# Motion Planning for Kinematic systems 

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#### Abstract

In this paper, we present a general theory of motion planning for kinematic systems. This theory has been developed for long by one of the authors in a previous series of papers. It is mostly based upon concepts from subriemannian geometry. Here, we summarize the results of the theory, and we improve on, by developping in details an intricated case: the ball with a trailer, which corresponds to a distribution with flag of type 2,3,5,6.

This paper is dedicated to Bernard Bonnard for his $\mathbf{6 0}{ }^{\text {th }}$ birthday.


Index Terms-Optimal control, Subriemannian geometry, robotics, motion planning

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References

## I. Introduction

Here we present the main lines of a theory of motion planning for kinematic systems, which is developped for about ten years in the papers [8], [9], [10], [11], [12], [13], [14]. One of the purposes of the paper is to survey the whole theory disseminated in these papers. But also we improve on the theory, by treating one more case, in which "the fourth order brackets are involved". We aslo improve on several previous results (periodicity of our optimal trajectories for instance). Potential application of this theory is motion planning for kinematic robots. We will show several basic examples here.

The theory starts from the seminal work of F. Jean, in the papers [16], [17], [18]. At the root of this point of view in robotics, there are also more applied authors like J.P. Laumond [20]. See also [25].

We consider kinematic systems that are given under the guise of a vector-distribution $\Delta$ over a $n$-dimensional manifold $M$. The rank of the distribution is $p$, and the corank $k=$ $n-p$. Motion planning problems will aways be local problems in an open neighborhood of a given finite path $\Gamma$ in $M$. Then we may always consider that $M=\mathbb{R}^{n}$. From a control point of view, a kinematic system can be specified by a control system, linear in the controls, typically denoted by $\Sigma$ :

$$
\begin{equation*}
(\Sigma) \dot{x}=\sum_{i=1}^{p} F_{i}(x) u_{i} \tag{1}
\end{equation*}
$$

where the $F_{i}$ 's are smooth $\left(C^{\infty}\right)$ vector fields that span the distribution $\Delta$. The standard controllability assumption is always assumed, i.e. the Lie algebra generated by the $F_{i}$ 's is transitive on $M$. Consequently, the distribution $\Delta$ is completely nonintegrable, and any smooth path $\Gamma:[0, T] \rightarrow M$ can be unifomly approximated by an admissible path $\gamma:[0, \theta] \rightarrow M$, i.e. a Lipschitz path, which is almost everywhere tangent to $\Delta$, i.e., a trajectory of (1).

This is precisely the abstract answer to the kinematic motion planning probem: it is possible to approximate uniformly nonadmissible paths by admissible ones. The purpose of this paper is to present a general constructive theory that solves

More precisely, in this class of problems, it is natural to try to minimize a cost of the following form:

$$
J(u)=\int_{0}^{\theta} \sqrt{\sum_{i=1}^{p}\left(u_{i}\right)^{2}} d t
$$

for several reasons: 1. the optimal curves do not depend on their parametrization, 2. the minimization of such a cost produces a metric space (the associated distance is called the subriemannian distance, or the Carnot-Caratheodory distance), 3. Minimizing such a cost is equivalent to minimize the following (called the energy of the path) quadratic cost $J_{E}(u)$, in fixed time $\theta$ :

$$
J_{E}(u)=\int_{0}^{\theta} \sum_{i=1}^{p}\left(u_{i}\right)^{2} d t
$$

The distance is defined as the minimum length of admissible curves connecting two points, and the length of the admissible curve corresponding to the control $u:[0, \theta] \rightarrow M$ is just $J(u)$.

In this presentation, another way to interpret the problem is as follows: the dynamics is specified by the distribution $\Delta$ (i.e. not by the vector fields $F_{i}$, but their span only). The cost is then determined by an Euclidean metric $g$ over $\Delta$, specified here by the fact that the $F_{i}$ 's form an orthonormal frame field for the metric.

At this point we would like to make a more or less philosophical comment: there is, in the world of nonlinear control theory, a permanent twofold critic against the optimal control approach: 1. the choice of the cost to be minimized is in general rather arbitrary, and 2. optimal control solutions may be non robust.

Some remarkable conclusions of our theory show the following: in reasonable dimensions and codimensions, the optimal trajectories are extremely robust, and in particular, do not depend at all (modulo certain natural transformations) on the choice of the metric, but on the distribution $\Delta$ only. Even stronger: they depend only on the nilpotent approximation along $\Gamma$ (a concept that will be defined later on, which is a good local approximation of the problem). For a lot of low values of the rank $p$ and corank $k$, these nilpotent approximations have no parameter (hence they are in a sense universal). The asymptotic optimal sysntheses (i.e. the phase portraits of the admissible trajectories that approximate up to a small $\varepsilon$ ) are also universal.

Given a motion planning problem, specified by a (nonadmissible) curve $\Gamma$, and a Subriemannian structure (1), we will consider two distinct concepts, namely: 1. The metric complexity $M C(\varepsilon)$ that measures asymptotically the length of the best $\varepsilon$-approximating admissible trajectories, and 2 . The interpolation entropy $E(\varepsilon)$, that measures the length of the best admissible curves that interpolate $\Gamma$ with pieces of length $\varepsilon$.

The first concept was introduced by F. Jean in his basic paper [16]. The second concept is closely related with the entropy of F. Jean in [17], which is more or less the same as the Kolmogorov's entropy of the path $\Gamma$, for the metric
structure induced by the Carnot-Caratheodory metric of the ambient space.

Also, along the paper, we will deal with generic problems only (but generic in the global sense, i.e. stable singularities are considered). That is, the set of motion planning problems on $\mathbb{R}^{n}$ is the set of couples $(\Gamma, \Sigma)$, embedded with the $C^{\infty}$ topology of uniform convergence over compact sets, and generic problems (or problems in general position) form an open-dense set in this topology. For instance, it means that the curve $\Gamma$ is always tranversal to $\Delta$ (except maybe at isolated points, in the cases $k=1$ only). Another example is the case of a surface of degeneracy of the Lie bracket distribution $[\Delta, \Delta]$ in the $n=3, k=1$ case. Generically, this surface (the Martinet surface) is smooth, and $\Gamma$ intersects it transversally at a finite number of points only.

Also, along the paper, we will illustrate our results with one of the following well known academic examples:

Example 1: the unicycle:

$$
\begin{equation*}
\dot{x}=\cos (\theta) u_{1}, \dot{y}=\sin (\theta) u_{1}, \dot{\theta}=u_{2} \tag{2}
\end{equation*}
$$

Example 2: the car with a trailer:

$$
\begin{equation*}
\dot{x}=\cos (\theta) u_{1}, \dot{y}=\sin (\theta) u_{1}, \dot{\theta}=u_{2}, \dot{\varphi}=u_{1}-\sin (\varphi) u_{2} \tag{3}
\end{equation*}
$$

Example 3: the ball rolling on a plane:

$$
\dot{x}=u_{1}, \dot{y}=u 2, \quad \dot{R}=\left[\begin{array}{ccc}
0 & 0 & u_{1}  \tag{4}\\
0 & 0 & u_{2} \\
-u_{1} & -u_{2} & 0
\end{array}\right] R
$$

where $(x, y)$ are the coordinates of the contact point between the ball and the plane, $R \in S O(3, \mathbb{R})$ is the right orthogonal matrix representing an othonormal frame attached to the ball.

Example 4: the ball with a trailer

$$
\begin{align*}
& \dot{x}=u_{1}, \dot{y}=u 2, \dot{R}=\left[\begin{array}{ccc}
0 & 0 & u_{1} \\
0 & 0 & u_{2} \\
-u_{1} & -u_{2} & 0
\end{array}\right] R  \tag{5}\\
& \dot{\theta}=-\frac{1}{L}\left(\cos (\theta) u_{1}+\sin (\theta) u_{2}\right)
\end{align*}
$$

Typical motion planning problems are: 1. for example 22, the parking problem: the non admissible curve $\Gamma$ is $s \rightarrow$ $(x(s), y(s), \theta(s), \varphi(s))=\left(s, 0, \frac{\pi}{2}, 0\right), 2$. for example (3), the full rolling with slipping problem, $\Gamma: s \rightarrow(x(s), y(s), R(s))$ $=(s, 0, I d)$, where $I d$ is the identity matrix. On figures 1 . 2 we show our approximating trajectories for both problems, that are in a sense universal. In figure 1, of course, the $x$-scale is much larger than the $y$-scale.

Up to now, our theory covers the following cases:
(C1) The distribution $\Delta$ is one-step bracket generating (i.e. $\operatorname{dim}([\Delta, \Delta]=n)$ except maybe at generic singularities,
(C2) The number of controls (the dimension of $\Delta$ ) is $p=2$, and $n \leq 6$.

The paper is organized as follows: In the next section $\Pi$. we introduce the basic concepts, namely the metric complexity,


Fig. 1. Parking of the car with a trailer


Fig. 2. Approximating rolling with slipping
the interpolation entropy, the nilpotent approximation along $\Gamma$, and the normal coordinates, that will be our basic tools.

Section III summarizes the main results of our theory, disseminated in our previous papers, with some complements and details. Section IV is the detailed study of the case $n=6$, $k=4$, which corresponds in particular to example 4, the ball with a trailer. In Section $V$, we state a certain number of remarks, expectations and conclusions.

