Distances and Riemannian metrics for multivariate spectral densities

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Abstract

We first introduce a class of divergence measures between power spectral density matrices. These are derived by comparing the suitability of different models in the context of optimal prediction. Distances between "infinitesimally close" power spectra are quadratic, and hence, they induce a differential-geometric structure. We study the corresponding Riemannian metrics and, for a particular case, provide explicit formulae for the corresponding geodesics and geodesic distances. The close connection between the geometry of power spectra and the geometry of the Fisher-Rao metric is noted.

I. INTRODUCTION

Distance measures between statistical models and between signals constitute some of the basic tools of Signal Processing, System Identification, and Control [1], [2]. Indeed, quantifying dissimilarities is the essence of detection, tracking, pattern recognition, model validation, signal classification, etc. Naturally, a variety of choices are readily available for comparing deterministic signals and systems. These include various L_p and Sobolev norms on signal spaces, and induced norms in spaces of systems. Statistical models on the other hand are not elements of a linear space. Their geometry is dictated by positivity constraints and hence, they lie on suitable cones or simplices. This is the case for covariances, histograms, probability distributions, or power spectra, as these need to be positive in a suitable sense. A classical theory for statistical models, having roots in the work of C.R. Rao and R.A. Fisher, is now known as "information geometry" [3], [4], [5], [6]. The present work aims at a geometric theory suitable for time-series modeled by power spectra. To this end, we follow a largely parallel route to that of information geometry (see [7]) in that a metric is now dictated by the dissimilarity of models in the context of prediction theory for second-order stochastic processes. The present work builds on [7], which focused on scalar time-series, and is devoted to power spectral densities of multivariable stochastic processes.

The need to compare two power spectra densities f_1, f_2 directly has led to a number of divergence measures which have been suggested at various times [1], [2]. Key among those are the Itakura-Saito distance

$$D_{\rm IS}(f_1, f_2) := \int_{-\pi}^{\pi} \left(\frac{f_1(\theta)}{f_2(\theta)} - \log \frac{f_1(\theta)}{f_2(\theta)} - 1 \right) \frac{d\theta}{2\pi}$$

and the logarithmic spectral deviation

$$\mathcal{D}_{\log}(f_1, f_2) := \sqrt{\int_{-\pi}^{\pi} \left| \log \frac{f_1(\theta)}{f_2(\theta)} \right|^2 \frac{d\theta}{2\pi}}$$

see e.g., [2, page 370]. The distance measures developed in [7] are closely related to both of these, and the development herein provides a multivariable counterpart. Indeed, the divergences that we list between matrix-valued power spectra are similar to the Itakura-Saito divergence and geodesics on the corresponding Riemannian manifolds of power spectra take the form of logarithmic integrals.

Distances between multivariable power spectra have only recently received any attention. In this direction we mention generalizations of the Hellinger and Itakura-Saito distances by Ferrante *et al.* [8], [9] and the use of the Umegaki-von Neumann relative entropy [10]. The goal of this paper is to generalize the geometric framework in [7] to the matrix-valued power spectra. We compare two power spectra in the context of linear prediction: a choice between the two is used to design an optimal filter which is then applied to a process corresponding to the second power spectrum. The "flatness" of the innovations process, as well as the degradation of the prediction error variance,

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when compared to the best possible, are used to quantify the mismatch between the two. This rationale provides us with natural divergence measures. We then identify corresponding Riemannian metrics that dictate the underlying geometry. For a certain case we compute closed-form expressions for the induced geodesics and geodesic distances. These provide a multivariable counterpart to the logarithmic intervals in [7] and the logarithmic spectral deviation [2, page 370]. It is noted that the geodesic distance has certain natural desirable properties; it is inverse-invariant and congruence-invariant. Moreover, the manifold of the multivariate spectral density functions endowed with this geodesic distance is a complete metric space. A discrete counter part of certain of these Riemannian metrics, on the manifold of positive definite matrices (equivalent to power spectra which are constant across frequencies), has been studied extensively in connection to the geometry of positive operators [11] and relates to the Rao-Fisher geometry on probability models restricted to the case of Gaussian random vectors.

Indeed, there is a deep connection between the Itakura-Saito distance and the Kullback-Leibler divergence between the corresponding probability models [2, page 371], [12] which provides a link to information geometry. Hence, the Riemannian geometry on power spectral densities in [7] as well as the multivariable structure presented herein is expected to have a strong connection also to the Fisher-Rao metric and the geometry of information. An interesting study in this direction which taps on an interpretation of the geometry of power spectra via the underlying probability structure and its connection to the Kullback-Leibler divergence is given in Yu and Mehta [13]. However, a transparent differential geometric explanation which highlights points of contact is still to be developed. Further key developments which parallel the framework reported herein and are focused on moment problems are presented in [8], [9].

The paper is organized as follows. In Section II we establish notation and overview the theory of the multivariate quadratic optimal prediction problem. In Section III we introduce alternative distance measures between multivariable power spectra which reflect mismatch in the context of one-step-ahead prediction. In Section IV we discuss Riemannian metrics that are induced by the divergence measures of the previous section. In Section V we discuss the geometry of positive matrices. In Section VI the geometric structure is analyzed and geodesics are identified. In Section VII we provide examples to highlight the nature of geodesics between power spectra and how these may compare to alternatives.

II. PRELIMINARIES ON MULTIVARIATE PREDICTION

Consider a multivariate discrete-time, zero mean, weakly stationary stochastic process $\{\mathbf{u}(k), k \in \mathbb{Z}\}\$ with $\mathbf{u}(k)$ taking values in $\mathbb{C}^{m \times 1}$. Throughout, boldface denotes random variables/vectors, \mathcal{E} denotes expectation, $\mathbf{j} = \sqrt{-1}$ the imaginary unit, and * the complex conjugate transpose. Let

$$R_k = \mathcal{E} \left\{ \mathbf{u}(\ell) \mathbf{u}^*(\ell - k) \right\} \text{ for } l, k \in \mathbb{Z}$$

denote the sequence of matrix covariances and $d\mu(\theta)$ be the corresponding matricial power spectral measure for which

$$R_k = \int_{-\pi}^{\pi} e^{-jk\theta} \frac{d\mu(\theta)}{2\pi}$$

For the most part, we will be concerned with the case of non-deterministic processes with an absolutely continuous power spectrum. Hence, unless we specifically indicate otherwise, $d\mu(\theta) = f(\theta)d\theta$ with $f(\theta)$ being a matrix-valued power spectral density (PSD) function. Further, for a non-deterministic process $\log(f(\theta))$ needs to be integrable, and this will be assumed throughout as well.

Our interest is in comparing PSD's and in studying possible metrics between such. The evident goal is to provide a means to quantify deviations and uncertainty in the spectral domain in a way that is consistent with particular applications. More specifically, we present metrizations of the space of PSD's which are dictated by optimal prediction and reflect dissimilarities that have an impact on the quality of prediction.

A. Geometry of multivariable processes

We will be considering least-variance linear prediction problems. To this end, we define $L_{2,\mathbf{u}}$ to be the closure of $m \times 1$ -vector-valued finite linear combinations of $\{\mathbf{u}(k)\}$ with respect to covergence in the mean [14, pg. 135]:

$$L_{2,\mathbf{u}} := \left\{ \sum_{\text{finite}} P_k \mathbf{u}(-k) : P_k \in \mathbb{C}^{m \times m}, \ k \in \mathbb{Z} \right\}.$$

Here, "bar" denotes closure. The indices in P_k and $\mathbf{u}(-k)$ run in opposite directions so as to simplify the notation later on where prediction is based on past observations. This space is endowed with both, a matricial inner product

$$\begin{split} \llbracket \sum_{k} P_{k} \mathbf{u}(-k), \sum_{k} Q_{k} \mathbf{u}(-k) \rrbracket := \\ \mathcal{E} \left\{ \left(\sum_{k} P_{k} \mathbf{u}(-k) \right) \left(\sum_{k} Q_{k} \mathbf{u}(-k) \right)^{*} \right\}, \end{split}$$

as well as a scalar inner product

$$\langle \sum_{k} P_{k} \mathbf{u}(-k), \sum_{k} Q_{k} \mathbf{u}(-k) \rangle := \operatorname{tr} \left[\sum_{k} P_{k} \mathbf{u}(-k), \sum_{k} Q_{k} \mathbf{u}(-k) \right]$$

Throughout, "tr" denotes the trace of a matrix. It is standard to establish the correspondence between

$$\mathbf{p} := p(\mathbf{u}) := \sum_{k} P_k \mathbf{u}(-k)$$
 and
 $p(z) := \sum_{k} P_k z^k$

with $z = e^{j\theta}$ for $\theta \in [-\pi, \pi]$. This is the Kolmogorov isomorphism between the "temporal" space $L_2(\mathbf{u})$ and "spectral" space $L_{2,d\mu}$,

$$\varphi : L_2(\mathbf{u}) \to L_{2,d\mu} : \sum_k P_k \mathbf{u}(-k) \mapsto \sum_k P_k z^k.$$

It is convenient to endow the latter space $L_{2,d\mu}$ with the matricial inner product

$$\llbracket p,q \rrbracket_{d\mu} := \int_{-\pi}^{\pi} \left(p(e^{\mathbf{j}\theta}) \frac{d\mu(\theta)}{2\pi} q(e^{\mathbf{j}\theta})^* \right)$$

as well as the scalar inner product

$$\langle p,q\rangle_{d\mu} := \operatorname{tr} \llbracket p,q \rrbracket_{d\mu}.$$

The additional structure due to the matricial inner product is often referred to as *Hilbertian* (as opposed to *Hilbert*) [15].

