaranteed for $u \in \Omega$. □

Once the gain Z_1 is computed, filter matrices H^i, Υ^i and E are given by (21), then J^i is derived from (9) and G is given by (11).

From the above results, the $\mathcal{H}_\infty$ filter design problem is reduced to find a parameter matrix Z_1 to stabilize the system (23) and to guarantee the $\mathcal{L}_2$ gain attenuation between w and e. Notice that in order to have the degrees of freedom provided by the gain matrix Z_1, matrix Φ in (22) must satisfy $\mathrm{Im}(Z_1) \not\subset \ker(\Phi)$ and particularly $\Phi \neq 0$. Generally the problem 1 may be solved even if $\Phi = 0$; in fact LMIs (30) can have a solution even if $\Phi = 0$. The following theorem gives a sufficient condition for $\Phi \neq 0$.

Theorem 2. *Let* Φ *be given by* (22) *and* q*,* p*,* m*,* r*,* n *be the dimensions of vectors* w*,* y*,* u*,* z *and* x *respectively. If* $q < p + (m+1)(p+r-n)$*, then* $\Phi \neq 0$*.* ■

Proof. $\mathrm{Im}((I_\alpha - \mathcal{F}\mathcal{F}^\dagger)^T) = \ker(\mathcal{F}^T)$ so

$$\dim(\mathrm{Im}(I_\alpha - \mathcal{F}\mathcal{F}^\dagger)) \geqslant \alpha - (m+1)n. \tag{31}$$

Since $\ker(I_\alpha - (E_2D(E_2D)^\dagger) = \mathrm{Im}(E_2D)$, we have

$$\dim(\ker(I_\alpha - (E_2D(E_2D)^\dagger)) = \mathrm{rank}(E_2D) \leqslant q. \tag{32}$$

With $\alpha = p + (m+1)(r+p)$, if $q < p + (m+1)(p+r-n)$ holds, then (31) and (32) yield $\mathrm{Im}(I_\alpha - \mathcal{F}\,\mathcal{F}^\dagger) \not\subset \ker\left(I_\alpha - E_2D(E_2D)^\dagger\right)$, which implies that $\Phi \neq 0$. □

Remark 1. *As* n*,* m*,* p *and* q *are fixed by the structure of the system, the only way to increase the possibilities to have* $\Phi \neq 0$ *is to increase* r*, i.e. increase the dimension of the filter.* ◇

3.2 Second case : filter design with the constraint $ED = 0$

Now, the matrix parameter Z is chosen such that $ED = 0$, then we have $\Gamma^i = 0$ (see (18)). From $ED = 0$ and equation (13), we obtain

$$\overline{\mathcal{N}}\ \overline{\mathcal{F}} = [\ 0\ \ A_L\ ] = \overline{A}_L \tag{33}$$

with $\overline{\mathcal{F}} = [\ \mathcal{D}\ \ \mathcal{F}\ ]$ where $\mathcal{D}^T = \left[\ D^T\ \ 0\ \ 0\ \right]$. Equation (33) has a solution $\overline{\mathcal{N}}$ if and only if

$$\operatorname{rank}\begin{bmatrix} 0 & A_L \\ \mathcal{D} & \mathcal{F} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathcal{D} & \mathcal{F} \end{bmatrix}. \tag{34}$$

Under condition (34), all the solutions $\overline{\mathcal{N}}$ are given by [19]

$$\overline{\mathcal{N}} = \overline{A}_L\overline{\mathcal{F}}^\dagger + Z_2(I_\alpha - \overline{\mathcal{F}}\,\overline{\mathcal{F}}^\dagger) \tag{35}$$

where Z_2 is an arbitrary matrix of appropriate dimension.