## II. BASIC CONCEPTS

In this section, we fix a generic motion planning problem $\mathcal{P}=(\Gamma, \Sigma)$. Also, along the paper there is a small parameter $\varepsilon$ (we want to approximate up to $\varepsilon$ ), and certain quantities $f(\varepsilon), g(\varepsilon)$ go to $+\infty$ when $\varepsilon$ tends to zero. We say that such quantities are equivalent $(f \simeq g)$ if $\lim _{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)}=1$. Also, $d$ denotes the subriemannian distance, and we consider the $\varepsilon$-subriemammian tube $T \varepsilon$ and cylinder $C \varepsilon$ around $\Gamma$ :

$$
\begin{aligned}
& T_{\varepsilon}=\{x \in M \mid d(x, \Gamma) \leq \varepsilon\} \\
& C_{\varepsilon}=\{x \in M \mid d(x, \Gamma)=\varepsilon\}
\end{aligned}
$$

## A. Entropy versus metric complexity

Definition 5: The metric complexity $M C(\varepsilon)$ of $\mathcal{P}$ is $\frac{1}{\varepsilon}$ times the minimum length of an admissible curve $\gamma_{\varepsilon}$ connecting the
endpoints $\Gamma(0), \Gamma(T)$ of $\Gamma$, and remaining in the tube $T_{\varepsilon}$.
Definition 6: The interpolation entropy $E(\varepsilon)$ of $\mathcal{P}$ is $\frac{1}{\varepsilon}$ times the minimum length of an admissible curve $\gamma_{\varepsilon}$ connecting the endpoints $\Gamma(0), \Gamma(T)$ of $\Gamma$, and $\varepsilon$-interpolating $\Gamma$, that is, in any segment of $\gamma_{\varepsilon}$ of length $\geq \varepsilon$, there is a point of $\Gamma$.

These quantities $M C(\varepsilon), E(\varepsilon)$ are functions of $\varepsilon$ which tends to $+\infty$ as $\varepsilon$ tends to zero. They are considered up to equivalence.The reason to divide by $\varepsilon$ is that the second quantity counts the number of $\varepsilon$-balls to cover $\Gamma$, or the number of pieces of length $\varepsilon$ to interpolate the full path. This is also the reason for the name "entropy".

Definition 7: An asymptotic optimal synthesis is a oneparameter family $\gamma_{\varepsilon}$ of admissible curves, that realizes the metric complexity or the entropy.

Our main purpose in the paper is twofold:

1. We want to estimate the metric complexity and the entropy, in terms of certain invariants of the problem. Actually, in all the cases treated in this paper, we will give eplicit formulas.
2. We shall exhibit explicit asymptotic optimal syntheses realizing the metric complexity or/and the entropy.

## B. Normal coordinates

Take a parametrized $p$-dimensional surface $S$, transversal to $\Delta$ (maybe defined in a neighborhood of $\Gamma$ only),

$$
S=\left\{q\left(s_{1}, \ldots, s_{p-1}, t\right) \in \mathbb{R}^{n}\right\}, \text { with } q(0, \ldots, 0, t)=\Gamma(t)
$$

Such a germ exists if $\Gamma$ is not tangent to $\Delta$. The exclusion of a neighborhood of an isolated point where $\Gamma$ is tangent to $\Delta$, (that is $\Gamma$ becomes "almost admissible"), will not affect our estimates presented later on (it will provide a term of higher order in $\varepsilon$ ). .

In the following, $\mathcal{C}_{\varepsilon}^{S}$ will denote the cylinder $\{\xi ; d(S, \xi)=$ $\varepsilon\}$.

Lemma 8: (Normal coordinates with respect to $S$ ). There are mappings $x: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}, y: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k-1}, w: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}$, such that $\xi=(x, y, w)$ is a coordinate system on some neighborhood of $S$ in $\mathbb{R}^{n}$, such that:
0. $S(y, w)=(0, y, w), \Gamma=\{(0,0, w)\}$

1. The restriction $\Delta_{\mid S}=\operatorname{ker} d w \cap_{i=1, . . k-1} \operatorname{ker} d y_{i}$, the metric $g_{\mid S}=\sum_{i=1}^{p}\left(d x_{i}\right)^{2}$,
2. $\mathcal{C}_{\varepsilon}^{S}=\left\{\xi \mid \sum_{i=1}^{p} x_{i}{ }^{2}=\varepsilon^{2}\right\}$,
3. geodesics of the Pontryagin's maximum principle ([21]) meeting the transversality conditions w.r.t. $S$ are the straight lines through $S$, contained in the planes $P_{y_{0}, w_{0}}=\{\xi \mid(y, w)=$ $\left.\left(y_{0}, w_{0}\right)\right\}$. Hence, they are orthogonal to $S$.

These normal coordinates are unique up to changes of coordinates of the form

$$
\begin{equation*}
\tilde{x}=T(y, w) x,(\tilde{y}, \tilde{w})=(y, w) \tag{6}
\end{equation*}
$$

where $T(y, w) \in O(p)$, the $p$-orthogonal group.

## C. Normal forms, Nilpotent approximation along $\Gamma$

1) Frames: Let us denote by $F=\left(F_{1}, \ldots, F_{p}\right)$ the orthonormal frame of vector fields generating $\Delta$. Hence, we will also write $\mathcal{P}=(\Gamma, F)$. If a global coordinate system $(x, y, w)$, not necessarily normal, is given on a neighborhood of $\Gamma$ in $\mathbb{R}^{n}$, with $x \in \mathbb{R}^{p}, y \in \mathbb{R}^{k-1}, w \in \mathbb{R}$, then we write:

$$
\begin{align*}
F_{j} & =\sum_{i=1}^{p} \mathcal{Q}_{i, j}(x, y, w) \frac{\partial}{\partial x_{i}}+\sum_{i=1}^{k-1} \mathcal{L}_{i, j}(x, y, w) \frac{\partial}{\partial y_{i}}  \tag{7}\\
& +\mathcal{M}_{j}(x, y, w) \frac{\partial}{\partial w} \\
j & =1, \ldots, p
\end{align*}
$$

Hence, the SR metric is specified by the triple $(\mathcal{Q}, \mathcal{L}, \mathcal{M})$ of smooth $x, y, w$-dependent matrices.
2) The general normal form: Fix a surface $S$ as in Section ?? and a normal coordinate system $\xi=(x, y, w)$ for a problem $\mathcal{P}$.

Theorem 9: (Normal form, [2]) There is a unique orthonormal frame $F=(\mathcal{Q}, \mathcal{L}, \mathcal{M})$ for $(\Delta, g)$ with the following properties:

1. $\mathcal{Q}(x, y, w)$ is symmetric, $\mathcal{Q}(0, y, w)=I d$ (the identity matrix),
2. $\mathcal{Q}(x, y, w) x=x$,
3. $\mathcal{L}(x, y, w) x=0, \mathcal{M}(x, y, w) x=0$.
4. Conversely if $\xi=(x, y, w)$ is a coordinate system satisfying conditions $1,2,3$ above, then $\xi$ is a normal coordinate system for the SR metric defined by the orthonormal frame $F$ with respect to the parametrized surface $\{(0, y, w)\}$.

Clearly, this normal form is invariant under the changes of normal coordinates (6).

Let us write:

$$
\begin{aligned}
\mathcal{Q}(x, y, w) & =I d+Q_{1}(x, y, w)+Q_{2}(x, y, w)+\ldots \\
\mathcal{L}(x, y, w) & =0+L_{1}(x, y, w)+L_{2}(x, y, w)+\ldots \\
\mathcal{M}(x, y, w) & =0+M_{1}(x, y, w)+M_{2}(x, y, w)+\ldots
\end{aligned}
$$

where $Q_{r}, L_{r}, M_{r}$ are matrices depending on $\xi=(x, y, w)$, the coefficients of which have order $k$ w.r.t. $x$ (i.e. they are in the $r^{t h}$ power of the ideal of $C^{\infty}(x, y, w)$ generated by the functions $\left.x_{r}, r=1, \ldots, n-p\right)$. In particular, $Q_{1}$ is linear in $x, Q_{2}$ is quadratic, etc... Set $u=$ $\left(u_{1}, \ldots, u_{p}\right) \in \mathbb{R}^{p}$. Then $\sum_{j=1}^{k-1} L_{1_{j}}(x, y, w) u_{j}=L_{1, y, w}(x, u)$ is quadratic in $(x, u)$, and $\mathbb{R}^{k-1}$-valued. Its $i^{t h}$ component is the quadratic expression denoted by $L_{1, i, y, w}(x, u)$. Similarly $\sum_{j=1}^{k-1} M_{1_{j}}(x, y, w) u_{j}=M_{1, y, w}(x, u)$ is a quadratic form in $(x, u)$. The corresponding matrices are denoted by $L_{1, i, y, w}$, $i=1, \ldots, k-1$, and $M_{1, y, w}$.

The following was proved in [2], [5] for corank 1:
Proposition 10: 1. $Q_{1}=0$,
2. $L_{1, i, y, w}, i=1, \ldots, p-1$, and $M_{1, y, w}$ are skew symmetric matrices.

A first useful very rough estimate in normal coordinates is the following:

Proposition 11: If $\xi=(x, y, w) \in T_{\varepsilon}$, then:

$$
\begin{aligned}
& \|x\|_{2} \leq \varepsilon \\
& \|y\|_{2} \leq k \varepsilon^{2}
\end{aligned}
$$

for some $k>0$.
At this point, we shall split the problems under consideration into two distinct cases: first the 2 -step bracket-generating case, and second, the 2-control case.
3) Two-step bracket-generating case: In that case, we set, in accordance to Proposition 11, that $x$ has weight 1 , and the $y_{i}$ 's and $w$ have weight 2. Then, the vector fields $\frac{\partial}{\partial x_{i}}$ have weight -1 , and $\frac{\partial}{\partial y_{i}}, \frac{\partial}{\partial w}$ have weight -2 .