Throughout, $p(e^{j\theta}) = \sum_k P_k e^{jk\theta}$, $q(e^{j\theta}) = \sum_k Q_k e^{jk\theta}$, where we use lower case p, q for matrix functions and upper case P_k, Q_k for their matrix coefficients. For non-deterministic processes with absolutely continuous spectral measure $d\mu(\theta) = f(\theta)d\theta$, we simplify the notation into

$$\llbracket p,q \rrbracket_f := \llbracket p,q \rrbracket_{fd\theta}, \text{ and } \\ \langle p,q \rangle_f := \langle p,q \rangle_{fd\theta}.$$

Least-variance linear prediction

$$\min\left\{\operatorname{tr} \mathcal{E}\{\mathbf{p}\mathbf{p}^*\}: \mathbf{p} = \mathbf{u}(0) - \sum_{k>0} P_k \mathbf{u}(-k), \ P_k \in \mathbb{C}^{m \times m}\right\}$$
(1)

can be expressed equivalently in the spectral domain

$$\min\left\{ \llbracket p, p \rrbracket_f : p(z) = I - \sum_{k>0} P_k z^k, \ P_k \in \mathbb{C}^{m \times m} \right\}$$
(2)

1

where the minimum is sought in the positive-definite sense, see [15, pg. 354], [14, pg. 143]. We use "*I*" to denote the identity matrix of suitable size. It holds that, although non-negative definiteness defines only a partial order on the cone of non-negative definite Hermitian matrices, a minimizer for (1) always exists. Of course this corresponds to a minimizer for (2). The existence of a minimizer is due to the fact that tr $\mathcal{E}\{pp^*\}$ is matrix-convex. Here $d\mu = f d\theta$ is an absolutely continuous measure and the quadratic form is not degenerate; see [16, Proposition 1] for a detailed analysis and a treatment of the singular case where μ is a discrete matrix-valued measure. Further, the minimizer of (1) coincides with the minimizer of

$$\min\left\{\langle p,p\rangle_f: p(z)=I-\sum_{k>0}P_k z^k, \ P_k\in\mathbb{C}^{m\times m}\right\}.$$
(3)

From here on, to keep notation simple, p(z) will denote the minimizer of such a problem, with f specified accordingly, and the minimal matrix of (1) will be denoted by Ω . That is,

$$\Omega := \llbracket p, p \rrbracket_f$$

while the minimal value of (3) is tr Ω . The minimizer p is precisely the image under the Kolmogorov isomorphism of the optimal *prediction error* **p** and Ω the *prediction-error variance*.

B. Spectral factors and optimal prediction

For a non-deterministic process the error variance Ω has full rank. Equivalently, the product of its eigenvalues is non-zero. The well-known Szegö-Kolmogorov formula [15, pg. 369]

$$\det \Omega = \exp\{\int_{-\pi}^{\pi} \log \det f(\theta) \frac{d\theta}{2\pi}\}$$
(4)

relates the product of the eigenvalues of the optimal one-step-ahead prediction error variance with the corresponding PSD. No expression is available in general that would relate f to Ω directly in the matricial case.

We consider only non-deterministic processes and hence we assume that

$$\log \det f(\theta) \in L_1[-\pi,\pi].$$

In this case, $f(\theta)$ admits a unique factorization

$$f(\theta) = f_+(e^{j\theta})f_+(e^{j\theta})^*, \tag{5}$$

with $f_+(e^{\mathbf{j}\theta}) \in \mathcal{H}_2^{m \times m}(\mathbb{D})$,

$$\det(f_+(z)) \neq 0 \text{ in } \mathbb{D} := \{z : |z| < 1\}$$

and normalized so that $f_+(0) = \Omega^{\frac{1}{2}}$. Throughout, $M^{\frac{1}{2}}$ denotes the Hermitian square root of a Hermitian matrix M. The factor f_+ is known as the *canonical (left) spectral factor*. In the case where f is a scalar function (m = 1) the canonical spectral factor is explicitly given by

$$f_{+}(z) = \exp\left\{\frac{1}{2}\int_{-\pi}^{\pi} \left(\frac{1+ze^{-j\theta}}{1-ze^{-j\theta}}\right)\log f(\theta)\frac{d\theta}{2\pi}\right\}, \quad |z| < 1,$$

As usual, $\mathcal{H}_2(\mathbb{D})$ denotes the Hardy space of functions which are analytic in the unit disk \mathbb{D} with square-integrable radial limits. Spectral factorization presents an "explicit" expression of the optimal prediction error in the form

$$p(z) = f_{+}(0)f_{+}^{-1}(z).$$
(6)

Thus, $p(z)^{-1}$ is a "normalized" (left) *outer* factor of f. The terminology "outer" refers to a (matrix-valued) function $g(e^{j\theta})$ for $\theta \in [-\pi, \pi]$ that can be extended into an analytic function in the open interior of the unit disc \mathbb{D} which is also invertible in \mathbb{D} . It is often standard not to differentiate between such a function in \mathbb{D} and the function on the boundary of radial-limits since these are uniquely defined from one another. In the engineering literature outer functions are also referred to as "minimum phase." Right-outer factors, where $f(\theta) = f_{+,\mathrm{right}}(e^{j\theta})^* f_{+,\mathrm{right}}(e^{j\theta})$ instead of (5) relate to a *post-diction* optimal estimation problem; in this, the present value of the process is estimated via linear combination of future values (see e.g., [16]). Only left factorizations will be used in the present paper.

III. COMPARISON OF PSD'S

We present two complementing viewpoints on how to compare two PSD's, f_1 and f_2 . In both, the optimal one-step-ahead predictor for one of the two stochastic processes, is applied to the other and compared to the corresponding optimal. The first is to consider how "white" the power spectrum of the innovations' process is. The second viewpoint is to compare how the error variance degrades with respect to the optimal predictor. Either principle provides a family of divergence measures and a suitable generalization of the Riemannian geometry of scalar PSD's given in [7]. There is a close relationship between the two.

A. Prediction errors and innovations processes

Consider two matrix-valued spectral density functions f_1 and f_2 . Since an optimal filter will be designed based on one of the two and then evaluated with respect to the other, some notation is in order.

First, let us use a subscript to distinguish between two processes $\mathbf{u}_i(k)$, $i \in \{1, 2\}$, having the f_i 's as the corresponding PSD's. They are assumed purely nondeterministic, vector-valued, and of compatible size. The optimal filters in the spectral domain are

$$p_i := \operatorname{argmin}\{\llbracket p, p \rrbracket_{f_i} \ p(0) = I, \\ \text{and } p \in \mathcal{H}_2^{m \times m}(\mathbb{D})\},\$$

and their respective error covariances

$$\Omega_i := \llbracket p_i, p_i \rrbracket_{f_i}$$

Now define

$$\Omega_{i,j} := \llbracket p_j, p_j \rrbracket_{f_i}.$$

Clearly, $\Omega_{i,j}$ is the variance of the prediction error when the filter p_j is used on a process having power spectrum f_i . Indeed, if we set

$$\mathbf{p}_{i,j} := \mathbf{u}_i(0) - P_{j,1}\mathbf{u}_i(-1) - P_{j,2}\mathbf{u}_i(-2) - \dots$$
(7)

the prediction-error covariance is

$$\llbracket \mathbf{p}_{i,j}, \mathbf{p}_{i,j} \rrbracket = \llbracket p_j, p_j \rrbracket_{f_i}.$$

The prediction error $\mathbf{p}_{i,j}$ can also be thought of as a time-process, indexed at time-instant $k \in \mathbb{Z}$,

$$\mathbf{p}_{ij}(k) := \mathbf{u}_i(k) - P_{j,1}\mathbf{u}_i(k-1) - P_{j,2}\mathbf{u}_i(k-2) - \dots$$

for $i, j \in \{1, 2\}$. This is an *innovations process*. Clearly, from stationarity,

$$\llbracket \mathbf{p}_{i,i}, \mathbf{p}_{i,i} \rrbracket = \Omega_i,$$

whereas

 $\llbracket \mathbf{p}_{i,j}, \mathbf{p}_{i,j} \rrbracket \geq \Omega_i,$

since in this case p_j is suboptimal for \mathbf{u}_i , in general.