Matrices E, H^i and Υ^i are then given by

$$E = E_{12} + Z_2E_{22},\ H^i = H^i_{12} + Z_2H^i_{22},\ \Upsilon^i = \Upsilon^i_{12} + Z_2\Upsilon^i_{22}, \tag{36}$$

where

$$\begin{aligned} E_{12} &= \overline{A}_L\overline{\mathcal{F}}^\dagger M_E, & H^i_{12} &= \overline{A}_L\overline{\mathcal{F}}^\dagger M_{H^i}, & \Upsilon^i_{12} &= \overline{A}_L\overline{\mathcal{F}}^\dagger M_{\Upsilon^i}, \\ E_{22} &= (I_\alpha - \overline{\mathcal{F}}\,\overline{\mathcal{F}}^\dagger)M_E, & H^i_{22} &= (I_\alpha - \overline{\mathcal{F}}\,\overline{\mathcal{F}}^\dagger)M_{H^i}, & \Upsilon^i_{22} &= (I_\alpha - \overline{\mathcal{F}}\,\overline{\mathcal{F}}^\dagger)M_{\Upsilon^i}. \end{aligned}$$

Using (36), the system (18) becomes

$$\begin{cases} \dot{\overline{e}} = (A_{u2}(u) + Z_2A_{z2}(u))\overline{e} + (B_{u2}(u) + Z_2B_{z2}(u))w \\ e = \overline{C}\overline{e} + \overline{D}_2w \end{cases} \tag{37}$$

where

$$A_{u2}(u) = H^0_{12} + \sum_{i=1}^{m} u^iH^i_{12}, \overline{C} = I_r, \tag{38a}$$

$$A_{z2}(u) = H^0_{22} + \sum_{i=1}^{m} u^iH^i_{22}, \overline{D}_2 = 0, \tag{38b}$$

$$B_{u2}(u) = LB - E_{12}CB - \left(\Upsilon^0_{12} + \sum_{i=1}^{m} u^i\Upsilon^i_{12}\right)D, \tag{38c}$$

$$B_{z2}(u) = -E_{22}CB - \left(\Upsilon^0_{22} + \sum_{i=1}^{m} u^i\Upsilon^i_{22}\right)D. \tag{38d}$$

From the above developments, the $\mathcal{H}_\infty$ filter synthesis is equivalent to the determination of Z_2 stabilizing the system (37) and ensuring the $\mathcal{L}_2$ gain attenuation between w and e. This can be obtained from theorem 1 by replacing the rank condition (15) by the rank condition (34) and system (23) by system (37).

Remark 2. *Similarly to the case where $ZE_2D = 0$, to have degrees of freedom introduced by gain matrix Z_2, we must have $(I_\alpha - \overline{\mathcal{F}}\,\overline{\mathcal{F}}^\dagger) \neq 0$.* ◇

4 LPV approach for the design of a "high gain"-like filter for bilinear systems

To show the efficiency of the LPV filter design given in section 3, we propose to apply the well-known high gain filter [11] to the bilinear system (1). This filter is based on the choice of a parameter called θ in the literature. In this section, we employ the expression "high gain"-like filter since we use an LPV approach to design θ in order to introduce the disturbance attenuation criterion given in definition 1.

In this section, like in [11], let us consider that system (1) is uniformly observable, then from [27], there exists $Q \in \mathbb{R}^{n\times n}$ (with $\det Q \neq 0$) such that the bilinear system (1) has the canonical form in the new basis, *i.e.* $\widetilde{x} = Qx$, that is, with $(A^i, C) \underset{Q}{\rightarrow} (\widetilde{A}^i, \widetilde{C})$, the companion form for $(\widetilde{A}^0, \widetilde{C})$ and the lower triangular form for $\widetilde{A}^i$ $(i = 1, \ldots, m)$. This form comes from the full rank of observability matrix (see [27]) and is especially adapted for the high gain observer. From [11] (see also [9, 8]), the high gain observer is given by

$$\begin{cases} \dot{\widehat{x}} = A^0\widehat{x} + \sum_{i=1}^{m} A^i u^i \widehat{x} + Ru - S_\infty^{-1}(\theta)C^T(C\widehat{x} - y) \\ \widehat{z} = L\widehat{x} \end{cases} \tag{39}$$

and is an exponential observer for system (1) where $S_\infty(\theta) = \overline{S}_\infty^T(\theta) = Q^{-1}\overline{S}_\infty(\theta)Q^{-T}$ is solution to the following Lyapunov equation (based on the companion form)

$$-\theta\overline{S}_\infty(\theta) - \overline{A}^T\overline{S}_\infty(\theta) - \overline{S}_\infty(\theta)\overline{A} + \widetilde{C}^T\widetilde{C} = 0, \tag{40}$$

and where $\overline{A} \in \mathbb{R}^{n\times n}$ is the antishift matrix and the parameter $\theta \in \mathbb{R}^{*+}$ is high enough.

