

Robust Self-Triggered Coordination With Ternary Controllers

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Abstract—This paper regards the coordination of networked systems, studied in the framework of hybrid dynamical systems. We design a coordination scheme which combines the use of ternary controllers with a self-triggered communication policy. The communication policy requires the agents to measure, at each sampling time, the difference between their states and those of their neighbors. The collected information is then used to update the control and determine the following sampling time. We show that the proposed scheme ensures finite-time convergence to a neighborhood of a consensus state: the coordination scheme does not require the agents to share a global clock, but allows them to rely on local clocks. We then study the robustness of the proposed self-triggered coordination system with respect to skewness in the agents' local clocks, to delays, and to limited precision in communication. Furthermore, we present two significant variations of our scheme. First, assuming a global clock to be available, we design a time-varying controller which asymptotically drives the system to consensus. The assumption of a global clock is then discussed, and relaxed to a certain extent. Second, we adapt our framework to a communication model in which each agent polls its neighbors separately, instead of polling all of them simultaneously. This communication policy actually leads to a self-triggered “gossip” coordination system.

Index Terms—Coordination, event-based control, gossip dynamics, hybrid systems, self-triggered control, ternary controllers.

I. INTRODUCTION

THE key issue in distributed and networked systems resides in ensuring performance with respect to a given control task (e.g., stability, coordination), in spite of communication constraints, which can be severe. In practice, although the system would naturally be described by continuous-time dynamics, the control law is only updated at discrete time instants: these can either be pre-specified (time-scheduled control), or be

determined by certain events that are triggered depending on the system's behavior. In a networked system, controls and triggering events regarding an agent must only depend on the states (or the outputs) of the agent's neighbors and of the agent itself. For this reason, there is special interest in self-triggered policies, in which communication and control actions are planned ahead, depending on the information available to each agent at a given time. Indeed, the implementation of an event-based policy, which requires continuous monitoring of a triggering condition which depends on the state of the agents' neighbors, may not be suitable for networked applications when sensing and communication resources are critical.

A. Summary of Contributions

The main contribution of this paper is the design of a new self-triggered consensus system. At each sampling time, a certain subset of “active” agents poll their neighbors obtaining measurements of the difference between their states and those of their neighbors: the available information is then used by the active agents to update their controls and compute their next update times. In our system, controls are constrained to belong to $\{-1, 0, +1\}$: the assumption of such coarsely quantized controllers is motivated by methodological reasons and reasons of opportunity.

It was shown in [2] that multi-agent systems achieve state agreement in finite time using the sign of local averages as control laws. This result indicates that to achieve agreement each agent only needs to keep track of the times when this local average reaches zero: this observation inspired us to the design of the sampling times of the agents. On the other hand, using ternary controllers considerably simplifies the agent dynamics, and this can be effectively exploited in designing a self-triggering policy. Our modeling and design approach leads naturally to a hybrid system which is defined in Section II. Next, in Section II-B we prove, by a Lyapunov analysis, that the hybrid system converges in finite time to a condition of “practical consensus”: that is, the solutions are within a neighborhood of a consensus, and the size of the neighborhood can be made arbitrarily small by decreasing a certain parameter of the controller quantizer. This parameter, which we denote by ε , represents the *sensitivity* of the quantizer: the smaller the ε , the more the system demands in terms of communication resources. We thus identify a trade-off between communication and coordination performance. This trade-off is precisely quantified: we provide ε -dependent estimates of the time taken by the solution to reach consensus and of the number of times the agents exchange information.

In self-triggered control, (pre)computation of the sampling times requires precise knowledge of the system's dynamics.

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Hence, uncertainty in the system can potentially disrupt the correct operation of the control algorithm. Nevertheless we show that the closed-loop system we propose is robust to a variety of uncertainties and disturbances which are relevant in networked systems such as imprecise clocks, delays, and limitations in data rates. This robustness can be enhanced by introducing a *conservativeness* parameter α in the triggering functions which determine the sampling times: the smaller the α , the shorter the intervals between sampling times. The robustness of the control algorithm is studied in Section III, by analyzing two extended models, both of which include the conservativeness parameter α .

In view of the need for predictions, it is also notable that our controllers do not need to be aware of any global information about the network (such as its algebraic connectivity or the number of agents). Furthermore, they only rely on relative measurements between neighbors: this feature contrasts with other approaches in the literature, which require a knowledge of absolute state information.

As an additional contribution, we show that a suitable time-varying controller, designed as a modification of the model introduced in Section II, can asymptotically drive the system to a consensus state. In this modified version, presented in Section IV, we introduce a time-dependent sensitivity threshold and a time-dependent gain parameter, which both decrease with time. In this framework, the time-dependent gain is used to scale down the ternary controllers used previously. In contrast with the previous one, this scheme does require the agents to access a global clock. We then propose two modifications of the algorithm that aim to mitigate the need for a global clock, either by changing the conservativeness parameter or by using some additional communication.

In the control scenarios considered in Section II–Section IV, every time an agent needs new information, it collects it from all of its neighbors simultaneously. In Section V, we show that this simultaneous action is not necessary. We design a self-triggered policy, in which the agents are free to poll their neighbors one by one, and find convergence results similar to those found before. This system involves variables, which correspond to the edges of the graph of the communication network, and these variables are updated synchronously by both agents insisting on an active edge. This feature makes the scheme the first example of self-triggered “gossip” coordination system. Correspondingly, in this protocol each node is subject to a sum of ternary controllers, each of them corresponding to an edge insisting on the node.