B. The color of innovations and PSD mismatch

We choose to normalize the innovations processes as follows:

$$\mathbf{h}_{i,j}(k) = \Omega_j^{-\frac{1}{2}} \mathbf{p}_{i,j}(k), \text{ for } k \in \mathbb{Z}.$$

The Kolmogorov isomorphism takes

$$\varphi : \mathbf{h}_{i,j}(k) \mapsto f_{j+}^{-1},$$

with the expectation/inner-product being that induced by f_i , and hence, the power spectral density of the process $\mathbf{h}_{i,j}(k)$ is

$$f_{\mathbf{h}_{ij}} = f_{j+}^{-1} f_i f_{j+}^{-*},$$

where $(\cdot)^{-*}$ is a shorthand for $((\cdot)^*)^{-1}$. When $f_i = f_j$, evidently $\{\mathbf{h}_k^{i,i}\}$ is a white noise process with covariance matrix equals to the identity.

Naturally, in an absolute sense, the mismatch between the two power spectra f_i, f_j can be quantified by the distance of $f_{\mathbf{h}_{ij}}$ to the identity. To this end we may consider any symmetrized expression:

$$\int_{-\pi}^{\pi} \mathrm{d}(f_{j+}^{-1}f_i f_{j+}^{-*}, I) \frac{d\theta}{2\pi} + \int_{-\pi}^{\pi} \mathrm{d}(f_{i+}^{-1}f_j f_{i+}^{-*}, I) \frac{d\theta}{2\pi}$$
(8)

for a suitable distance $d(\cdot, \cdot)$ between positive definite matrices. In general, it is deemed desirable that distances between power spectra are invariant to scaling (as is the case when distances depend on ratios of spectra, [2]). Researchers and practitioners alike have insisted on such a property, especially for speech and image systems, due to an apparent agreement with subjective qualities of sound and images. It is thus interesting to seek a multivariable analogues inherent in the above comparison.

Due to the non-negative definiteness of power spectra, a convenient option is to take "d" as the trace:

$$\int_{-\pi}^{\pi} \operatorname{tr} \left(f_{j+}^{-1} f_i f_{j+}^{-*} - I \right) + \operatorname{tr} \left(f_{i+}^{-1} f_j f_{i+}^{-*} - I \right) \frac{d\theta}{2\pi}.$$

This indeed defines a distance measure since $(x + x^{-1} - 2)$ is a non-negative function for $0 < x \in \mathbb{R}$ that vanishes only when x = 1. Thus, we define

$$D_1(f_1, f_2) := \int_{-\pi}^{\pi} \operatorname{tr} \left(f_2^{-1} f_1 + f_1^{-1} f_2 - 2I \right) \frac{d\theta}{2\pi}.$$
(9a)

Interestingly, $D_1(f_1, f_2)$ can be re-written as follows:

$$D_1(f_1, f_2) = \int_{-\pi}^{\pi} \|f_1^{-1/2} f_2^{1/2} - f_1^{1/2} f_2^{-1/2}\|_{Fr}^2 \frac{d\theta}{2\pi}$$
(9b)

where $||M||_{Fr}^2 := \operatorname{tr} MM^*$ denotes the square of the Frobenius norm¹. It can be readily verified starting from the right hand side of (9b) and simplifying this to match (9a). It is now be easily seen that $D_1(f_i, f_j)$ has a number of desirable properties listed in the following proposition.

Proposition 1: Consider f_i, f_j being PSD's of non-deterministic processes and $g(e^{j\theta})$ an arbitrary outer matrixvalued function in $\mathcal{H}_2^{m \times m}(\mathbb{D})$. The following hold:

(i)
$$D_1(f_i, f_j) \ge 0$$
.

(ii) $D_1(f_i, f_j) = 0$ iff $f_i = f_j$ (a.e.).

(iii)
$$D_1(f_i, f_i) = D_1(f_i, f_i).$$

- (iv) $D_1(f_i, f_j) = D_1(f_i^{-1}, f_j^{-1}).$ (v) $D_1(f_i, f_j) = D_1(g_{f_i}g^*, g_{f_j}g^*).$

Proof: Properties (i-iv) follow immediately from (9b) while the invariance property (v) is most easily seen by employing (9a).

C. Suboptimal prediction and PSD mismatch

We now attempt to quantify how suboptimal the performance of a filter is when this is based on the incorrect choice between the two alternative PSD's. To this end, we consider the error covariance and compare it to that of the optimal predictor. A basic inequality between these error covariances is summarized in the following proposition.

Proposition 2: Under our earlier standard assumptions, for $i, j \in \{1, 2\}$ and $\Omega_i, \Omega_j > 0$, it holds that

$$\Omega_{i,j} \ge \Omega_i. \tag{10a}$$

Further, the above holds as an equality iff $p_i = p_j$.

Proof: It follows from the optimality of p_i since

$$[\![p_j, p_j]\!]_{f_i} \ge [\![p_i, p_i]\!]_{f_i} = \Omega_i.$$

 $[\]sqrt{\operatorname{tr} MM^*}$ is also referred to also as the Hilbert-Schmidt norm.

Corollary 3: The following hold:

$$\Omega_i^{-\frac{1}{2}}\Omega_{i,j}\Omega_i^{-\frac{1}{2}} \ge I \tag{10b}$$

$$\det(\Omega_{i,j}) \geq \det(\Omega_i) \tag{10c}$$

$$\operatorname{tr}(\Omega_{i,j}) \geq \operatorname{tr}(\Omega_i)$$
 (10d)

$$\Omega_j^{-\frac{1}{2}}\Omega_{i,j}\Omega_j^{-\frac{1}{2}} \geq \Omega_j^{-\frac{1}{2}}\Omega_i\Omega_j^{-\frac{1}{2}}.$$
(10e)

Further, each " \geq " holds as equality iff $p_i = p_j$.

Thus, a mismatch between the two spectral densities can be quantified by the strength of the above inequalities. To this end, we may consider a number of alternative "divergence measures". First we consider:

$$D_2(f_i, f_j) := \log \det \left(\Omega_i^{-\frac{1}{2}} \Omega_{i,j} \Omega_i^{-\frac{1}{2}} \right).$$
(11)

Equivalent options leading to the same Riemannian structure are:

$$\frac{1}{m}\operatorname{tr}(\Omega_i^{-\frac{1}{2}}\Omega_{i,j}\Omega_i^{-\frac{1}{2}}) - 1, \text{ and}$$
(12a)

$$\det(\Omega_i^{-\frac{1}{2}}\Omega_{i,j}\Omega_i^{-\frac{1}{2}}) - 1.$$
(12b)

Using the generalized Szegö-Kolmogorov expression (4) we readily obtain that

$$D_{2}(f_{i}, f_{j}) = \log \det \left(\int_{-\pi}^{\pi} f_{j+}^{-1} f_{i} f_{j+}^{-*} \frac{d\theta}{2\pi} \right) - \int_{-\pi}^{\pi} \log \det \left(f_{j+}^{-1} f_{i} f_{j+}^{-*} \right) \frac{d\theta}{2\pi}$$

$$= \operatorname{tr} \left(\log \int_{-\pi}^{\pi} f_{j+}^{-1} f_{i} f_{j+}^{-*} \frac{d\theta}{2\pi} - \int_{-\pi}^{\pi} \log f_{j+}^{-1} f_{i} f_{j+}^{-*} \frac{d\theta}{2\pi} \right).$$
(13)

This expression takes values in $[0, \infty]$, and is zero if and only if the normalized spectral factors $p^{-1} = \Omega^{-1/2} f_+$ are identical for the two spectra. Further, it provides a natural generalization of the divergence measures in [7] and of the Itakura distance to the case of multivariable spectra. It satisfies "congruence invariance." This is stated next.

Proposition 4: Consider two PSD's f_i, f_j of non-deterministic processes and $g(e^{j\theta})$ an outer matrix-valued function in $\mathcal{H}_2^{m \times m}(\mathbb{D})$. The following hold:

(i) $D_2(f_i, f_j) \ge 0.$

(ii)
$$D_2(f_i, f_j) = 0$$
 iff $p_i = p_j$.

(iii) $D_2(f_i, f_j) = D_2(gf_ig^*, gf_jg^*).$

Proof: Properties (i-ii) follow immediately from (11) while the invariance property (iii) is most easily seen be employing (13). To this end, first note that gf_+ obviously constitutes the spectral factor of gfg^* . Substituting the corresponding expressions in (13) establishes the invariance.

D. Alternative divergence measures

Obviously, a large family of divergence measures between two matrix-valued power spectra can be obtained based on (8). For completeness, we suggest representative possibilities some of which have been independently considered in recent literature.