Inside a tube $T_{\varepsilon}$, we write our control system as a term of order -1 , plus a residue, that has a certain order w.r.t. $\varepsilon$. Here, $O\left(\varepsilon^{k}\right)$ means a smooth term bounded by $c \varepsilon^{k}$. We have, for a trajectory remaining inside $T_{\varepsilon}$ :

$$
\begin{align*}
\dot{x} & =u+O\left(\varepsilon^{2}\right)  \tag{8}\\
\dot{y}_{i} & =\frac{1}{2} x^{\prime} L^{i}(w) u+O\left(\varepsilon^{2}\right) ; \quad i=1, \ldots, k-1 \\
\dot{w} & =\frac{1}{2} x^{\prime} M(w) u+O\left(\varepsilon^{2}\right)
\end{align*}
$$

where $L^{i}(w), M(w)$ are skew-symmetric matrices depending smoothly on $w$.

Remark 12: In 8, (1), the term $O\left(\varepsilon^{2}\right)$ can seem surprising. One should wait for $O(\varepsilon)$. It is due to (1) in Proposition 10

In that case, we define the Nilpotent Approximation $P$ along $\Gamma$ of the problem $\mathcal{P}$ by keeping only the term of order -1 :

$$
\begin{align*}
\dot{x} & =u  \tag{9}\\
\dot{y}_{i} & =\frac{1}{2} x^{\prime} L^{i}(w) u ; \quad i=1, \ldots, p-1  \tag{P}\\
\dot{w} & =\frac{1}{2} x^{\prime} M(w) u
\end{align*}
$$

Consider two trajectories $\xi(t), \hat{\xi}(t)$ of $\mathcal{P}$ and $\hat{\mathcal{P}}$ corresponding to the same control $u(t)$, issued from the same point on $\Gamma$, and both arclength-parametrized (which is equivalent to $\|u(t)\|=1)$. For $t \leq \varepsilon$, we have the following estimates:
$\|x(t)-\hat{x}(t)\| \leq c \varepsilon^{3},\|y(t)-\hat{y}(t)\| \leq c \varepsilon^{3},\|w(t)-\hat{w}(t)\| \leq c \varepsilon^{3}$,
for a suitable constant $c$.
Remark 13: It follows that the distance (either $d$ or $\hat{d}$-the distance associated with the nilpotent approximation) between $\xi(t), \hat{\xi}(t)$ is smaller than $\varepsilon^{1+\alpha}$ for some $\alpha>0$.

This fact comes from the estimate just given, and the standard ball-box Theorem ([15]). It will be the key point to reduce the motion planning problem to the one of its nilpotent approximation along $\Gamma$.
4) The 2-control case:
5) Normal forms: In that case, we have the following general normal form, in normal coordinates. It was proven first in [1], in the corank1 case. The proof holds in any corank, without modification.

Consider Normal coordinates with respect to any surface $\mathcal{S}$. There are smooth functions, $\beta(x, y, w), \gamma_{i}(x, y, w), \delta(x, y, w)$, such that $\mathcal{P}$ can be written as (on a neighborhood of $\Gamma$ ):

$$
\begin{align*}
\dot{x}_{1} & =\left(1+\left(x_{2}\right)^{2} \beta\right) u_{1}-x_{1} x_{2} \beta u_{2}  \tag{11}\\
\dot{x}_{2} & =\left(1+\left(x_{1}\right)^{2} \beta\right) u_{2}-x_{1} x_{2} \beta u_{1} \\
\dot{y}_{i} & =\gamma_{i}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \quad \dot{w}=\delta\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right),
\end{align*}
$$

where moreover $\beta$ vanishes on the surface $\mathcal{S}$.
The following normal forms can be obtained, on the tube $T_{\varepsilon}$, by just changing coordinates in $\mathcal{S}$ in certain appropriate way. It means that a trajectory $\xi(t)$ of $\mathcal{P}$ remaining in $T_{\varepsilon}$ satisfies:

Generic $4-2$ case (see [12]):

$$
\begin{aligned}
\dot{x}_{1} & =u_{1}+0\left(\varepsilon^{3}\right), \dot{x}_{2}=u_{2}+0\left(\varepsilon^{3}\right) \\
\dot{y} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{2}\right) \\
\dot{w} & =\delta(w) x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{3}\right) .
\end{aligned}
$$

We define the nilpotent approximation as:

$$
\begin{aligned}
\left(\hat{\mathcal{P}}_{4,2}\right) \quad \dot{x}_{1} & =u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \\
\dot{w} & =\delta(w) x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) .
\end{aligned}
$$

Again, we consider two trajectories $\xi(t), \hat{\xi}(t)$ of $\mathcal{P}$ and $\hat{\mathcal{P}}$ corresponding to the same control $u(t)$, issued from the same point on $\Gamma$, and both arclength-parametrized (which is equivalent to $\|u(t)\|=1$ ). For $t \leq \varepsilon$, we have the following estimates:

$$
\begin{equation*}
\|x(t)-\hat{x}(t)\| \leq c \varepsilon^{4},\|y(t)-\hat{y}(t)\| \leq c \varepsilon^{3},\|w(t)-\hat{w}(t)\| \leq c \varepsilon^{4} \tag{12}
\end{equation*}
$$

Which implies that, for $t \leq \varepsilon$, the distance ( $d$ or $\hat{d}$ ) between $\xi(t)$ and $\hat{\xi}(t)$ is less than $\varepsilon^{1+\alpha}$ for some $\alpha>0$, and this will be also the keypoint to reduce our problem to the Nilpotent approximation.

Generic 5-2 case (see [13]):

$$
\begin{aligned}
\dot{x}_{1} & =u_{1}+0\left(\varepsilon^{3}\right), \dot{x}_{2}=u_{2}+0\left(\varepsilon^{3}\right) \\
\dot{y} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{2}\right) \\
\dot{z} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{3}\right) \\
\dot{w} & =\delta(w) x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{3}\right) .
\end{aligned}
$$

We define the nilpotent approximation as:

$$
\begin{aligned}
\left(\hat{\mathcal{P}}_{5,2}\right) \quad \dot{x}_{1} & =u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \\
\dot{z} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \\
\dot{w} & =\delta(w) x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) .
\end{aligned}
$$

The estimates necessary to reduce to Nilpotent approximation are:

$$
\begin{align*}
\|x(t)-\hat{x}(t)\| & \leq c \varepsilon^{4},\|y(t)-\hat{y}(t)\| \leq c \varepsilon^{3}  \tag{13}\\
\|z(t)-\hat{z}(t)\| & \leq c \varepsilon^{4},\|w(t)-\hat{w}(t)\| \leq c \varepsilon^{4}
\end{align*}
$$

Generic 6-2 case (proven in Appendix):

$$
\begin{align*}
\dot{x}_{1} & =u_{1}+0\left(\varepsilon^{3}\right), \dot{x}_{2}=u_{2}+0\left(\varepsilon^{3}\right)  \tag{14}\\
\dot{y} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{2}\right) \\
\dot{z}_{1} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{3}\right) \\
\dot{z}_{2} & =x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{3}\right) \\
\dot{w} & =Q_{w}\left(x_{1}, x_{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{4}\right),
\end{align*}
$$

where $Q_{w}\left(x_{1}, x_{2}\right)$ is a quadratic form in $x$ depending smoothly on $w$.

We define the nilpotent approximation as:
$\left(\hat{\mathcal{P}}_{6,2}\right)$

$$
\begin{align*}
\dot{x}_{1} & =u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)  \tag{15}\\
\dot{z}_{1} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \dot{z}_{2}=x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \\
\dot{w} & =Q_{w}\left(x_{1}, x_{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)
\end{align*}
$$

The estimates necessary to reduce to Nilpotent approximation are:

$$
\begin{align*}
\|x(t)-\hat{x}(t)\| & \leq c \varepsilon^{4},\|y(t)-\hat{y}(t)\| \leq c \varepsilon^{3}  \tag{16}\\
\|z(t)-\hat{z}(t)\| & \leq c \varepsilon^{4},\|w(t)-\hat{w}(t)\| \leq c \varepsilon^{5}
\end{align*}
$$

In fact, the proof given in Appendix, of the reduction to this normal form, contains the other cases 4-2 and 5-2.
6) Invariants in the 6-2 case, and the ball with a trailer:

Let us consider a one form $\omega$ that vanishes on $\Delta^{\prime \prime}=$ $[\Delta,[\Delta, \Delta]]$. Set $\alpha=d \omega_{\mid \Delta}$, the restriction of $d \omega$ to $\Delta$. Set $H=\left[F_{1}, F_{2}\right], I=[F 1, H], J=\left[F_{2}, H\right]$, and consider the $2 \times 2$ matrix $A(\xi)=\left(\begin{array}{cc}d \omega\left(F_{1}, I\right) & d \omega\left(F_{2}, I\right) \\ d \omega\left(F_{1}, J\right) & d \omega\left(F_{2}, J\right)\end{array}\right)$.

Due tu Jacobi Identity, $A(\xi)$ is a symmetric matrix. It is also equal to $\left(\begin{array}{ll}\omega\left(\left[F_{1}, I\right]\right) & \omega\left(\left[F_{2}, I\right]\right) \\ \omega\left(\left[F_{1}, J\right]\right) & \omega\left(\left[F_{2}, J\right]\right)\end{array}\right)$, using the fact that $\omega([X, Y])=d \omega(X, Y)$ in restriction to $\Delta^{\prime \prime}$.

Let us consider a gauge transformation, i.e. a feedback that preserves the metric (i.e. a change of othonormal frame $\left(F_{1}, F_{2}\right)$ obtained by setting $\tilde{F}_{1}=\cos (\theta(\xi)) F_{1}+\sin (\theta(\xi)) F_{2}$, $\left.\tilde{F}_{2}=-\sin (\theta(\xi)) F_{1}+\cos (\theta(\xi)) F_{2}\right)$.

It is just a matter of tedious computations to check that the matrix $A(\xi)$ is changed for $\tilde{A}(\xi)=R_{\theta} A(\xi) R_{-\theta}$. On the other hand, the form $\omega$ is defined modulo muttiplication by a nonzero function $f(\xi)$, and the same holds for $\alpha$, since $d(f \omega)=f d \omega+d f \wedge \omega$, and $\omega$ vanishes over $\Delta^{\prime \prime}$. The following lemma follows:

Lemma 14: The ratio $r(\xi)$ of the (real) eigenvalues of $A(\xi)$ is an invariant of the structure.