An analytic methodology to find θ_0 "high enough" (such for all $\theta > \theta_0$ the stability of the filtering error is ensured) is given in the proof of theorem 3 in [11]. Some $\theta < \theta_0$ can ensure the stability, moreover the analytic approach does not enable to give a criteria to choose θ with respect to disturbance attenuation. In [11] and [8] it is proved that, given θ_1, if $S_\infty^{-1}(\theta_1)$ is a gain ensuring an exponential stability then for all $\theta > \theta_1$, $S_\infty^{-1}(\theta)$ is a gain ensuring an exponential stability.

Moreover the dynamics of the observation error $\widetilde{e} = x - \widehat{x}$ can be seen as an LPV system

$$\begin{cases} \dot{\widetilde{e}} = \mathcal{A}(u,\theta)\widetilde{e} + \mathcal{B}(\theta)w \\ e = L\widetilde{e} \end{cases} \tag{41}$$

where u is considered as a varying "parameter" and

$$\mathcal{A}(u,\theta) = A^0 + \sum_{i=1}^{m} A^i u^i - S_\infty^{-1}(\theta)C^TC \tag{42a}$$

$$\mathcal{B}(\theta) = B - S_\infty^{-1}(\theta)C^TD. \tag{42b}$$

Now we look for a value of the tuning parameter θ, guaranteeing both the quadratic stability of the observation error and the optimal disturbance attenuation (see definition 1). To do that, we use an LPV approach based on lemma 1 which requires the following notations

$$\widehat{\mathcal{A}}(\rho,\theta) = A^0 + \sum_{i=1}^{m} \rho^i A^i - S_\infty^{-1}(\theta)C^TC = \mathcal{A}(u,\theta). \tag{43}$$

Using notations of (27a), (27b) and (43), the following theorem ensures the quadratic stability of the "high gain"-like filter (39) and the $\mathcal{L}_2$ gain attenuation from w to e.

Theorem 3. *For a given θ, if there exist matrices $P^i \in \mathbb{R}^{n\times n}$ (for $i = 0, \ldots, m$), $F \in \mathbb{R}^{n\times n}$, such that, for $j = 1, \ldots, \nu$, $\widetilde{P}(\overline{\beta}^j) = \widetilde{P}(\overline{\beta}^j)^T > 0$ and*

$$\begin{bmatrix} (1,1) & (1,2) & F\mathcal{B}(\theta) \\ * & -F - F^T & F\mathcal{B}(\theta) \\ * & * & -\gamma^2 I_q \end{bmatrix} < 0 \tag{44}$$

with

$$\begin{aligned} (1,1) &= \overline{P}(\overline{\beta}^j) + F\widehat{\mathcal{A}}(\overline{\beta}^j, \theta) + \widehat{\mathcal{A}}^T(\overline{\beta}^j, \theta)F^T + L^T L, \\ (1,2) &= \widetilde{P}(\overline{\beta}^j) - F + \widehat{\mathcal{A}}(\overline{\beta}^j, \theta)^T F^T \end{aligned}$$

and $\overline{\beta}^j \in \mathcal{S}$, then the system (39) *is an exponential observer for the system* (1) *and the mapping from the disturbance input w to the filtering error e has $\mathcal{L}_2$ gain less than a given scalar γ (see definition 1).* •