1) Frobenius distance: If we use the Frobenius norm in (8) we obtain

$$D_{\rm F}(f_1, f_2) := \frac{1}{2} \sum_{i,j} \int_{-\pi}^{\pi} \|f_{j+}^{-1} f_i f_{j+}^{-*} - I\|_{\rm Fr}^2 \frac{d\theta}{2\pi}$$
(14a)

where $\sum_{i,j}$ designates the "symmetrized sum" taking $(i,j) \in \{(1,2), (2,1)\}$. It's straightforward to see that all of

$$f_{j+}^{-1}f_if_{j+}^{-*}, f_j^{-\frac{1}{2}}f_if_j^{-\frac{1}{2}} \text{ and } f_j^{-1}f_i$$

share the same eigenvalues for any $\theta \in [-\pi, \pi]$. Thus,

$$\|f_{j+}^{-1}f_if_{j+}^{-*} - I\|_{\mathrm{Fr}}^2 = \|f_j^{-\frac{1}{2}}f_if_j^{-\frac{1}{2}} - I\|_{\mathrm{Fr}}^2,$$

and

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$$D_{\rm F}(f_1, f_2) = \frac{1}{2} \sum_{i,j} \int_{-\pi}^{\pi} \|f_j^{-\frac{1}{2}} f_i f_j^{-\frac{1}{2}} - I\|_{\rm Fr}^2 \frac{d\theta}{2\pi}.$$
 (14b)

Obviously (14b) is preferable over (14a) since no spectral factorization is involved.

2) Hellinger distance: A generalization of the Hellinger distance has been proposed in [9] for comparing multivariable spectra. Briefly, given two positive definite matrices f_1 and f_2 one seeks factorizations $f_i = g_i g_i^*$ so that the integral over frequencies of the Frobenius distance $||g_1 - g_2||_{Fr}^2$ between the factors is minimal. The factorization does not need to correspond to analytic factors. When one of the two spectra is the identity, the optimization is trivial and the Hellinger distance becomes

$$\int_{-\pi}^{\pi} \|f^{\frac{1}{2}} - I\|_{\mathrm{Fr}}^{2} \frac{d\theta}{2\pi}.$$

A variation of this idea is to compare the normalized innovation spectra $(f_{j+}^{-1}f_if_{j+}^{-*})^{\frac{1}{2}}$, for $i, j \in \{1, 2\}$, to the identity. We do this in a symmetrized fashion so that together with symmetry the metric inherits the inverseinvariance property. Thus, we define

$$D_{\rm H}(f_1, f_2) := \sum_{i,j} \int_{-\pi}^{\pi} \| (f_{j+}^{-1} f_i f_{j+}^{-*})^{\frac{1}{2}} - I \|_{\rm Fr}^2 \frac{d\theta}{2\pi}$$

$$= \sum_{i,j} \int_{-\pi}^{\pi} \| (f_j^{-\frac{1}{2}} f_i f_j^{-\frac{1}{2}})^{\frac{1}{2}} - I \|_{\rm Fr}^2 \frac{d\theta}{2\pi}.$$
(15)

The second equality follows by the fact that $f_{j+}f_{j}^{-\frac{1}{2}}$ is a frequency-dependent unitary matrix. 3) *Multivariable Itakura-Saito distance:* The classical Itakura-Saito distance can be readily generalized by taking

$$d(f, I) = tr(f - \log f - I).$$

The values are always positive for $I \neq f > 0$ and equal to zero when f = I. Thus, we may define

$$D_{IS}(f_1, f_2) = \int_{-\pi}^{\pi} d(f_{2+}^{-1} f_1 f_{2+}^{-*}, I) \frac{d\theta}{2\pi}$$

$$= \int_{-\pi}^{\pi} \left(\operatorname{tr}(f_2^{-1} f_1) - \log \det(f_2^{-1} f_1) - m \right) \frac{d\theta}{2\pi}.$$
(16)

The Itakura-Saito distance has its origins in maximum likelihood estimation for speech processing and is related to the Kullback-Leibler divergence between the probability laws of two Gaussian random processes [2], [12]. More recently, [8] introduced the matrix-version of the Itakura-Saito distance for solving the state-covariance matching problem in a multivariable setting.

4) Log-spectral deviation: It has been argued that a logarithmic measure of spectral deviations is in agreement with perceptive qualities of sound and for this reason it has formed the basis for the oldest distortion measures considered [2]. In particular, the L_2 distance between the logarithms of power spectra is referred to as "Log-spectral deviation" or the "logarithmic energy." A natural multivariable version is to consider

$$d(f, I) = \|\log(f)\|_{Fr}^2$$

This expression is already symmetrized, since $d(f, I) = d(f^{-1}, I)$ by virtue of the fact that the eigenvalues of $\log(f)$ and those of $\log(f^{-1})$ differ only in their sign. Thereby,

$$\|\log(f_{j+}^{-1}f_if_{j+}^{-*})\|_{\mathrm{Fr}}^2 = \|\log(f_{i+}^{-1}f_jf_{i+}^{-*})\|_{\mathrm{Fr}}^2$$

Thus we define

$$D_{\text{Log}}(f_1, f_2) := \int_{-\pi}^{\pi} \|\log(f_{1+}^{-1} f_2 f_{1+}^{-*})\|_{\text{Fr}}^2 \frac{d\theta}{2\pi}$$

$$= \int_{-\pi}^{\pi} \|\log(f_1^{-\frac{1}{2}} f_2 f_1^{-\frac{1}{2}})\|_{\text{Fr}}^2 \frac{d\theta}{2\pi}.$$
(17)

This represents a multivariable version of the log-spectral deviation (see [2, page 370]). Interestingly, as we will see later on, $D_{Log}(f_1, f_2)$ possesses several useful properties and, in fact, its square root turns out to be precisely a geodesic distance in a suitable Riemannian geometry.

IV. RIEMANNIAN STRUCTURE ON MULTIVARIATE SPECTRA

Consider a "small" perturbation $f + \Delta$ away from a nominal power spectral density f. All divergence measures that we have seen so far are continuous in their arguments and, in-the-small, can be approximated by a quadratic form in Δ which depends continuously on f. This is what is referred to as a *Riemannian metric*. The availability of a metric gives the space of power spectral densities its properties. It dictates how perturbations in various directions compare to each other. It also provides additional important concepts: geodesics, geodesic distances, and curvature. Geodesics are paths of smallest length connecting the start to the finish; this length is the geodesic distance. Thus, geodesics in the space of power spectral densities represent deformations from a starting power spectral density f_0 to an end "point" f_1 . Curvature on the other hand is intimately connected with approximation and convexity of sets.

In contrast to a general divergence measure, the geodesic distance obeys the triangular inequality and thus, it is a metric (or, a pseudo-metric when by design it is unaffected by scaling or other group of transformations). Geodesics are also natural structures for modeling changes and deformations. In fact, a key motivation behind the present work is to model time-varying spectra via geodesic paths in a suitable metric space. This viewpoint provides a non-parametric model for non-stationary spectra, analogous to a spectrogram, but one which takes into account the inherent geometry of power spectral densities.

Thus, in the sequel we consider infinitesimal perturbations about a given power spectral density function. We explain how these give rise to nonnegative definite quadratic forms. Throughout, we assume that all functions are smooth enough so that the indicated integrals exist. This can be ensured if all spectral density functions are bounded with bounded derivatives and inverses. Thus, we will restrict our attention to the following class of PDF's:

$$\mathcal{F} := \{ f \mid m \times m \text{ positive definite, differentiable} \\ \text{on } [-\pi, \pi], \text{ with continuous derivative} \}.$$

In the above, we identify the end points of $[-\pi, \pi]$ since f is thought of as a function on the unit circle. Since the functions f are strictly positive definite and bounded, tangent directions of \mathcal{F} consists of admissible perturbations Δ . These need only be restricted to be differentiable with square integrable derivative, hence the tangent space at any $f \in \mathcal{F}$ can be identified with

$$\mathcal{D} := \{ \Delta \mid \text{differentiable on } [-\pi, \pi] \\ \text{with continuous derivative} \}.$$

A. Geometry based on the "flatness" of innovations spectra

We first consider the divergence D_1 in (9a-9b) which quantifies how far the PSD of the normalized innovations process is from being constant and equal to the identity. The induced Riemannian metric takes the form

$$g_{1,f}(\Delta) := \int_{-\pi}^{\pi} \|f^{-1/2} \Delta f^{-1/2}\|_{Fr}^2 \frac{d\theta}{2\pi}.$$
(18a)

Proposition 5: Let $(f, \Delta) \in \mathcal{F} \times \mathcal{D}$ and $\epsilon > 0$. Then, for ϵ sufficiently small,

$$D_1(f, f + \epsilon \Delta) = g_{1,f}(\epsilon \Delta) + O(\epsilon^3).$$

Proof: First note that

$$\operatorname{tr} \left(f(f+\epsilon\Delta)^{-1} \right) = \operatorname{tr} \left(f^{1/2} (I+f^{-1/2}\epsilon\Delta f^{-1/2})^{-1} f^{-1/2} \right)$$
$$= \operatorname{tr} \left(I+f^{-1/2}\epsilon\Delta f^{-1/2} \right)^{-1}$$
$$\operatorname{tr} \left(f(f+\epsilon\Delta)^{-1} \right) = m - \operatorname{tr} (f^{-1/2}\epsilon\Delta f^{-1/2})$$
$$+ \|f^{-1/2}\epsilon\Delta f^{-1/2}\|_{\operatorname{Fr}}^2 + O(\epsilon^3).$$

Likewise,

$$tr(f + \epsilon \Delta)f^{-1} = m + tr(\epsilon \Delta f^{-1})$$
$$= m + tr(f^{-1/2}\epsilon \Delta f^{-1/2})$$

Therefore,

$$D_1(f, f+\epsilon\Delta) = \operatorname{tr} \int_{-\pi}^{\pi} (f(f+\epsilon\Delta)^{-1} + (f+\epsilon\Delta)f^{-1} - 2I) \frac{d\theta}{2\pi}$$
$$= \int_{-\pi}^{\pi} ||f^{-1/2}\epsilon\Delta f^{-1/2}||_{\operatorname{Fr}}^2 \frac{d\theta}{2\pi} + O(\epsilon^3).$$

Obviously, an alternative expression for $g_{1,f}$ that requires neither spectral factorization nor the computation of the Hermitian square root of f, is the following:

$$g_{1,f}(\Delta) := \int_{-\pi}^{\pi} \operatorname{tr}\left(f^{-1}\Delta f^{-1}\Delta\right) \frac{d\theta}{2\pi}.$$
(18b)

It is interesting to also note that any of (14), (15), (16), and (17) leads to the same Riemannian metric.