Let us now consider the normal form (14), and compute the form $\omega=\omega_{1} d x_{1}+\ldots+\omega_{6} d w$ along $\Gamma$ (that is, where $x, y, z=0$ ). Computing all the brackets show that $\omega_{1}=\omega_{2}=$
$\ldots=\omega_{5}=0$. This shows also that in fact, along $\Gamma, A(\xi)$ is just the matrix of the quadratic form $Q_{w}$. We get the following:

Lemma 15: The invariant $r(\Gamma(t))$ of the problem $\mathcal{P}$ is the same as the invariant $\hat{r}(\Gamma(t))$ of the nilpotent approximation along $\Gamma$.

Let us compute the ratio $r$ for the ball with a trailer, Equation (5]. We denote by $A_{1}, A_{2}$ the two right-invariant vector fields over $S o(3, \mathbb{R})$ appearing in 5 . We have:

$$
\begin{aligned}
F_{1} & =\frac{\partial}{\partial x_{1}}+A_{1}-\frac{1}{L} \cos (\theta) \frac{\partial}{\partial \theta} \\
F_{2} & =\frac{\partial}{\partial x_{2}}+A_{2}-\frac{1}{L} \sin (\theta) \frac{\partial}{\partial \theta} \\
{\left[A_{1}, A_{2}\right] } & =A_{3},\left[A_{1}, A_{3}\right]=-A_{2},\left[A_{2}, A_{3}\right]=A_{1} .
\end{aligned}
$$

Then, we compute the brackets: $H=A_{3-\frac{1}{L^{2}}} \frac{\partial}{\partial \theta}$,
 $-A_{3-\frac{1}{L^{4}}} \frac{\partial}{\partial \theta},\left[F_{1}, J\right]=0=\left[F_{2}, I\right],\left[F_{2}, J\right]=-A_{3-\frac{1}{L^{4}} \frac{\partial}{\partial \theta}}$. Then:

Lemma 16: For the ball with a trailer, the ratio $r(\xi)=1$.
These two last lemmas are a key point in the section IV: theyl imply in particular that the system of geodesics of the nilpotent approximation is integrable in Liouville sense, as we shall see.

## III. Results

In this section, we summarize and comment most of the results obtained in the papers [8], [9], [10], [12], [13], [14].

## A. General results

We need the concept of an $\varepsilon$-modification of an asymptotic optimal synthesis.

Definition 17: Given a one parameter family of (absolutely continuous, arclength parametrized) admissible curves $\gamma_{\varepsilon}$ : $\left[0, T_{\gamma_{\varepsilon}}\right] \rightarrow \mathbb{R}^{n}$, an $\varepsilon$-modification of $\gamma_{\varepsilon}$ is another one parameter family of (absolutely continuous, arclength parametrized) admissible curves $\tilde{\gamma}_{\varepsilon}:\left[0, T_{\tilde{\gamma}_{\varepsilon}}\right] \rightarrow \mathbb{R}^{n}$ such that for all $\varepsilon$ and for some $\alpha>0$, if $\left[0, T_{\gamma_{\varepsilon}}\right]$ is splitted into subintervals of length $\varepsilon,[0, \varepsilon],[\varepsilon, 2 \varepsilon],[2 \varepsilon, 3 \varepsilon], \ldots$ then:

1. $\left[0, T_{\tilde{\gamma}_{\varepsilon}}\right]$ is splitted into corresponding intervals, $\left[0, \varepsilon_{1}\right]$, $\left[\varepsilon_{1}, \varepsilon_{1}+\varepsilon_{2}\right],\left[\varepsilon_{1}+\varepsilon_{2}, \varepsilon_{1}+\varepsilon_{2}+\varepsilon_{3}\right], \ldots$ with $\varepsilon \leq \varepsilon_{i}<\varepsilon\left(1+\varepsilon^{\alpha}\right)$, $i=1,2, \ldots$,
2. for each couple of an interval $I_{1}=\left[\tilde{\varepsilon}_{i}, \tilde{\varepsilon}_{i}+\varepsilon\right]$, (with $\left.\tilde{\varepsilon}_{0}=0, \tilde{\varepsilon}_{1}=\varepsilon_{1}, \tilde{\varepsilon}_{2}=\varepsilon_{1}+\varepsilon_{2}, \ldots\right)$ and the respective interval $I_{2}=[i \varepsilon,(i+1) \varepsilon], \frac{d}{d t}(\tilde{\gamma})$ and $\frac{d}{d t}(\gamma)$ coincide over $I_{2}$, i.e.:
$\frac{d}{d t}(\tilde{\gamma})\left(\tilde{\varepsilon}_{i}+t\right)=\frac{d}{d t}(\gamma)(i \varepsilon+t)$, for almost all $t \in[0, \varepsilon]$.
Remark 18: This concept of an $\varepsilon$-modification is for the following use: we will construct asymptotic optimal syntheses for the nilpotent approximation $\hat{\mathcal{P}}$ of problem $\mathcal{P}$. Then, the asymptotic optimal syntheses have to be slightly modified in order to realize the interpolation constraints for the original (non-modified) problem. This has to be done "slightly" for the length of paths remaining equivalent.

In this section it is always assumed but not stated that we consider generic problems only. One first result is the following:

Theorem 19: In the cases 2 -step bracket generating, 4-2, 52, 6-2, (without singularities), an asymptotic optimal synthesis [relative to the entropy] for $\mathcal{P}$ is obtained as an $\varepsilon$-modification of an asymptotic optimal synthesis for the nilpotent approximation $\hat{\mathcal{P}}$. As a consequence the entropy $E(\varepsilon)$ of $\mathcal{P}$ is equal to the entropy $\hat{E}(\varepsilon)$ of $\hat{\mathcal{P}}$.

This theorem is proven in [12]. However, we can easily get an idea of the proof, using the estimates of formulas 10,12 , 13, 16.

All these estimates show that, if we apply an $\varepsilon$-interpolating strategy to $\hat{\mathcal{P}}$, and the same controls to $\mathcal{P}$, at time $\varepsilon$ (or length $\varepsilon$-since it is always possible to consider arclengthparametrized trajectories), the enpoints of the two trajectories are at subriemannian distance (either $d$ or $\hat{d}$ ) of order $\varepsilon^{1+\alpha}$, for some $\alpha>0$. Then the contribution to the entropy of $\mathcal{P}$, due to the correction necessary to interpolate $\Gamma$ will have higher order.

Also, in the one-step bracket-generating case, we have the following equality:

Theorem 20: (one step bracket-generating case, corank $k \leq$ 3) The entropy is equal to $2 \pi$ times the metric complexity: $E(\varepsilon)=2 \pi M C(\varepsilon)$.

The reason for this distinction between corank less or more than 3 is very important, and will be explained in the section III-C

Another very important result is the following logarithmic lemma, that describes what happens in the case of a (generic) singularity of $\Delta$. In the absence of such singularities, as we shall see, we shall always have formulas of the following type, for the entropy (the same for the metric complexity):

$$
\begin{equation*}
E(\varepsilon) \simeq \frac{1}{\varepsilon^{p}} \int_{\Gamma} \frac{d t}{\chi(t)} \tag{17}
\end{equation*}
$$

where $\chi(t)$ is a certain invariant along $\Gamma$. When the curve $\Gamma(t)$ crosses tranversally a codimension- 1 singularity (of $\Delta^{\prime}$, or $\left.\Delta^{\prime \prime}\right)$, the invariant $\chi(t)$ vanishes. This may happen at isolated points $t_{i}, \quad i=1, \ldots r$. In that case, we always have the following:

Theorem 21: (logarithmic lemma). The entropy (resp. the metric complexity) satisfies:

$$
E(\varepsilon) \simeq-2 \frac{\ln (\varepsilon)}{\varepsilon^{p}} \sum_{i=1}^{r} \frac{1}{\rho\left(t_{i}\right)}, \quad \text { where } \rho(t)=\left|\frac{d \chi(t)}{d t}\right|
$$

On the contrary, there are also generic codimension 1 singularities where the curve $\Gamma$, at isolated points, becomes tangent to $\Delta$, or $\Delta^{\prime}, \ldots$ At these isolated points, the invariant $\chi(t)$ of Formula 17 tends to infinity. In that case, the formula 17 remains valid (the integral converges).

## B. Generic distribution in $\mathbb{R}^{3}$

This is the simplest case, and is is important, since many cases just reduce to it. Let us describe it in details.

Generically, the 3-dimensional space $M$ contains a 2dimensional singularity (called the Martinet surface, denoted by $\mathcal{M}$ ). This singularity is a smooth surface, and (except at


Fig. 3. 3-dimensional contact case
isolated points on $\mathcal{M}$ ), the distribution $\Delta$ is not tangent to $\mathcal{M}$. Generically, the curve $\Gamma$ crosses $\mathcal{M}$ transversally at a finite number of isolated points $t_{i}, i=1, \ldots, r$. These points are not the special isolated points where $\Delta$ is tangent to $\mathcal{M}$ (this would be not generic). They are called Martinet points. This number $r$ can be zero. Also, there are other isolated points $\tau_{j}$, $j=1, \ldots, l$, at which $\Gamma$ is tangent to $\Delta$ (which means that $\Gamma$ is almost admissible in a neighborhood of $\tau_{j}$ ). Out of $\mathcal{M}$, the distribution $\Delta$ is a contact distribution (a generic property).
Let $\omega$ be a one-form that vanishes on $\Delta$ and that is 1 on $\dot{\Gamma}$, defined up to multiplication by a function which is 1 along $\Gamma$. Along $\Gamma$, the restriction 2 -form $d \omega_{\mid \Delta}$ can be made into a skew-symmetric endomorphism $A(\Gamma(t))$ of $\Delta$ (skew symmetric with respect to the scalar product over $\Delta$ ), by duality: $<A(\Gamma(t)) X, Y>=d \omega(X, Y)$. Let $\chi(t)$ denote the moduli of the eigenvalues of $A(\Gamma(t))$. We have the following:
Theorem 22: 1. If $r=0, M C(\varepsilon) \simeq \frac{2}{\varepsilon^{2}} \int_{\Gamma} \frac{d t}{\varkappa(t)}$. At points where $\chi(t) \rightarrow+\infty$, the formula is convergent.
2. If $r \neq 0, M C(\varepsilon) \simeq-2 \frac{\ln (\varepsilon)}{\varepsilon^{2}} \sum_{i=1}^{r} \frac{1}{\rho\left(t_{i}\right)}, \quad$ where $\rho(t)=$ $\left|\frac{d \chi(t)}{d t}\right|$.
3. $E(\varepsilon)=2 \pi M C(\varepsilon)$.