Proof. For the given θ, using (25), (26) and (27), matrix inequality

$$\begin{bmatrix} (1,1) & (1,2) & F\mathcal{B}(\theta) \\ * & -F - F^T & F\mathcal{B}(\theta) \\ * & * & -\gamma^2 I_q \end{bmatrix} < 0 \tag{45}$$

with

$$\begin{aligned} (1,1) &= \dot{P}(u) + F\mathcal{A}(u, \theta) + \mathcal{A}(u, \theta)^T F^T + L^T L, \\ (1,2) &= P(u) - F + \mathcal{A}(u, \theta)^T F^T, \end{aligned}$$

holds if the LMIs (44) are satisfied for the ν vertices of the convex polytope $\mathcal{P}$ (see (28)), *i.e.* for each element $\overline{\beta}^j$ of $\mathcal{S}$ given by equation (29) (see [1]). Therefore applying lemma 1 to the LPV system (41) (with $\mathbb{A}(u) = \mathcal{A}(u, \theta)$, $\mathbb{B}(u) = \mathcal{B}(\theta)$, $\mathbb{C}(u) = L$ and $\mathbb{D}(u) = 0$ in (3), for all $u \in \Omega$) ensures that the system (41) is quadratically stable for $w = 0$ and the mapping from the disturbance input w to the filtering error e has $\mathcal{L}_2$ gain less than the scalar γ. This proves the theorem. □

Remark 3. *In [11], the canonical form associated to uniform observability is explicitly used to prove the stability of the high gain observer. In the "high gain"-like filter presented below, the stability of the filtering error is guaranteed by LMIs* (44)*, thus the requirement of uniform observability becomes unnecessary.* ◇

Remark 4. *Theorems 1 and 3 remain valid for nonlinearities of form $xf(u)$ in system* (1)*, if $f(u)$ and its derivatives are bounded.* ◇

5 Illustrative example

To illustrate our results, let us consider the following bilinear system

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} ux \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w \\ z = x \end{cases} \tag{46}$$

where x and y represent respectively the state and the output. The problem is to estimate z, minimizing the influence of the disturbance w on the filtering error e. Signals u and $\dot{u}$ are bounded as follows : $-1 \leqslant u \leqslant 1$ and $-10 \leqslant \dot{u} \leqslant 10$.

The states x_1 and x_2 are given in figure 1. The filtering errors e_1 and e_2, the control input u and the disturbance w are given in figure 2. For the lack of space, without developing, we give the disturbance attenuation gain obtained of the filter (5) obtained with constraint (19), $\gamma = 4.225$ and with the constraint $ED = 0$, $\gamma = 5.09$, respectively.

In this example, matrix D is of full row rank, so the constraint $ED = 0$ implies that $E = 0$, whereas the constraint $ZE_2D = 0$ gives more degrees of freedom in the filter design (in this case, the filter matrix E can be different from zero). Contrary to this numerical example, if matrix D is not of full row rank, the gain Z is not constrained to verify $ZE_2D = 0$ and the relation $ED = 0$ can be used in the filter design with $E \neq 0$.

The plot of $\gamma_{\min}(\theta)$ obtained in LMIs (44) enables us to determine graphically the value θ_{opt}. Figure 3 shows that the minimal of $\gamma_{\min}$ is 6.4 for $\theta = \theta_{\text{opt}} = 3.5$ and thus $(S_\infty^{-1}(\theta_{\text{opt}})C^T)^T = [\,7 \;\; 12.5\,]^T$. Notice that the methodology proposed in [11] yields a lower bound for $\theta_0 = 19.12$. In this way, this figure has permitted to make the choice of θ taking the disturbance attenuation into account. For a too small θ, the filtering error is not stable. For values of θ near the minimal value guaranteeing the stability of the filtering error, the optimal $\gamma_{\min}$ is very high (γ tends to ∞ when θ tends to the limit value enabling the stability of the filtering error).

6 Conclusion

This paper has presented a computationally tractable solution to the $\mathcal{H}_\infty$ functional filtering problem via an LPV approach for bilinear systems. By choosing appropriate Lyapunov functions, sufficient conditions for asymptotic stability and $\mathcal{H}_\infty$ disturbance attenuation have been provided in terms of LMIs. The proposed designs are shown to be efficient via a numerical example. The different design procedures which are presented enable to enlarge to bilinear systems filtering the applicability of the LMI-LPV approach.