B. Geometry based on suboptimality of prediction

The paradigm in [7] for a Riemannian structure of scalar power spectral densities was originally built on the degradation of predictive error variance, as this is reflected in the strength of the inequalities of Proposition 2. In this section we explore the direct generalization of that route. Thus, we consider the quadratic form which \mathcal{F} inherits from the relevant divergence D₂, defined in (11). The next proposition shows that this defines the corresponding metric:

$$g_{2,f}(\Delta) := \operatorname{tr} \int_{-\pi}^{\pi} (f_{+}^{-1} \Delta f_{+}^{-*})^{2} \frac{d\theta}{2\pi} - \operatorname{tr} \left(\int_{-\pi}^{\pi} f_{+}^{-1} \Delta f_{+}^{-*} \frac{d\theta}{2\pi} \right)^{2} = g_{1,f}(\Delta) - \operatorname{tr} \left(\int_{-\pi}^{\pi} f_{+}^{-1} \Delta f_{+}^{-*} \frac{d\theta}{2\pi} \right)^{2}.$$
(19)

Proposition 6: Let $(f, \Delta) \in \mathcal{F} \times \mathcal{D}$ and $\epsilon > 0$. Then, for ϵ sufficiently small,

$$D_2(f, f + \epsilon \Delta) = \frac{1}{2}g_{2,f}(\epsilon \Delta) + O(\epsilon^3).$$

Proof: In order to simplify the notation let

$$\Delta_{\epsilon} := f_+^{-1} \epsilon \Delta f_+^{-*}.$$

Since Δ , f are both bounded, $|\operatorname{tr}(\Delta_{\epsilon}^k)| = O(\epsilon^k)$ as well as $|\operatorname{tr}(\int_{-\pi}^{\pi} \Delta_{\epsilon} \frac{d\theta}{2\pi})^k| = O(\epsilon^k)$. Using a Taylor series expansion,

$$\operatorname{tr} \log \left(\int_{-\pi}^{\pi} f_{+}^{-1} (f + \epsilon \Delta) f_{+}^{-*} \frac{d\theta}{2\pi} \right)$$

=
$$\operatorname{tr} \log \left(I + \int_{-\pi}^{\pi} \Delta_{\epsilon} \frac{d\theta}{2\pi} \right)$$

=
$$\operatorname{tr} \left(\int_{-\pi}^{\pi} \Delta_{\epsilon} \frac{d\theta}{2\pi} \right) - \frac{1}{2} \operatorname{tr} \left(\int_{-\pi}^{\pi} \Delta_{\epsilon} \frac{d\theta}{2\pi} \right)^{2} + O(\epsilon^{3}),$$

while

$$\operatorname{tr}\left(\int_{-\pi}^{\pi} \log(f_{+}^{-1}(f+\epsilon\Delta)f_{+}^{-*})\frac{d\theta}{2\pi}\right)$$
$$=\int_{-\pi}^{\pi} \operatorname{tr}\log(I+\Delta_{\epsilon})\frac{d\theta}{2\pi}$$
$$=\int_{-\pi}^{\pi} \operatorname{tr}(\Delta_{\epsilon}-\frac{1}{2}\Delta_{\epsilon}^{2})\frac{d\theta}{2\pi}+O(\epsilon^{3}).$$

Thus

$$D_2(f, f + \epsilon \Delta) = \frac{1}{2} \operatorname{tr} \left(\int_{-\pi}^{\pi} \Delta_{\epsilon}^2 d\theta - \left(\int_{-\pi}^{\pi} \Delta_{\epsilon} \frac{d\theta}{2\pi} \right)^2 \right) + O(\epsilon^3).$$

Evidently, $g_{2,f}$ and $g_{1,f}$ are closely related. The other choices of D similarly yield either $g_{1,f}$, as noted earlier, or $g_{2,f}$. In fact, $g_{2,f}$ can be derived based on (12).

We remark a substantial difference between $g_{1,f}$ and $g_{2,f}$. In contrast to $g_{2,f}$, evaluation of $g_{1,f}$ does not require computing f_+ . However, on the other hand, both $g_{1,f}$, and $g_{2,f}$ are similarly unaffected by consistent scaling of fand Δ .

V. GEOMETRY ON POSITIVE MATRICES

As indicated earlier, a Riemannian metric $g(\Delta)$ on the space of Hermitian $m \times m$ matrices is a family of quadratic forms originating from inner products that depend smoothly on the Hermitian "foot point" M —the standard Hilbert-Schmidt metric $g_{HS}(\Delta) = \langle \Delta, \Delta \rangle := tr(\Delta^2)$ being one such. Of particular interest are metrics on the space of positive definite matrices that ensure the space is complete and geodesically complete². For our purposes, matrices typically represent covariances. To this end a standard recipe for constructing a Riemannian metric is to begin with an information potential, such as the Boltzmann entropy of a Gaussian distribution with zero mean and covariance M,

$$S(M) := -\frac{1}{2}\log(\det(M)) + \text{constant},$$

and define an inner product via its Hessian

$$\langle X, Y \rangle_M := \frac{\partial^2}{\partial x \partial y} S(M + xX + yY)|_{x=0,y=0}$$

= tr(M⁻¹XM⁻¹Y).

The Riemannian metric so defined,

$$g_M(\Delta) := \operatorname{tr}(M^{-1}\Delta M^{-1}\Delta)$$
$$= \|M^{-\frac{1}{2}}\Delta M^{-\frac{1}{2}}\|_{\operatorname{Fr}}^2,$$

is none other than the Fisher-Rao metric on Gaussian distributions expressed in the space of the corresponding covariance matrices.

The relationship of the Fisher-Rao metric on Gaussian distributions with the metric $g_{1,f}$ in (18b) is rather evident. Indeed, g_M coincides with $g_{1,f}$ for power spectra which are constant across frequencies, i.e., taking f = M to be a constant Hermitian positive definite matrix.

It is noted that $g_M(\Delta)$ remains invariant under congruence, that is,

$$g_M(\Delta) = g_{TMT^*}(T\Delta T^*)$$

for any square invertible matrix-function T. This is a natural property to demand since it implies that the distance between covariance matrices does not change under coordinate transformations. The same is inherited by $g_{1,f}$ for power spectra. It is for this reason that g_M has in fact been extensively studied in the context of general C^* -algebras and their positive elements; we refer to [11, pg. 201-235] for a nice exposition of relevant material and for further references. Below we highlight certain key facts that are relevant to this paper. But first, and for future reference, we recall a standard result in differential geometry.

Proposition 7: Let \mathcal{M} be a Riemannian manifold with $\|\Delta\|_M^2$ denoting the Riemannian metric at $M \in \mathcal{M}$ and Δ a tangent direction at M. For each pair of points $M_0, M_1 \in \mathcal{M}$ consider the path space

$$\Theta_{M_0,M_1} := \{M_\tau : [0,1] \to \mathcal{M} : M_\tau \text{ is a piecewise smooth} \}$$

path connecting the two given points}.

Denote by $\dot{M}_{\tau} := dM_{\tau}/d\tau$. The arc-length

$$\int_0^1 \|\dot{M}_\tau\|_M d\tau$$

²A space is complete when Cauchy sequences converge to points in the space. It is geodesically complete when the definition domain of geodesics extends to the complete real line \mathbb{R} ; i.e., extrapolating the path beyond the end points remains always in the space.

as well as the "action/energy" functional

$$\int_0^1 \|\dot{M}_\tau\|_M^2 d\tau$$

attain a minimum at a common path in Θ_{f_0,f_1} . Further, the minimal value of the arclength is the square root of the minimal value of the energy functional, and on a minimizing path the "speed" $\|\dot{M}_{\tau}\|_M$ remains constant for $\tau \in [0,1]$.

Proof: See [17, pg. 137].