Let us describe the asymptotic optimal syntheses. They are shown on Figures 3, 4

Figure 3 concerns the case $r=0$ (everywhere contact type). The points where the distribution $\Delta$ is not transversal to $\Gamma$ are omitted (they again do not change anything). Hence $\Delta$ is also transversal to the cylinders $C_{\varepsilon}$, for $\varepsilon$ small. Therefore, $\Delta$ defines (up to sign) a vector field $X_{\varepsilon}$ on $C \varepsilon$, tangent to $\Delta$, that can be chosen of length 1 . The asymptotic optimal synthesis consists of: 1. Reaching $C_{\varepsilon}$ from $\Gamma(0)$, 2. Follow a trajectory of $X_{\varepsilon}, 3$. Join $\Gamma(t)$. The steps 1 and 3 cost $2 \varepsilon$, which is neglectible w.r.t. the full metric complexity. To get the optimal synthesis for the interpolation entropy, one has to make the same construction, but starting from a subriemannian cylinder $C_{\varepsilon}^{\prime}$ tangent to $\Gamma$.

In normal coordinates, in that case, the $x$-trajectories are just circles, and the corresponding optimal controls are just trigonometric functions, with period $\frac{2 \pi}{\varepsilon}$.


Fig. 4. 3-dimensional Martinet case

Figure 4 concerns the case $r \neq 0$ (crossing Martinet surface). At a Martinet point, the vector-field $X_{\varepsilon}$ has a limit cycle, which is not tangent to the distribution. The asymptotic optimal strategy consists of: a. following a trajectory of $X_{\varepsilon}$ till reaching the height of the center of the limit cycle, $b$. crossing the cylinder, with a neglectible cost $2 \varepsilon$, c. Following a trajectory of the opposite vector field $-X_{\varepsilon}$. The strategy for entropy is similar, but using the tangent cylinder $C_{\varepsilon}^{\prime}$.

## C. The one-step bracket-generating case

For the corank $k \leq 3$, the situation is very similar to the 3 -dimensional case. It can be competely reduced to it. For details, see [10].

At this point, this strange fact appears: there is the limit corank $k=3$. If $k>3$ only, new phenomena appear. Let us explain now the reason for this
Let us consider the following mapping $\mathcal{B}_{\xi}: \Delta_{\xi} \times \Delta_{\xi} \rightarrow$ $T_{x} M / \Delta_{\xi},(X, Y) \rightarrow[X, Y]+\Delta_{\xi}$. It is a well defined tensor mapping, which means that it actually applies to vectors (and not to vector fields, as expected from the definition). This is due to the following formula, for a one-form $\omega$ : $d \omega(X, Y)=$ $\omega([X, Y])+\omega(Y) X-\omega(X) Y$. Let us call $I_{\xi}$ the image by $\mathcal{B}_{\xi}$ of the product of two unit balls in $\Delta_{\xi}$. The following holds:

Theorem 23: For a generic $\mathcal{P}$, for $k \leq 3$, the sets $I_{\Gamma(t)}$ are convex.
This theorem is shown in [10], with the consequences that we will state just below.
This is no more true for $k>3$, the first catastrophic case being the case 10-4 (a $p=4$ distribution in $\mathbb{R}^{10}$ ). The intermadiate cases $k=4,5$ in dimension 10 are interesting, since on some open subsets of $\Gamma$, the convexity property may hold or not. These cases are studied in the paper [13].

The main consequence of this convexity property is that everything reduces (out of singularities where the logarithmic lemma applies) to the 3-dimensional contact case, as is shown in the paper [10]. We briefly summarize the results.

Consider the one forms $\omega$ that vanish on $\Delta$ and that are 1 on $\dot{\Gamma}$, and again, by the duality w.r.t. the metric over $\Delta$, define $d \omega_{\Delta \Delta}(X, Y)=<A X, Y>$, for vector fields $X, Y$ in $\Delta$. Now, we have along $\Gamma$, a $(k-1)$-parameter affine family of skew symmetric endomorphisms $A_{\Gamma(t)}$ of
$\Delta_{\Gamma(t)}$. Say, $A_{\Gamma(t)}(\lambda)=A_{\Gamma(t)}^{0}+\sum_{i=1}^{k-1} \lambda_{i} A_{\Gamma(t)}^{i}$. Set $\chi(t)=$ $\inf _{\lambda}\left\|A_{\Gamma(t)}(\lambda)\right\|=\left\|A_{\Gamma(t)}\left(\lambda^{*}(t)\right)\right\|$.

Out of isolated points of $\Gamma$ (that count for nothing in the metric complexity or in the entropy), the $t$-one parameter family $A_{\Gamma(t)}\left(\lambda^{*}(t)\right)$ can be smoothly block-diagonalized (with $2 \times 2$ bloks), using a gauge transformation along $\Gamma$. After this gauge transformation, the 2-dimensional eigenspace corresponding to the largest (in moduli) eigenvalue of $A_{\Gamma(t)}\left(\lambda^{*}(t)\right)$, corresponds to the two first coordinates in the distribution, and to the 2 first controls. In the asymptotic optimal synthesis, all other controls are put to zero [here the convexity property is used], and the picture of the asymptotic optimal synthesis is exactly that of the 3 -dimensional contact case. We still have the formulas:

$$
M C(\varepsilon) \simeq \frac{2}{\varepsilon^{2}} \int_{\Gamma} \frac{d t}{\varkappa(t)}, \quad E(\varepsilon)=2 \pi M C(\varepsilon)
$$

The case $k>3$ was first treated in [12] in the $10-$ dimensional case, and was completed in general in [14].

In that case, the situation does not reduce to the 3dimensional contact case: the optimal controls, in the asymptotic optimal synthesis for the nilpotent approximation are still trigonometric controls, but with different periods that are successive integer multiples of a given basic period. New invariants $\lambda_{\theta(t)}^{j}$ appear, and the formula for the entropy is:

$$
E(\varepsilon) \simeq \frac{2 \pi}{\varepsilon^{2}} \int_{0}^{T} \frac{\sum_{j=1}^{r} j \lambda_{\theta}^{j}}{\sum_{j=1}^{r}\left(\lambda_{\theta}^{j}\right)^{2}} d \theta
$$

the optimal controls being of the form:

$$
\begin{align*}
u_{2 j-1}(t) & =-\sqrt{\frac{j \lambda_{\theta(t)}^{j}}{\sum_{j=1}^{r} j \lambda_{\theta(t)}^{j}}} \sin \left(\frac{2 \pi j t}{\varepsilon}\right),  \tag{18}\\
u_{2 j}(t) & =\sqrt{\frac{j \lambda_{\theta(t)}^{j}}{\sum_{j=1}^{r} j \lambda_{\theta(t)}^{j}}} \cos \left(\frac{2 \pi j t}{\varepsilon}\right), \quad j=1, \ldots, r
\end{align*}
$$

$$
u_{2 r+1}(t)=0 \text { if } p \text { is odd }
$$

These last formulas hold in the free case only (i.e. the case where the corank $k=\frac{p(p-1)}{2}$, the dimension of he second homogeneous component of the free Lie-algebra with $p$ generators). The non free case is more complicated (see [14]).

To prove all the results in this section, one has to proceed as follows: 1. use the theorem of reduction to nilpotent approximation (19], and 2. use the Pontriaguin'smaximum principle on the normal form of the nilpotent approximation, in normal coordinates

## D. The 2 -control case, in $\mathbb{R}^{4}$ and $\mathbb{R}^{5}$.

These cases correspond respectively to the car with a trailer (Example 2) and the ball on a plate (Example 33.
We use also the theorem 19 of reduction to Nilpotent approximation, and we consider the normal forms $\hat{\mathcal{P}}_{4,2}, \hat{\mathcal{P}}_{5,2}$