References

[1] P. Apkarian, P. Gahinet, and J. Biannic, "Affine parameter-dependent Lyapunov functions for real parametric uncertainty," in *Proceedings of the IEEE Conference on Decision and Control*, Lake Buena Vista, USA, 1994, pp. 2026–2031.

[2] G. Bornard, N. Couenne, and F. Celle, "Regularly persistent observers for bilinear systems," in *New Trends in Nonlinear Control Theory*, ser. Lecture Notes in Control and Information Sciences, J. Descusse, M. Fliess, A. Isidori, and D. Leborgne, Eds. Springer-Verlag, 1988, vol. 122, pp. 130–140.

[3] K. Busawon and J. De Leon-Morales, "An improved high gain observer for single input uniformly observable systems," in *Proceedings of the European Control Conference*, Karlsruhe, Germany, 1999.

[4] S. Chughtai and N. Munro, "LMI-based gain-scheduled control," in *Proceedings of the Control Conference*, University of Bath, UK, 2004.

[5] M. Darouach, "Existence and design of functional observers for linear systems," *IEEE Transactions on Automatic Control*, vol. 45, pp. 940–943, 2000.

[6] M. Darouach, M. Zasadzinski, and M. Hayar, "Reduced-order observer design for descriptor systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 41, pp. 1068–1072, 1996.

[7] I. Derese, P. Stevens, and E. Noldus, "Observers for bilinear systems with bounded input," *International Journal of Systems Science*, vol. 10, pp. 649–668, 1979.

[8] F. Deza, D. Bossanne, E. Busvelle, J. Gauthier, and D. Rakotopara, "Exponential observers for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 38, pp. 482–484, 1993.

[9] F. Deza, E. Busvelle, J. Gauthier, and D. Rakotopara, "High gain estimation for nonlinear systems," *Systems & Control Letters*, vol. 18, pp. 295–299, 1992.

[10] E. Fridman and U. Shaked, "A descriptor system approach to $\mathcal{H}_\infty$ control of linear time-delay systems," *IEEE Transactions on Automatic Control*, vol. 47, pp. 253–270, 2002.

[11] J. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems, applications to bioreactors," *IEEE Transactions on Automatic Control*, vol. 37, pp. 875–880, 1992.

[12] B. Gérard, H. Souley Ali, M. Zasadzinski, and M. Darouach, "LPV approach for $\mathcal{H}_\infty$ reduced order filters for bilinear systems," in *Proceedings of the European Control Conference*, Kos, Greece, 2007.

[13] O. Grasselli and A. Isidori, "An existence theorem for observers of bilinear systems," *IEEE Transactions on Automatic Control*, vol. 26, pp. 1299–1300, 1981.

[14] S. Hara and K. Furuta, "Minimal order state observers for bilinear systems," *International Journal of Control*, vol. 24, pp. 705–718, 1976.

[15] R. Hermann and A. Krener, "Nonlinear controllability and observability," *IEEE Transactions on Automatic Control*, vol. 22, pp. 728–740, 1977.

[16] R. Mohler, *Nonlinear systems : Applications to Bilinear Control*. Englewood Cliffs, New Jersey: Prentice Hall, 1991, vol. 2.

[17] S. Raghavan and J. Hedrick, "Observer design for a class of nonlinear systems," *International Journal of Control*, vol. 59, pp. 515–528, 1994.

[18] R. Rajamani, "Observer for Lipschitz nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 397–401, 1998.

[19] C. Rao and S. Mitra, *Generalized Inverse of Matrices and its Applications*. New York: Wiley, 1971.

[20] M. Rodrigues, H. Hammouri, C. Mechmeche, and N. Benhadj Braiek, "A high gain observer based LMI approach," in *Proceedings of the Triennal IFAC World Congress*, Seoul, Korea, 2008.

[21] W. Rugh, *Nonlinear System Theory*, ser. Johns Hopkins Series in Information Sciences and Systems. Baltimore: Johns Hopkins University Press, 1981.