The insight behind the statement of the proposition is as follows. The arclength is evidently unaffected by a reparametrization of a geodesic connecting the two points. The "energy" functional on the other hand, is minimized for a specific parametrization of geodesic where the velocity stays constant. Thus, the two are intimately related. The proposition will be applied first to paths between matrices, but in the next section it will also be invoked for geodesics between power spectra.

Herein we are interested in geodesic paths M_{τ} , $\tau \in [0, 1]$, connecting positive definite matrices M_0 to M_1 and in computing the corresponding geodesic distances

$$d_{g}(M_{0}, M_{1}) = \int_{0}^{1} \|M_{\tau}^{-1/2} \frac{dM_{\tau}}{d\tau} M_{\tau}^{-1/2}\|_{Fr} d\tau.$$

Recall that a geodesic M_{τ} is the shortest path on the manifold connecting the beginning to the end.

Theorem 8: Given Hermitian positive matrices M_0, M_1 the geodesic between them with respect to g_M is unique (modulo re-parametrization) and given by

$$M_{\tau} = M_0^{1/2} (M_0^{-1/2} M_1 M_0^{-1/2})^{\tau} M_0^{1/2}, \qquad (20)$$

for $0 \le \tau \le 1$. Further, it holds that

$$d_g(M_0, M_\tau) = \tau d_g(M_0, M_1), \text{ for } \tau \in [0, 1],$$

and the geodesic distance is

$$d_{g}(M_{0}, M_{1}) = \|\log(M_{0}^{-1/2}M_{1}M_{0}^{-1/2})\|_{Fr}.$$

Proof: A proof is given in [11, Theorem 6.1.6, pg. 205]. However, since this is an important result for our purposes and for completeness, we provide an independent short proof relying on Pontryagin's minimum principle.

We first note that, since g_M is congruence invariant, the path $TM_{\tau}T^*$ is a geodesic between TM_0T^* and TM_1T^* , for any invertible matrix T. Further, the geodesic length is independent of T. Thus, we set

$$T = M_0^{-\frac{1}{2}},$$

and seek a geodesic path between

$$X_0 = I \text{ and } X_1 = M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}}.$$
 (21)

Appealing to Proposition 7 we seek

$$\min\{\int_0^1 \operatorname{tr}(X_\tau^{-1}U_\tau X_\tau^{-1}U_\tau)d\tau,$$
subject to $\dot{X}_\tau = U_\tau$, and X_0, X_1 specified}. (22)

Now, (22) is a standard optimal control problem. The value of the optimal control must annihilate the variation of the Hamiltonian with respect to the "control" U_{τ}

$$\operatorname{tr}(X_{\tau}^{-1}U_{\tau}X_{\tau}^{-1}U_{\tau}) + \operatorname{tr}(\Lambda_{\tau}U_{\tau}).$$

Here, Λ_{τ} represents the co-state (i.e., Lagrange multiplier functions). The variation is

$$\operatorname{tr}(2X_{\tau}^{-1}U_{\tau}X_{\tau}^{-1}\delta_{U}+\Lambda_{\tau}\delta_{U})$$

and this being identically zero for all δ_U implies that

$$U_{\tau} = -\frac{1}{2} X_{\tau} \Lambda_{\tau} X_{\tau}. \tag{23}$$

Similarly, the co-state equation is obtained by considering the variation with respect to X. This gives

$$\dot{\Lambda}_{\tau} = 2X_{\tau}^{-1}U_{\tau}X_{\tau}^{-1}U_{\tau}X_{\tau}^{-1}$$

Substitute the expression for U_{τ} into the state and the co-state equations to obtain

$$\dot{X}_{\tau} = -\frac{1}{2}X_{\tau}\Lambda_{\tau}X_{\tau}$$
$$\dot{\Lambda}_{\tau} = \frac{1}{2}\Lambda_{\tau}X_{\tau}\Lambda_{\tau}.$$

Note that

$$X_{\tau}\Lambda_{\tau} + X_{\tau}\Lambda_{\tau} = 0,$$

identically, for all τ . Hence, the product $X_{\tau}\Lambda_{\tau}$ is constant. Set

$$X_{\tau}\Lambda_{\tau} = -2C. \tag{24}$$

The state equation becomes

$$X_{\tau} = CX_{\tau}.$$

The solution with initial condition $X_0 = I$ is

$$X_{\tau} = \exp(C\tau).$$

Matching (21) requires that $\exp(C) = X_1 = M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}}$. Thus, $X_{\tau} = (M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}})^{\tau}$ and the geodesic is as claimed. Further,

$$C = \log(M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}})$$

while $U_{\tau} = CX_{\tau}$ from (24) and (23). So finally, for the minimizing choice of U_{τ} we get that the cost

$$\int_0^\tau \operatorname{tr}(X_\tau^{-1} U_\tau X_\tau^{-1} U_\tau) d\tau = \int_0^\tau \operatorname{tr}(C^2) d\tau$$
$$= \tau \|\log(M_0^{-1/2} M_1 M_0^{-1/2})\|_{\operatorname{Fr}}^2$$

as claimed.

Remark 9: It's important to point out the lower bound

 $d_{g}(M_{0}, M_{1}) \ge \|\log M_{0} - \log M_{1}\|_{Fr}$ (25)

on the geodesic distance which holds with equality when M_0 and M_1 commute. This is known as the exponential metric increasing property [11, page 203] and will be used later on. \Box

The mid point of the geodesic path in (20) is what is known as the geometric mean of the two matrices M_0 and M_1 . This is commonly denoted by

$$M_{\frac{1}{2}} := M_0 \sharp M_1$$

Similar notation, with the addition of a subscript τ , will be used to designate the complete geodesic path

$$M_{\tau} = M_0 \sharp_{\tau} M_1 := M_0^{1/2} (M_0^{-1/2} M_1 M_0^{-1/2})^{\tau} M_0^{1/2}$$

(see [11]). A number of useful properties can be easily verified:i) Congruence invariance: for any invertible matrix T,

$$d_g(M_0, M_1) = d_g(TM_0T^*, TM_1T^*)$$

ii) Inverse invariance:

$$d_g(M_0, M_1) = d_g(M_0^{-1}, M_1^{-1})$$

iii) The metric satisfies the semiparallelogram law.

iv) The space of positive definite matrices metrized by $\rm d_g$ is complete; that is, any Cauchy sequence of positive definite matrices converges to a positive definite matrix.

v) Given any three "points" M_0 , M_1 , M_2 ,

$$\mathrm{d}_{\mathrm{g}}(M_0 \sharp_\tau M_1, M_0 \sharp_\tau M_2) \le \tau \,\mathrm{d}_{\mathrm{g}}(M_1, M_2),$$

which implies that geodesics diverge at least as fast as "Euclidean geodesics".

Remark 10: Property v) implies that the Riemannian manifold of positive definite matrices with metric d_g has nonpositive sectional curvature [18, pg. 39–40]. The nonpositive sectional curvature of a simply connected complete Riemannian manifold has several important geometric consequences. It implies the existence and uniqueness of a geodesic connecting any two points on the manifold [18, pg. 3–4]. Convex sets on such a manifold are defined by the requirement that geodesics between any two points in the set lie entirely in the set [18, pg. 67]. Then, "projections" onto the set exist in that there is always a closest point within convex set to any given point. Evidently, such a property should be valuable in applications, such as speaker identification or speech recognition based on a database of speech segments; e.g., models may be taken as the "convex hull" of prior sample spectra and the metric distance of a new sample compared to how far it resides from a given such convex set. Another property of such a manifold is that the center of mass of a set of points is contained in the closure of its convex hull [18, pg. 68]; this property has been used to define the geometric means of symmetric positive matrices in [19]. \Box

VI. GEODESICS AND GEODESIC DISTANCES

Power spectral densities are families of Hermitian matrices parametrized by the frequency θ , and as such, can be thought of as positive operators on a Hilbert space. Geometries for positive operators have been extensively studied for some time now, and power spectral densities may in principle be studied with similar tools. However, what it may be somewhat surprising is that the geometries obtained earlier, based on the innovations flatness and optimal prediction, have points of contact with this literature. This was seen in the correspondence between the metrics that we derived.

In the earlier sections we introduced two metrics, g_1 and g_2 . Although there is a close connection between the two, as suggested by (19), it is only for the former that we are able to identify geodesics and compute the geodesic lengths, based on the material in Section V. We do this next.