Fig. 5. The dance of minimum entropy, for 3rd bracket.
of Section II-C5 In both cases, we change the variable $w$ for $\tilde{w}$ such that $d \tilde{w}=\frac{d w}{\delta(w)}$. We look for arclengthparametrized trajectories of the nilpotent approximation (i.e. $\left(u_{1}\right)^{2}+\left(u_{2}\right)^{2}=1$ ), that start from $\Gamma(0)$, and reach $\Gamma$ in fixed time $\varepsilon$, maximizing $\int_{0}^{\varepsilon} \dot{w}(\tau) d \tau$. Abnormal extremals do no come in the picture, and optimal curves correspond to the hamiltonian

$$
H=\sqrt{\left(P F_{1}\right)^{2}+\left(P F_{2}\right)^{2}}
$$

where $P$ is the adjoint vector. It turns out that, in our normal coordinates, the same trajectories are optimal for both the 4-2 and the 5-2 case (one has just to notice that the solution of the $4-2$ case meets the extra interpolation condition corresponding to the 5-2 case).
Setting as usual $u_{1}=\cos (\varphi)=P F_{1}, u_{2}=\sin (\varphi)=$ $P F_{2}$, we get $\dot{\varphi}=P\left[F_{1}, F_{2}\right], \ddot{\varphi}=-P\left[F_{1},\left[F_{1}, F_{2}\right]\right] P F_{1}-$ $P\left[F_{2},\left[F_{1}, F_{2}\right]\right] P F_{2}$.
At this point, we have to notice that only the components $P_{x_{1}}, P_{x_{2}}$ of the adloint vector $P$ are not constant (the hamiltonian in the nilpotent approximation depends only on the $x$ variables), therefore, $P\left[F_{1},\left[F_{1}, F_{2}\right]\right]$ and $P\left[F_{2},\left[F_{1}, F_{2}\right]\right]$ are constant (the third brackets are also constant vector fields). Hence, $\ddot{\varphi}=\alpha \cos (\varphi)+\beta \sin (\varphi)=\alpha \dot{x}_{1}+\beta \dot{x}_{2}$ for appropriate constants $\alpha, \beta$. It follows that, for another constant $k$, we have, for the optimal curves of the nilpotent approximation, in normal coordinates $x_{1}, x_{2}$ :

$$
\begin{aligned}
\dot{x}_{1} & =\cos (\varphi), \dot{x}_{2}=\sin (\varphi), \\
\dot{\varphi} & =k+\lambda x_{1}+\mu x_{2}
\end{aligned}
$$

Remark 24: 1. It means that we are looking for curves in the $x_{1}, x_{2}$ plane, whose curvature is an affine function of the position,
2. In the two-step bracket generating case (contact case), otimal curves were circles, i.e. curves of constant curvature,
3. the conditions of $\varepsilon$-interpolation of $\Gamma$ say that these curves must be periodic (there will be more details on this point in the next section), that the area of a loop must be zero $(y(\varepsilon)=0)$, and finally (in the 5-2 case) that another moment must be zero.

It is easily seen that such a curve, meeting these interpolation conditions, must be an elliptic curve of elastica-type. The periodicity and vanishing surface requirements imply that it is the only periodic elastic curve shown on Figure 5 , parametrized in a certain way.

The formulas are, in terms of the standard Jacobi elliptic functions:

$$
\begin{aligned}
& u_{1}(t)=1-2 d n\left(K\left(1+\frac{4 t}{\varepsilon}\right)\right)^{2} \\
& u_{2}(t)=-2 d n\left(K\left(1+\frac{4 t}{\varepsilon}\right)\right) \operatorname{sn}\left(K\left(1+\frac{4 t}{\varepsilon}\right)\right) \sin \left(\frac{\varphi_{0}}{2}\right)
\end{aligned}
$$

where $\varphi_{0=130^{\circ}}$ (following [22], p. 403) and $\varphi_{0}=130,692^{\circ}$ following Mathematica ${ }^{\circledR}$, with $k=\sin \left(\frac{\varphi_{0}}{2}\right)$ and $K(k)$ is the quarter period of the Jacobi elliptic functions. The trajectory on the $x_{1}, x_{2}$ plane, shown on Figure 5, has equations:

$$
\begin{aligned}
& x_{1}(t)=-\frac{\varepsilon}{4 K}\left[\frac{-4 K t}{\varepsilon}+2\left(\operatorname{Eam}\left(\frac{4 K t}{\varepsilon}+K\right)-\operatorname{Eam}(K)\right)\right] \\
& x_{2}(t)=k \frac{\varepsilon}{2 K} \operatorname{cn}\left(\frac{4 K t}{\varepsilon}+K\right)
\end{aligned}
$$

On the figure 2, one can clearly see, at the contact point of the ball with the plane, a trajectory which is a "repeated small deformation" of this basic trajectory.

The formula for the entropy is, in both the 4-2 and 5-2 cases:

$$
E(\varepsilon)=\frac{3}{2 \sigma \varepsilon^{3}} \int_{\Gamma} \frac{d t}{\delta(t)}
$$

where $\sigma$ is a universal constant, $\sigma \approx 0.00580305$.
Details of computations on the 4-2 case can be found in [12], and in [13] for the 5-2 case.

## IV. The BaLl with A TRAILER

We start by using Theorem 19, to reduce to the nilpotent approximation along $\Gamma$ :

$$
\begin{align*}
&\left(\hat{\mathcal{P}}_{6,2}\right) \dot{x}_{1}  \tag{19}\\
&=u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \\
& \dot{z}_{1}=x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \dot{z}_{2}=x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \\
& \dot{w}=Q_{w}\left(x_{1}, x_{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) .
\end{align*}
$$

By Lemma 16, we can consider that

$$
\begin{equation*}
Q_{w}\left(x_{1}, x_{2}\right)=\delta(w)\left(\left(x_{1}\right)^{2}+\left(x_{2}\right)^{2}\right) \tag{20}
\end{equation*}
$$

where $\delta(w)$ is the main invariant. In fact, it is the only invariant for the nilpotent approximation along $\Gamma$. Moreover, if we reparametrize $\Gamma$ by setting $d w:=\frac{d w}{\delta(w)}$, we can consider that $\delta(w)=1$.

Then, we want to maximaize $\int \dot{w} d t$ in fixed time $\varepsilon$, with the interpolation conditions: $x(0)=0, y(0)=0, z(0)=$ $0, w(0)=0, x(\varepsilon)=0, y(\varepsilon)=0, z(\varepsilon)=0$.

From Lemma 27 in the appendix, we know that the optimal trajectory is smooth and periodic, (of period $\varepsilon$ ).

Clearly, the optimal trajectory has also to be a length minimizer, then we have to consider the usual hamiltonian for length: $H=\frac{1}{2}\left(\left(P . F_{1}\right)^{2}+\left(P . F_{2}\right)^{2}\right)$, in which $P=\left(p_{1}, \ldots, p_{6}\right)$ is the adjoint vector. It is easy to see that the abnormal extremals do not come in the picture (cannot be optimal with our additional interpollation conditions), and in fact, we will show that the hamiltonian system corresponding to the hamiltonian $H$ is integrable.

Remark 25: This integrability property is no more true in the general 6-2 case. It holds only for the ball with a trailer.

As usual, we work in Poincaré coordinates, i.e. we consider level $\frac{1}{2}$ of the hamiltonian $H$, and we set:

$$
u_{1}=P F=\sin (\varphi), \quad u_{2}=P G=\cos (\varphi)
$$

Differentiating twice, we get

$$
\dot{\varphi}=P[F, G], \ddot{\varphi}=-P F F G \cdot P F-P G F G \cdot P G
$$

where $F F G=[F,[F, G]]$ and $G F G=[G,[F, G]]$. We set $\lambda=-P F F G, \mu=-P G F G$. We get that:

$$
\begin{equation*}
\ddot{\varphi}=\lambda \sin (\varphi)+\mu \cos (\varphi) \tag{21}
\end{equation*}
$$

Now, we compute $\dot{\lambda}$ and $\dot{\mu}$. We get, with similar notations as above for the brackets (we bracket from the left):

$$
\begin{aligned}
& \dot{\lambda}=P F F F G \cdot P F+P G F F G \cdot P G \\
& \dot{\mu}=P F G F G \cdot P F+P G G F G \cdot P G
\end{aligned}
$$

and computing the brackets, we see that $G F F G=F G F G=$ 0 . Also, since the hamiltonian does not depend on $y, z, w$, we get that $p_{3}, p_{4}, p_{5}, p_{6}$ are constants. Computing the brackets $F F G$ and $G F G$, we get that

$$
\lambda=\frac{3}{2} p_{4}+p_{6} x_{1}, \quad \mu=\frac{3}{2} p_{5}+p_{6} x_{2}
$$

and then, $\dot{\lambda}=p_{6} \sin (\varphi)$ and $\dot{\mu}=p_{6} \cos (\varphi)$. Then, by 21, $\ddot{\varphi}=\frac{\lambda \dot{\lambda}}{p_{6}}+\frac{\mu \dot{\mu}}{p_{6}}$, and finally:

$$
\begin{align*}
\dot{x}_{1} & =\sin (\varphi), \quad \dot{x}_{2}=\cos (\varphi)  \tag{22}\\
\dot{\varphi} & =K+\frac{1}{2 p_{6}}\left(\lambda^{2}+\mu^{2}\right) \\
\dot{\lambda} & =p_{6} \sin (\varphi), \quad \dot{\mu}=p_{6} \cos (\varphi)
\end{align*}
$$

Setting $\omega=\frac{\lambda}{p_{6}}, \delta=\frac{\mu}{p_{6}}$, we obtain:

$$
\begin{aligned}
& \dot{\omega}=\sin (\varphi), \quad \dot{\delta}=\cos (\varphi) \\
& \dot{\varphi}=K+\frac{p_{6}}{2}\left(\omega^{2}+\delta^{2}\right)
\end{aligned}
$$

It means that the plane curve $(\omega(t), \delta(t))$ has a curvature which is a quadratic function of the distance to the origin. Then, the optimal curve $\left(x_{1}(t), x_{2}(t)\right)$ projected to the horizontal plane of the normal coordinates has a curvature which is a quadratic function of the distance to some point. Following the lemma 23) in the appendix, this system of equations is integrable.

Summarizing all the results, we get the following theorem.
Theorem 26: (asymptotic optimal synthesis for the ball with a trailer) The asymptotic optimal synthesis is an $\varepsilon$ modification of the one of the nilpotent approximation, which has the following properties, in projection to the horizontal plane $\left(x_{1}, x_{2}\right)$ in normal coordinates:

1. It is a closed smooth periodic curve, whose curvature is a quadratic function of the position, and a function of the square distance to some point,
2. The area and the $2^{\text {nd }}$ order moments $\int_{\Gamma} x_{1}\left(x_{2} d x_{1}-\right.$ $\left.x_{1} d x_{2}\right)$ and $\int_{\Gamma} x_{2}\left(x_{2} d x_{1}-x_{1} d x_{2}\right)$ are zero.