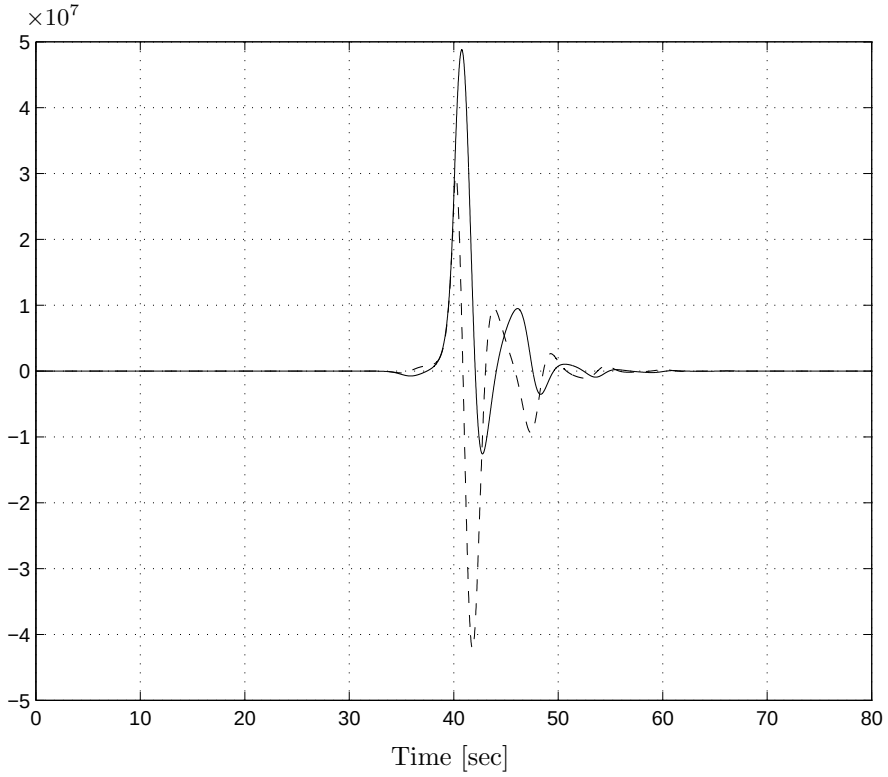

Figure 1: States x_1 (solid line) and x_2 (dashed line)

[22] M. Seron, J. Braslavsky, and G. Goodwin, *Fundamental Limitations in Filtering and Control*. New York: Springer-Verlag, 1997.

[23] H. Souley Ali, M. Zasadzinski, H. Rafaralahy, and M. Darouach, "Robust $\mathcal{H}_\infty$ reduced order functional filtering for uncertain bilinear systems," *Automatica*, vol. 42, pp. 405–413, 2006.

[24] B. Tibken and E. Hofer, "Systematic observer design for bilinear systems," in *IEEE International Symposium on Circuits and Systems*, Portland, USA, 1989.

[25] A. van der Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control," *IEEE Transactions on Automatic Control*, vol. 37, pp. 770–784, 1992.

[26] W. Wang and C. Kao, "Estimator design for bilinear systems with bounded inputs," *Journal of the Chinese Institute of Engineering*, vol. 14, pp. 157–163, 1991.

[27] D. Williamson, "Observation of bilinear systems with application to biological control," *Automatica*, vol. 13, pp. 243–254, 1977.

[28] M. Zasadzinski, H. Rafaralahy, C. Mechmeche, and M. Darouach, "On disturbance decoupled observer for a class of bilinear systems," *Transactions. of the ASME, Journal of Dynamic Systems, Measurement, and Control*, vol. 120, pp. 371–377, 1998.