Theorem 11: There exists a unique geodesic path f_{τ} with respect to $g_{1,f}$, connecting any two spectra $f_0, f_1 \in \mathcal{F}$. The geodesic path is

$$f_{\tau} = f_0^{1/2} (f_0^{-1/2} f_1 f_0^{-1/2})^{\tau} f_0^{1/2}, \tag{26}$$

for $0 \le \tau \le 1$. The geodesic distance is

$$d_{g_1}(f_0, f_1) = \sqrt{\int_{-\pi}^{\pi} \|\log f_0^{-1/2} f_1 f_0^{-1/2}\|_{Fr}^2 \frac{d\theta}{2\pi}}$$

Proof: As before, in view of Proposition 7, instead of the geodesic length we may equivalently consider minimizing the energy/action functional

$$E = \int_0^1 \int_{-\pi}^{\pi} \|f_{\tau}^{-1/2} \dot{f}_{\tau} f_{\tau}^{-1/2}\|_{Fr}^2 \frac{d\theta}{2\pi} d\tau$$
$$= \int_{-\pi}^{\pi} \int_0^1 \|f_{\tau}^{-1/2} \dot{f}_{\tau} f_{\tau}^{-1/2}\|_{Fr}^2 d\tau \frac{d\theta}{2\pi}.$$

Clearly, this can be minimized point-wise in θ invoking Theorem 8. Now, inversion as well as the fractional power of symmetric (strictly) positive matrices represent continuous and differentiable maps. Hence, it can be easily seen that, because f_0 , f_1 are in \mathcal{F} so is

$$f_{\tau} = f_0^{1/2} (f_0^{-1/2} f_1 f_0^{-1/2})^{\tau} f_0^{1/2}.$$

Therefore, this path is the sought minimizer of

$$\int_0^1 \|f_\tau^{-1/2} \dot{f}_\tau f_\tau^{-1/2}\|_{\mathrm{Fr}}^2 d\tau$$

and the geodesic length is as claimed.

Corollary 12: Given any f_0 , f_1 , $f_2 \in \mathcal{F}$, the function $d_{g_1}(f_0 \sharp_{\tau} f_1, f_0 \sharp_{\tau} f_2)$ is convex on τ .

Proof: The proof is a direct consequence of the convexity of the metric $d_q(\cdot, \cdot)$.

The importance of the statement in the corollary is that the metric space has nonpositive curvature. Other properties are similarly inherited. For instance, d_{g_1} satisfies the semi-parallelogram law.

Next we explain that the closure of the space of positive differentiable power spectra, under g_1 , is simply power spectra that are squarely log integrable. This is not much of a surprise in view of the metric and the form of the geodesic distance. Thus, the next proposition shows that the completion, denoted by "bar," is in fact

$$\overline{\mathcal{F}} := \{ f \mid m \times m \text{ positive definite a.e.,} \\ \text{on } [-\pi, \pi], \ \log f \in L_2[-\pi, \pi] \}.$$
(27)

It should be noted that the metric d_{g_1} is not equivalent to an L_2 -based metric $\|\log(f_1) - \log(f_2)\|_2$ for the space. Here,

$$||h||_2 := \sqrt{\int_{-\pi}^{\pi} ||h||_{\mathrm{Fr}}^2 \frac{d\theta}{2\pi}}.$$

In fact, using the latter $\overline{\mathcal{F}}$ has zero curvature while, using d_{g_1} , $\overline{\mathcal{F}}$ becomes a space with non-positive (non-trivial) curvature.

Proposition 13: The completion of \mathcal{F} under d_{g_1} is as indicated in (27).

Proof: Clearly, for $f \in \mathcal{F}$, $\log f \in L_2[-\pi, \pi]$ since f is continuous on the closed interval and positive definite. Further, the logarithm maps positive differentiable matrix-functions to positive differentiable ones, bijectively. Our proof of $\overline{\mathcal{F}}$ being the completion of \mathcal{F} is carried out in three steps. First we will show that the limit of every Cauchy sequence in \mathcal{F} belongs to $\overline{\mathcal{F}}$. Next we argue that every point in $\overline{\mathcal{F}}$ is the limit of a sequence in \mathcal{F} , which together with the first step shows that \mathcal{F} is dense in $\overline{\mathcal{F}}$. Finally, we need to show that $\overline{\mathcal{F}}$ is complete with d_{g_1} .

First, consider a Cauchy sequence $\{f_n\}$ in \mathcal{F} which converges to f. Hence, there exists an N, such that for any $k \ge N$, $d_{g_1}(f_k, f) < 1$. Using the triangular inequality for d_{g_1} , we have that

$$\mathrm{d}_{\mathrm{g}_1}(I,f) \le \mathrm{d}_{\mathrm{g}_1}(I,f_N) + \mathrm{d}_{\mathrm{g}_1}(f_N,f),$$

or, equivalently,

$$\|\log f\|_2 < \|\log f_N\|_2 + 1.$$

Since $\|\log f_N\|_2$ is finite, $f \in \overline{\mathcal{F}}$.

Next, for any point f in $\overline{\mathcal{F}}$ which is not continuous, we show that it is the limit of a sequence in \mathcal{F} . Let $h = \log f$, then $h \in L_2[-\pi, \pi]$. Since the set of differentiable functions $C^1[-\pi, \pi]$ is dense in $L_2[-\pi, \pi]$, there exists a sequence $\{h_n \in C^1[-\pi, \pi]\}$ which converges to h in the L_2 norm. Using Theorem 3 in [20, pg. 86], there exists a subsequence $\{h_{n_k}\}$ which converges to h almost everywhere in $[-\pi, \pi]$, i.e.,

$$||h_{n_k}(\theta) - h(\theta)||_{\text{Fr}} \to 0$$
 a.e., as $n_k \to \infty$.

Since the exponential map is continuous [21, pg. 430], $||e^{h_{n_k}(\theta)} - e^h(\theta)||_{Fr}$ converges to 0 almost everywhere as well. Using the sub-multiplicative property of the Frobenius norm, we have that

$$\|I - e^{-h(\theta)} e^{h_{n_k}(\theta)}\|_{\mathrm{Fr}} \le \|e^{-h(\theta)}\|_{\mathrm{Fr}} \|e^{h_{n_k}(\theta)} - e^{h}(\theta)\|_{\mathrm{Fr}}$$

where the right side of the above inequality goes to zero. Thus the spectral radius of $(I - e^{-h(\theta)}e^{h_{n_k}(\theta)})$ goes to zero [22, pg. 297]. Hence, all the eigenvalues $\lambda_i(e^{-h(\theta)}e^{h_{n_k}(\theta)})$, $1 \le i \le m$, converge to 1 as $k \to \infty$. Then,

 $f_{n_k} = e^{h_{n_k}} \in \mathcal{F}$ and

$$d_{g_1}(f_{n_k}, f) = \sqrt{\int_{-\pi}^{\pi} \|\log f^{-1/2} f_{n_k} f^{-1/2}\|_{Fr}^2 \frac{d\theta}{2\pi}} \\ = \sqrt{\int_{-\pi}^{\pi} \sum_{i=1}^{m} \log^2 \lambda_i (f^{-1} f_{n_k}) \frac{d\theta}{2\pi}} \\ = \sqrt{\int_{-\pi}^{\pi} \sum_{i=1}^{m} \log^2 \lambda_i (e^{-h} e^{h_{n_k}}) \frac{d\theta}{2\pi}}.$$

Since $\log \lambda_i(e^{-h}e^{h_{n_k}}) \to 0$ a.e., for $1 \le i \le m$, $d_{g_1}(f_{n_k}, f) \to 0$ as well. Therefore, f is the limit of $\{f_{n_k}\}$.

Finally we show that $\overline{\mathcal{F}}$ is complete under d_{g_1} . Let $\{f_n\}$ be a Cauchy sequence in $(\overline{\mathcal{F}}, d_{g_1})$, and let $h_n = \log f_n$. Using the inequality (25), we have

$$d_{g_1}(f_k, f_l) \ge \sqrt{\int_{-\pi}^{\pi} \|h_k - h_l\|_{\mathrm{Fr}}^2 \frac{d\theta}{2\pi}}$$

Thus $\{h_n\}$ is also a Cauchy sequence in $L_2[-\pi,\pi]$, which is a complete metric space. As a result, $\{h_n\}$ converges to a point h in $L_2[-\pi,\pi]$. Following the similar procedure as in the previous step, there exists a subsequence $\{f_{n_k}\}$ which converges to $f = e^h \in \overline{\mathcal{F}}$. This completes our proof.