Fig. 6. The dance of minimum entropy for the ball with a trailer
3. The entropy is given by the formula: $E(\varepsilon)=\frac{\sigma}{\varepsilon^{4}} \int_{\Gamma} \frac{d w}{\delta(w)}$, where $\delta(w)$ is the main invariant from 20, and $\sigma$ is a universal constant.

In fact we can go a little bit further to integrate explicitely the system 22. Set $\bar{\lambda}=\cos (\varphi) \lambda-\sin (\varphi) \mu, \bar{\mu}=\sin (\varphi) \lambda+$ $\cos (\varphi) \mu$. we get:

$$
\begin{aligned}
& \frac{d \bar{\lambda}}{d t}=-\bar{\mu}\left(K+\frac{1}{2 p_{6}}\left(\bar{\lambda}^{2}+\bar{\mu}^{2}\right)\right), \\
& \frac{d \bar{\mu}}{d t}=p_{6}+\bar{\lambda}\left(K+\frac{1}{2 p_{6}}\left(\bar{\lambda}^{2}+\bar{\mu}^{2}\right)\right) .
\end{aligned}
$$

This is a 2 dimensional (integrable) hamiltonian system. The hamiltonian is:

$$
H_{1}=-p_{6} \bar{\lambda}-\frac{2 p_{6}}{4}\left(K+\frac{1}{2 p_{6}}\left(\bar{\lambda}^{2}+\bar{\mu}^{2}\right)\right)^{2}
$$

This hamiltonian system is therefore integrable, and solutions can be expressed in terms of hyperelliptic functions. A liitle numerics now allows to show, on figure 6, the optimal $x$ trajectory in the horizontal plane of the normal coordinates.

On the figure 7 , we show the motion of the ball with a trailer on the plane (motion of the contact point between the ball and the plane).Here, the problem is to move along the $x$ axis, keeping constant the frame attached to the ball and the angle of the trailer.

## V. Expectations and conclusions

Some movies of minimum entropy for the ball rolling on a plane and the ball with a trailer are visible on the website $* * * * * * * * * * * * * * * * * * * * * * * * * * * *$.

## A. Universality of some pictures in normal coordinates

Our first conclusion is the following: there are certain universal pictures for the motion planning problem, in corank less or equal to 3 , and in rank 2 , with 4 brackets at most (could be 5 brackets at a singularity, with the logarithmic lemma).


Fig. 7. Parking the ball with a trailer


Fig. 8. The universal movements in normal coordinates

These figures are, in the two-step bracket generating case: a circle, for the third bracket, the periodic elastica, for the $4^{t h}$ bracket, the plane curve of the figure 6

They are periodic plane curves whose curvature is respectively: a constant, a linear function of of the position, a quadratic function of the position.

This is, as shown on Figure 8, the clear beginning of a series.

## B. Robustness

As one can see, in many cases ( 2 controls, or corank $k \leq 3$ ), our strategy is extremely robust in the following sense: the asymptotic optimal syntheses do not depend, from the qualitative point of view, of the metric chosen. They depend only on the number of brackets needed to generate the space.

## C. The practical importance of normal coordinates

The main practical problem of implementation of our strategy comes with the $\varepsilon$-modifications. How to compute them,
how to implement? In fact, the $\varepsilon$-modifications count at higher order in the entropy. But, if not applied, they may cause deviations that are not neglectible. The high order w.r.t. $\varepsilon$ in the estimates of the error between the original system and its nilpotent approximation (Formulas 10, 12, 13, ??) make these deviations very small. It is why the use of our concept of a nilpotent approximation along $\Gamma$, based upon normal coordinates is very efficient in practice.

On the other hand, when a correction appears to be needed (after a noneglectible deviation), it corresponds to brackets of lower order. For example, in the case of the ball with a trailer ( $4^{t h}$ bracket), the $\varepsilon$-modification corresponds to brackets of order 2 or 3 . The optimal pictures corresponding to these orders can still be used to perform the $\varepsilon$-modifications.

## D. Final conclusion

This approach, to approximate optimally nonadmissible paths of nonholomic systems, looks very efficient, and in a sense, universal. Of course, the theory is not complete, but the cases under consideration (first, 2-step bracket-generating, and second, two controls) correspond to many practical situations. But there is still a lot of work to do to in order to cover all interesting cases. However, the methodology to go ahead is rather clear.

## VI. APPENDIX

## A. Appendix 1: Normal form in the 6-2 case

We start from the general normal form (11) in normal coordinates:

$$
\begin{aligned}
\dot{x}_{1} & =\left(1+\left(x_{2}\right)^{2} \beta\right) u_{1}-x_{1} x_{2} \beta u_{2}, \\
\dot{x}_{2} & =\left(1+\left(x_{1}\right)^{2} \beta\right) u_{2}-x_{1} x_{2} \beta u_{1}, \\
\dot{y}_{i} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \gamma_{i}(y, w), \\
\dot{w} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \delta(y, w)
\end{aligned}
$$

We will make a succession of changes of parametriztion of the surrface $\mathcal{S}$ (w.r.t. which normal coordinates were constructed). These coordinate changes will always preserve tha fact that $\Gamma(t)$ is the point $x=0, y=0, w=t$.

Remind that $\beta$ vanishes on $\mathcal{S}$, and since $x$ has order 1 , we can already write on $T_{\varepsilon}: \dot{x}=u+O\left(\varepsilon^{3}\right)$. One of the $\gamma_{i}$ 's (say $\gamma_{1}$ ) has to be nonzero (if not, $\Gamma$ is tangent to $\Delta^{\prime}$ ). Then, $y_{1}$ has order 2 on $T_{\varepsilon}$. Set for $i>1, \tilde{y}_{i}=y_{i}-\frac{\gamma_{i}}{\gamma_{1}}$. Differentiating, we get that $\frac{d \tilde{y}_{i}}{d t}=\dot{y}_{i}-\frac{\gamma_{i}}{\gamma_{1}} \dot{y}_{1}+O\left(\varepsilon^{2}\right)$, and $z_{1}=\tilde{y}_{2}, z_{2}=\tilde{y}_{3}$ have order 3. We set also $w:=w-\frac{\delta}{\gamma_{1}}$, and we are at the following point:

$$
\begin{aligned}
\dot{x} & =u+O\left(\varepsilon^{3}\right), \quad \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \gamma_{1}(w)+O\left(\varepsilon^{2}\right), \\
\dot{z}_{i} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) L_{i}(w) \cdot x+O\left(\varepsilon^{3}\right) \\
\dot{w} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \delta(w) \cdot x+O\left(\varepsilon^{3}\right)
\end{aligned}
$$

where $L_{i}(w) \cdot x, \delta(w) . x$ are liner in $x$. The function $\gamma_{1}(w)$ can be put to 1 in the same way by setting $y:=\frac{y}{\gamma_{1}(w)}$. Now let
$T(w)$ be an invertible $2 \times 2$ matrix. Set $\tilde{z}=T(w) z$. It is easy to see that we can chose $T(w)$ for we get:

$$
\begin{aligned}
\dot{x} & =u+O\left(\varepsilon^{3}\right), \quad \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)+O\left(\varepsilon^{2}\right), \\
\dot{z}_{i} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) x_{i}+O\left(\varepsilon^{3}\right), \\
\dot{w} & =\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \delta(w) \cdot x+O\left(\varepsilon^{3}\right),
\end{aligned}
$$

Another change of the form: $w:=w+L(w) \cdot x$, where $L(w) . x$ is linear in $x$ kills $\delta(w)$ and brings us to $\dot{w}=\left(\frac{x_{2}}{2} u_{1}-\right.$ $\left.\frac{x_{1}}{2} u_{2}\right) O\left(\varepsilon^{2}\right)$. This $O\left(\varepsilon^{2}\right)$ can be of the form $Q_{w}(x)+h(w) y+$ $O\left(\varepsilon^{3}\right)$ where $Q_{w}(x)$ is quadratic in $x$. If we kill $h(w)$, we get the expected result. This is done with a change of coordinates of the form: $w:=w+\varphi(w) \frac{y^{2}}{2}$.

## B. Appendix 2: Plane curves whose curvature is a function of the distance to the origin

This result was known already, see [24]. However we provide here a very simple proof.

Consider a plane curve $(x(t), y(t))$, whose curvature is a function of the distance from the origin, i.e.:

$$
\begin{equation*}
\dot{x}=\cos (\varphi), \dot{y}=\sin (\varphi), \dot{\varphi}=k\left(x^{2}+y^{2}\right) \tag{23}
\end{equation*}
$$

Equation 23 is integrable.
Proof: Set $\bar{x}=x \cos (\varphi)+y \sin (\varphi), \bar{y}=-x \sin (\varphi)+$ $y \cos (\varphi)$. Then $k\left(\bar{x}^{2}+\bar{y}^{2}\right)=k\left(x^{2}+y^{2}\right)$. Just computing, one gets:

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=1+\bar{y} k\left(\bar{x}^{2}+\bar{y}^{2}\right)  \tag{24}\\
& \frac{d \bar{y}}{d t}=-\bar{x} k\left(\bar{x}^{2}+\bar{y}^{2}\right)
\end{align*}
$$

We just show that 24 is a hamiltonian system. Since we are in dimension 2, it is always Liouville-integrable. Then, we are looking for solutions of the system of PDE's:

$$
\begin{aligned}
& \frac{\partial H}{\partial \bar{x}}=1+\bar{y} k\left(\bar{x}^{2}+\bar{y}^{2}\right) \\
& \frac{\partial H}{\partial \bar{y}}=-\bar{x} k\left(\bar{x}^{2}+\bar{y}^{2}\right)
\end{aligned}
$$

But the Schwartz integrability conditions are satisfied: $\frac{\partial^{2} H}{\partial \bar{x} \partial \bar{y}}=$ $\frac{\partial^{2} H}{\partial \bar{y} \partial \bar{x}}=2 \bar{x} \bar{y} k^{\prime}$.
C. Appendix 3: periodicity of the optimal curves in the 6-2 case

Proof: We consider the nilpotent approximation $\hat{\mathcal{P}}_{6,2}$ given in formula 15 .

$$
\begin{align*}
\left(\hat{\mathcal{P}}_{6,2}\right) \quad \dot{x}_{1} & =u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)  \tag{25}\\
\dot{z}_{1} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \dot{z}_{2}=x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \\
\dot{w} & =Q_{w}\left(x_{1}, x_{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)
\end{align*}
$$

We consider the particular case of the ball with a trailer. Then, according to Lemma 16, the ratio $r(\xi)=1$.