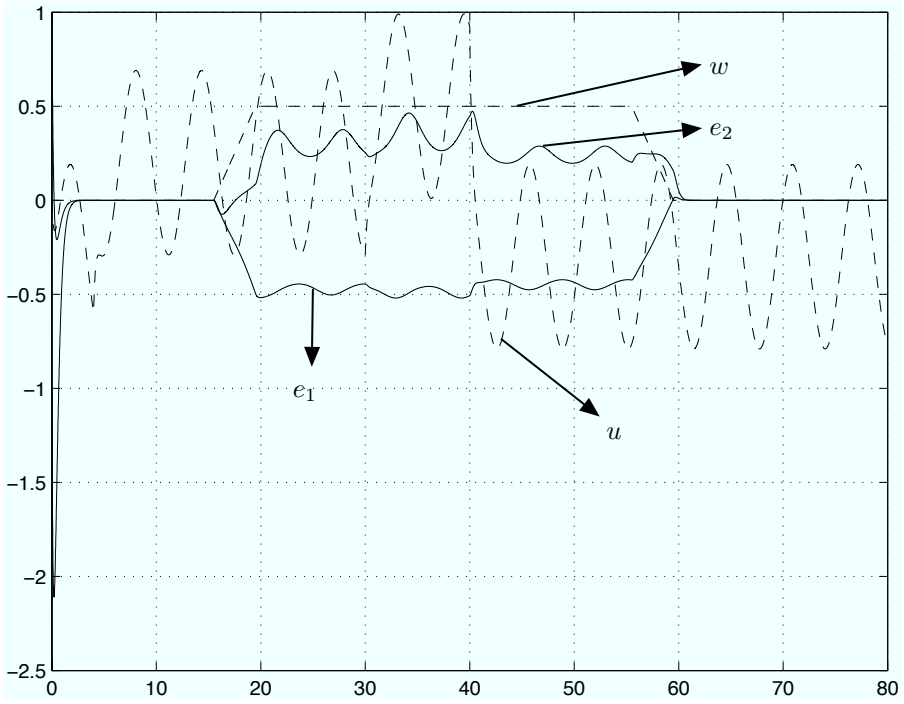

Figure 2: Filtering errors e_1 and e_2, control input u and disturbance w

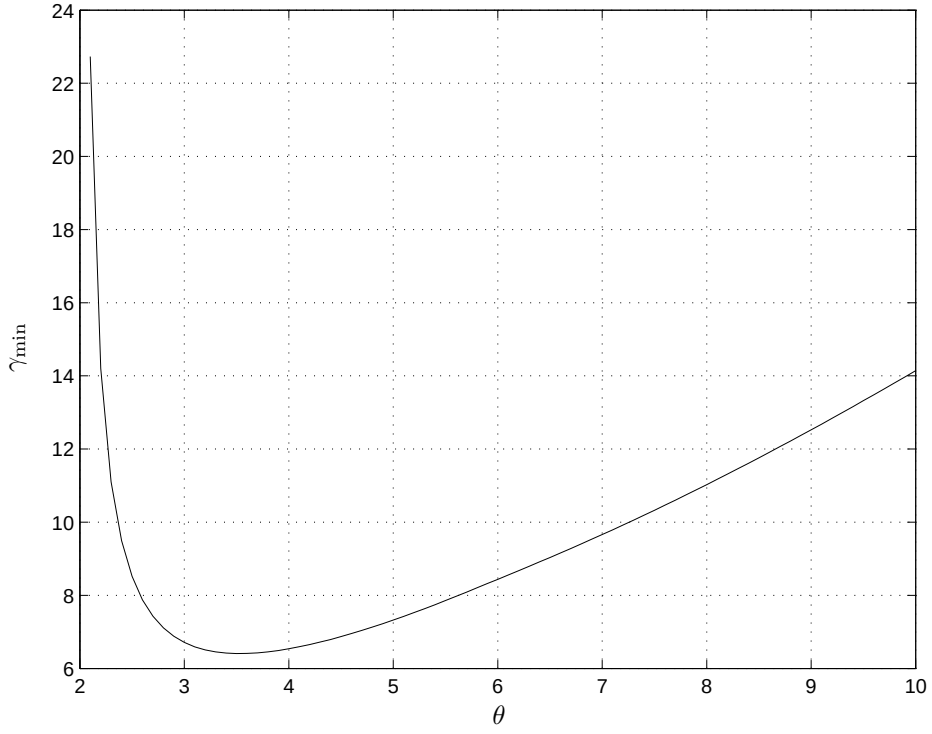

Figure 3: Curve $\gamma_{\min}(\theta)$