Remark 14: Geodesics of $g_{2,f}$ for scalar power spectra were constructed in [7]. At the present time, a multivariable generalization appears to be a daunting task. The main obstacle is of course non-commutativity of matricial density functions and the absence of an integral representation of analytic spectral factors in terms of matrix-valued power spectral densities. In this direction we point out that some of the needed tools are in place. For instance, a square matrix-valued function which is analytic and non-singular in the unit disc \mathbb{D} , admits a logarithm which is also analytic in \mathbb{D} . To see this, consider such a matrix-function, say $f_+(z)$. The matrix logarithm is well defined locally in a neighborhood of any $z_0 \in \mathbb{D}$ via the Cauchy integral

$$g(z) = \frac{1}{2\pi i} \int_{L_{z_0}} \ln(\zeta) (\zeta I - f_+(z))^{-1} d\zeta.$$

Here, L_{z_0} is a closed path in the complex plane that encompasses all of the eigenvalues of $f_+(z_0)$ and does not separate the origin from the point at ∞ . The Cauchy integral gives a matrix-function g(z) which is analytic in a sufficiently small neighborhood of z_0 in the unit disc \mathbb{D} —the size of the neighborhood being dictated by the requirement that the eigenvalues stay within L_{z_0} , and $\exp(g(z)) = f_+(z)$. To define the logarithm consistently over \mathbb{D} we need to ensure that we always take the same principle value. This is indeed the case if we extend g(z) via analytic continuation: since $f_+(z)$ is not singular anywhere in \mathbb{D} and the unit disc is simply connected, the values for g(z) will be consistent, i.e., any path from z_0 to an arbitrary $z \in \mathbb{D}$ will lead to the same value for g(z). Thus, one can set $\log(f_+) = g$ and understand this to be a particular version of the logarithm. Similarly, powers of f_+ can also be defined using Cauchy integrals,

$$\frac{1}{2\pi i} \int_{L_{z_0}} \zeta^{\tau} (\zeta I - f_+(z))^{-1} d\zeta$$

for $\tau \in [0, 1]$, first in a neighborhood of a given $z_0 \in \mathbb{D}$, and then by analytic continuation to the whole of \mathbb{D} . As with the logarithm, there may be several versions. Geodesics for $g_{2,f}$ appear to be require paths in the space of cannonical spectral factors for the corresponding matricial densities, such as $f_{\tau+} = f_{0+}(f_{0+}^{-1}f_{1+})^{\tau}_+$. However, the correct expression remains elusive at present. \Box

VII. EXAMPLES

We first demonstrate geodesics connecting two power spectral densities that correspond to all-pole models, i.e., two autoregressive (AR) spectra. The geodesic path between them does not consist of AR-spectra, and it can be considered as a non-parametric model for the transition. The choice of AR-spectra for the end points is only for convenience. As discussed earlier, the aim of the theory is to serve as a tool in non-parametric estimation, path following, morphing, etc., in the spectral domain.

A scalar example:

Consider the two power spectral denisities

$$f_i(\theta) = \frac{1}{|a_i(e^{j\theta})|^2}, \ i \in \{0, 1\},$$

where

$$a_0 = (z^2 - 1.96\cos(\frac{\pi}{5}) + 0.98^2)(z^2 - 1.7\cos(\frac{\pi}{3}) + 0.85^2)$$
$$(z^2 - 1.8\cos(\frac{2\pi}{3}) + 0.9^2),$$
$$a_1 = (z^2 - 1.96\cos(\frac{2\pi}{15}) + 0.98^2)(z^2 - 1.5\cos(\frac{7\pi}{30}) + 0.75^2)$$
$$(z^2 - 1.8\cos(\frac{5\pi}{8}) + 0.9^2).$$

Their roots are marked by \times 's and \circ 's respectively, in Figure 2, and shown with respect to the unit circle in the complex plane. We consider and compare the following three ways of interpolating power spectra between f_0 and f_1 .



Fig. 1. Plots of $\log f_0(\theta)$ (upper) and $\log f_1(\theta)$ (lower) for $\theta \in [0, \pi]$.



Fig. 2. Locus of the roots of $a_{\tau}(z)$ for $\tau \in [0, 1]$.

First, a parametric approach where the AR-coefficient are interpolated:

$$f_{\tau,\mathrm{AR}}(\theta) = \frac{1}{|a_{\tau}(e^{\mathbf{j}\theta})|^2},\tag{28a}$$

with $a_{\tau}(z) = (1 - \tau)a_0(z) + \tau a_1(z)$. Clearly, there is a variety of alternative options (e.g., to interpolate partial reflection coefficients, etc.). However, our choice is intended to highlight the fact that in a parameter space, admissible models may not always form a convex set. This is evidently the case here as the path includes factors that become "unstable." The locus of the roots of $a_{\tau}(z) = 0$ for $\tau \in [0, 1]$ is shown in Figure 2.

Then we consider a linear segment connecting the two spectra:

$$f_{\tau,\text{linear}} = (1 - \tau)f_0 + \tau f_1.$$
 (28b)

Again, this is to highlight the fact that the space of power spectra is not linear, and in this case, extrapolation beyond the convex linear combination of the two spectra leads to inadmissible function (as the path leads outside of the cone of positive functions). Finally, we provide the g_1 -geodesic between the two

 $f_{\tau,\text{geodesic}} = f_0 (\frac{f_1}{f_0})^{\tau}.$ (28c)

We compare $f_{\tau,AR}$, $f_{\tau,\text{linear}}$ and $f_{\tau,\text{geodesic}}$ for $\tau \in \{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}\}$. We first note that in plotting $\log f_{\tau,AR}$ in Figure 3, that $f_{\frac{2}{3},AR}$ is not shown since it is not admissible. Likewise $\log f_{\tau,\text{linear}}$ in Figure 4 breaks up for $\tau = \frac{4}{3}$, since

Fig. 3. $\log f_{\tau,AR}(\theta)$ for $\tau = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$ (blue), $\tau = 0, 1$ (red).

 $f_{\frac{4}{3},\text{linear}}$ becomes negative for a range of frequencies –dashed curve indicates the absolute value of the logarithm when this takes complex values. The plot of log $f_{\tau,\text{geodesic}}$ is defined for all the τ and shown in Figure 5. It is worth





Fig. 5. $\log f_{\tau,\text{geodesic}}(\theta)$ for $\tau = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$ (blue), $\tau = 0, 1$ (red).

pointing out how two apparent "modes" in $f_{\tau,\text{linear}}$ and $f_{\tau,\text{geodesic}}$ are swapping their dominance, which does not occur when following $f_{\tau,\text{AR}}$.



A multivariable example:

Consider the two matrix-valued power spectral densities

$$f_{0} = \begin{bmatrix} 1 & 0\\ 0.1e^{j\theta} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{|a_{0}(e^{j\theta})|^{2}} & 0\\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.1e^{-j\theta}\\ 0 & 1 \end{bmatrix}$$
$$f_{1} = \begin{bmatrix} 1 & 0.1e^{j\theta}\\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0\\ 0 & \frac{1}{|a_{1}(e^{j\theta})|^{2}} \end{bmatrix} \begin{bmatrix} 1 & 0\\ 0.1e^{-j\theta} & 1 \end{bmatrix}.$$

Typically, these reflect the dynamic relationship between two time series; in turn these may represent noise input/output of dynamical systems or measurements across independent array of sensors, etc. The particular example reflects the typical effect of an energy source shifting its signature from one of two sensors to the other as, for instance, a possible scatterer moves with respect to the two sensors.

Below f_0 and f_1 are shown in Fig. 6 and Fig. 7, respectively. Since the value of a power spectral density f, at each point in frequency, is a Hermitian matrix, our convention is to show in the (1,1), (1,2) and (2,2) subplots the log-magnitude of the entries f(1,1), f(1,2) (which is the same as f(2,1)) and f(2,2), respectively. Then, since only f(1,2) is complex (and the complex conjugate of f(2,1)), we plot its phase in the (2,1) subplot.



Fig. 6. Subplots (1,1), (1,2) and (2,2) show $\log f_0(1,1), \log |f_0(1,2)|$ (same as $\log |f_0(2,1)|$) and $\log f_0(2,2)$. Subplot (2,1) shows $\arg(f_0(2,1))$.



Fig. 7. Subplots (1,1), (1,2) and (2,2) show $\log f_1(1,1), \log |f_1(1,2)|$ (same as $\log |f_1(2,1)|$) and $\log f_0(2,2)$. Subplot (2,1) shows $\arg(f_1(2,1))$.



Fig. 8. Subplots (1,1), (1,2) and (2,2) show $\log f_{\tau}(1,1), \log |f_{\tau}(1,2)|$ (same as $\log |f_{\tau}(2,1)|$) and $\log f_{\tau}(2,2)$. Subplot (2,1) shows $\arg(f_{\tau}(2,1))$, for $\tau \in [0,1]$.

Three dimensional surface show the geodesic connecting f_0 to f_1 in Figure 8. Here, $f_{\tau,\text{geodesic}}$ is drawn using

$$f_{\tau,\text{geodesic}} = f_0^{\frac{1}{2}} (f_0^{-\frac{1}{2}} f_1 f_0^{-\frac{1}{2}})^{\tau} f_0^{\frac{1}{2}}.$$

It is interesting to observe the smooth shift of the energy across frequency and directionality.

VIII. CONCLUSIONS

The aim of this study has been to develop multivariable divergence measures and metrics for matrix-valued power spectral densities. These are expected to be useful in quantifying uncertainty in the spectral domain, detecting events in non-stationary time series, smoothing and spectral estimation in the context of vector valued stochastic processes. The spirit of the work follows closely classical accounts going back to [1], [2] and proceeds along the lines of [7]. Early work in signal analysis and system identification has apparently focused only on divergence measures between scalar spectral densities, and only recently have such issues on multivariable power spectra attracted attention [8], [9]. Further, this early work on scalar power spectra was shown to have deep roots in statistical inference, the Fisher-Rao metric, and Kullback-Leibler divergence [6], [2, page 371], [7], [13]. Thus, it is expected that interesting connections between the geometry of multivariable power spectra and information geometry will be established as well.

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