It follows that the last equation can be rewritten $\dot{w}=$ $\delta(w)\left(\left(x_{1}\right)^{2}+\left(x_{2}\right)^{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)$ for some never vanishing function $\delta(w)$ (vanishing would contradict the full rank of $\left.\Delta^{(4)}\right)$. We can change the coordinate $w$ for $\tilde{w}$ such that $d \tilde{w}=\frac{d w}{\delta(w)}$.

We get finally:

$$
\begin{align*}
\left(\hat{\mathcal{P}}_{6,2}\right) \quad \dot{x}_{1} & =u_{1}, \dot{x}_{2}=u_{2}, \dot{y}=\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)  \tag{26}\\
\dot{z}_{1} & =x_{2}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right), \dot{z}_{2}=x_{1}\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right) \\
\dot{w} & =\left(\left(x_{1}\right)^{2}+\left(x_{2}\right)^{2}\right)\left(\frac{x_{2}}{2} u_{1}-\frac{x_{1}}{2} u_{2}\right)
\end{align*}
$$

This is a right invariant system on $\mathbb{R}^{6}$ with cooordinates $\xi=(\varsigma, w)=(x, y, z, w)$, for a certain Nilpotent Lie group structure over $\mathbb{R}^{6}$ (denoted by $G$ ). It is easily seen (just expressing right invariance) that the group law is ot the form $\left(\varsigma_{2}, w_{2}\right)\left(\varsigma_{1}, w_{1}\right)=\left(\varsigma_{1} * \varsigma_{2}, w_{1}+w_{2}+\Phi\left(\varsigma_{1}, \varsigma_{2}\right)\right)$, where $*$ is the multiplication of another Lie group structure on $\mathbb{R}^{5}$, with coordinates $\varsigma\left(\right.$ denoted by $\left.G_{0}\right)$. In fact, $G$ is a central extension of $\mathbb{R}$ by $G_{0}$.

Lemma 27: The trajectories of 26 that maximize $\int \dot{w} d t$ in fixed time $\varepsilon$, with interpolating conditions $\varsigma(0)=\varsigma(\varepsilon)=0$, have a periodic projection on $\varsigma$ (i.e. $\varsigma(t)$ is smooth and periodic of period $\varepsilon$ ).

Remark 28: 1. Due to the invariance with respect to the $w$ coordinate of (26), it is equivalent to consider the problem with the more restrictive terminal conditions $\varsigma(0)=\varsigma(\varepsilon)=0$, $w(0)=0$,
2. The scheme of this proof works also to show periodicity in the 4-2 and 5-2 cases.

The idea for the proof was given to us by A. Agrachev.
Proof: Let $\left(\varsigma, w_{1}\right),\left(\varsigma, w_{2}\right)$ be initial and terminal points of an optimal solution of our problem. By right translation by $\left(\varsigma^{-1}, 0\right)$, this trajectory is mapped into another trajectory of the system, with initial and terminal points $\left(0, w_{1}+\Phi\left(\varsigma, \varsigma^{-1}\right)\right)$ and $\left(0, w_{1}+\Phi\left(\varsigma, \varsigma^{-1}\right)\right)$. Hence, this trajectory has the same value of the cost $\int \dot{w} d t$. We see that the optimal cost is in fact independant of the $\varsigma$-coordinate of the initial and terminal condition.

Therefore, the problem is the same as maximizing $\int \dot{w} d t$ but with the (larger) endpoint condition $\varsigma(0)=\varsigma(\varepsilon)$ (free). Now, we can apply the general transversality conditions of Theorem 12.15 page 188 of [4]. It says that the initial and terminal covectors $\left(p_{\varsigma}^{1}, p_{w}^{1}\right)$ and $\left(p_{\varsigma}^{2}, p_{w}^{2}\right)$ are such that $p_{\varsigma}^{1}=$ $p_{\varsigma}^{2}$. This is enough to show periodicity.

## REFERENCES

[1] A.A. Agrachev, H.E.A. Chakir, J.P. Gauthier, Subriemannian Metrics on $\mathrm{R}^{3}$, in Geometric Control and Nonholonomic Mechanics, Mexico City 1996, pp. 29-76, Proc. Can. Math. Soc. 25, 1998.
[2] A.A. Agrachev, J.P. Gauthier, Subriemannian Metrics and Isoperimetric Problems in the Contact Case, in honor L. Pontriaguin, 90th birthday commemoration, Contemporary Maths, Tome 64, pp. 5-48, 1999 (Russian). English version: journal of Mathematical sciences, Vol 103, ${ }^{\circ} 6$, pp. 639-663.
[3] A.A. Agrachev, J.P. Gauthier, On the subanalyticity of Carnot Caratheodory distances, Annales de l'Institut Henri Poincaré, AN 18, 3 (2001), pp. 359-382.
[4] A.A. Agrachev, Y. Sachkov, Contro Theory from the geometric view point, Springer Verlag Berlin Heidelberg, 2004.
[5] G. Charlot Quasi Contact SR Metrics: Normal Form in $\mathbb{R}^{2 n}$, Wave Front and Caustic in $\mathbb{R}^{4}$; Acta Appl. Math., 74, $\mathrm{N}^{\circ} 3$, pp. 217-263, 2002.
[6] H.E.A. Chakir, J.P. Gauthier, I.A.K. Kupka, Small Subriemannian Balls on $\mathrm{R}^{3}$, Journal of Dynamical and Control Systems, Vol 2, $\mathrm{N}^{\circ} 3$, , pp. 359-421, 1996.
[7] F.H. Clarke, Optimization and nonsmooth analysis, John Wiley \& Sons, 1983.
[8] J.P. Gauthier, F.Monroy-Perez, C. Romero-Melendez, On complexity and Motion Planning for Corank one SR Metrics, 2004, COCV; Vol 10, pp. 634-655.
[9] J.P. Gauthier, V. Zakalyukin, On the codimension one Motion Planning Problem, JDCS, Vol. 11, $\mathrm{N}^{\circ}$ 1, January 2005, pp.73-89.
[10] J.P. Gauthier, V. Zakalyukin, On the One-Step-Bracket-Generating Motion Planning Problem, JDCS, Vol. 11, $\mathrm{N}^{\circ}$ 2, april 2005, pp. 215-235.
[11] J.P. Gauthier, V. Zakalyukin, Robot Motion Planning, a wild case, Proceedings of the Steklov Institute of Mathematics, Vol 250, pp.5669, 2005.
[12] J.P. Gauthier, V. Zakalyukin, On the motion planning problem, complexity, entropy, and nonholonomic interpolation, Journal of dynamical and control systems, Vol. 12, $\mathrm{N}^{\circ} 3$, July 2006.
[13] J.P. Gauthier, V. Zakalyukin, Entropy estimations for motion planning problems in robotics, Volume In honor of Dmitry Victorovich Anosov, Proceedings of the Steklov Mathematical Institute, Vol. 256, pp. 62-79, 2007.
[14] JP Gauthier, B. Jakubczyk, V. Zakalyukin, Motion planning and fastly oscillating controls, SIAM Journ. On Control and Opt, Vol. 48 (5), pp. 3433-3448, 2010.
[15] M. Gromov, Carnot Caratheodory Spaces Seen from Within, Eds A. Bellaiche, J.J. Risler, Birkhauser, pp. 79-323, 1996.
[16] F. Jean, Complexity of Nonholonomic Motion Planning, International Journal on Control, Vol 74, $\mathrm{N}^{\circ} 8$, pp 776-782, 2001.
[17] F. Jean, Entropy and Complexity of a Path in SR Geometry, COCV, Vol 9, pp. 485-506, 2003.
[18] F. Jean, E. Falbel, Measures and transverse paths in SR geometry, Journal d'Analyse Mathématique, Vol. 91, pp. 231-246, 2003.
[19] T. Kato, Perturbation theory for linear operators, Springer Verlag 1966, pp. 120-122.
[20] J.P. Laumond, (editor), Robot Motion Planning and Control, Lecture notes in Control and Information Sciences 229, Springer Verlag, 1998.
[21] L. Pontryagin, V. Boltyanski, R. Gamkelidze, E. Michenko, The Mathematical theory of optimal processes, Wiley, New-York, 1962.
[22] A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, Dover, New-York, 1944.
[23] H.J. Sussmann, G. Lafferriere, Motion planning for controllable systems without drift; In Proceedings of the IEEE Conference on Robotics and Automation, Sacramento, CA, April 1991. IEEE Publications, New York, 1991, pp. 109-148.
[24] D.A. Singer, Curves whose curvature depend on the distance from the origin, the American mathematical Monthly, 1999, vol. 106, no9, pp. 835-841.
[25] H.J. Sussmann, W.S. Liu, Lie Bracket extensions and averaging: the single bracket generating case; in Nonholonomic Motion Planning, Z. X. Li and J. F. Canny Eds., Kluwer Academic Publishers, Boston, 1993, pp. 109-148.