B. Literature Review

The reference literature for this paper includes quantized and self-triggered controls for distributed systems. Many papers have studied quantization issues in coordination, including [3]–[6]: specifically, binary controllers are used to stabilize consensus in [2]. In a centralized nonlinear setting, the use of ternary controllers in connection with quantized communication has been investigated in [7].

Since the seminal work in [7], the control community has been interested in investigating event-based and self-triggered control policies, as opposed to time-scheduled (or time-triggered) policies. In this framework, we note that robustness

issues—with respect to parameter uncertainties, delays, and communication losses—are studied in [9]–[12]. Relevant papers focusing on networked systems include [13]–[19]. The work in [20] is also related, as it presents a hybrid coordination dynamics requiring communication only when specific thresholds are met. Recent closely related work includes the solution of coordination problems using self-triggered broadcast communication in [21]. Compared to this reference, the present manuscript proposes a different communication policy, which is based on polling the neighbors upon need, instead of broadcasting to them.

An approach which involves polling neighbors has also been considered in the recent paper [22]. Our contribution differs from [22] in a number of aspects, including the following ones. First, our approach relies on relative measurements and not on absolute ones. Second, in [22] the computation of the next sampling time by an agent requires information not only from the agent’s neighbors, but also from the neighbors of the agent’s neighbors (i.e., two-hop neighbors). Third, while in [22] zero execution time is allowed (this happens when an agent’s local average converges to zero in finite time), in our approach inter-execution times are guaranteed to be bounded away from zero, and the lower bounds are explicitly characterized. Another work on event-based coordination is [23], that uses copies of the neighbors’s dynamical models to generate the triggering events. Furthermore, we note that a broad class of self-triggered asynchronous consensus systems can be studied by the tools in [24]: however, this framework does not accommodate to the use of ternary controllers and does not include our setup.

Self-triggered policies can also be used for other control tasks, such as deployment of robotic networks: in [25], the authors exploit the knowledge of the speed of the deploying robots in order to design the self-triggering policy. A similar idea features in our work.

Notation: We denote by $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ the sets of real, positive, and nonnegative numbers, respectively; by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers.

II. SYSTEM DEFINITION AND MAIN RESULT

We assume to have a set of nodes $I = \{1, \dots, n\}$ and an undirected¹ connected graph $G = (I, E)$ with E a set of unordered pairs of nodes, called edges. We denote by L the Laplacian matrix of G , which is a symmetric matrix. For each node $i \in I$, we denote by $\mathcal{N}_i$ the set of its neighbors, and by d_i its degree, that is, the cardinality of $\mathcal{N}_i$.

We consider the following hybrid dynamics on a triplet of n -dimensional variables involving the *consensus variable* x , the *controls* u , and the *local clock* variables θ . All these variables are defined for time $t \geq 0$. Controls are assumed to belong to $\{-1, 0, +1\}$. The specific quantizer of choice is $\text{sign}_\varepsilon : \mathbb{R} \rightarrow \{-1, 0, +1\}$, defined according to

$$\text{sign}_\varepsilon(z) = \begin{cases} \text{sign}(z) & \text{if } |z| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where $\varepsilon > 0$ is a *sensitivity* parameter.

¹We note that this assumption entails communication in both directions between pairs of connected nodes. However, our communication protocol—described in Protocol A—does not require synchronous bidirectional communication.

The system $(x, u, \theta) \in \mathbb{R}^{3n}$ satisfies the following continuous evolution

$$\begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -1 \end{cases} \quad (2)$$

except for every t such that the set $\mathcal{S}(\theta, t) = \{i \in I : \theta_i = 0\}$ is non-empty. At such time instants the system satisfies the following discrete evolution

$$\begin{cases} x_i(t^+) = x_i(t) & \forall i \in I \\ u_i(t^+) = \begin{cases} \text{sign}_\varepsilon(\text{ave}_i(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ u_i(t) & \text{otherwise} \end{cases} \\ \theta_i(t^+) = \begin{cases} f_i(x(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \end{cases} \quad (3)$$

where for every $i \in I$ the map $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ is defined by

$$f_i(x) = \max \left\{ \frac{1}{4d_i} \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right|, \frac{\varepsilon}{4d_i} \right\} \quad (4)$$

and for brevity of notation we let

$$\text{ave}_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)). \quad (5)$$

The intuition behind the design of the above controller is the following: as we shall verify later, (4) is such that at each time t and for each $i \in I$, the sign of $u_i(t)$ is consistent with the sign of the “ideal” coordination control $\text{ave}_i(t)$, i.e., $u_i(t)\text{ave}_i(t) \geq 0$. This consistency is key to ensure the desired convergence properties.

It is worth to remark that, although an absolute time variable t is used in the system’s definition, and in the analysis which follows, the agents implementing Protocol A do *not* need to be aware of such an absolute time. Instead, they rely on their local clocks θ_i . Actually, the jump times of each variable θ_i naturally define a sequence of local switching times, which we denote by $\{t_k^i\}_{k \in \mathbb{Z}_{\geq 0}}$, by taking

$$t_{k+1}^i = t_k^i + f^i(x(t_k^i)) \quad \text{for all } i \in I.$$

Then, we immediately argue that, by (4), for every $i \in I$, the sequence of local switching times $\{t_k^i\}_{k \in \mathbb{Z}_{\geq 0}}$ has the following “dwell time” property: for every $k \geq 0$

$$t_{k+1}^i - t_k^i \geq \frac{\varepsilon}{4d_{\max}}. \quad (6)$$

This property is very relevant for the applications, as it ensures that information exchange needs not to be arbitrarily fast. Mathematically, it guarantees the existence, completeness, and uniqueness of classical solutions to the system. Initial conditions can be chosen as $x(0) = \bar{x} \in \mathbb{R}^n$, $u(0) \in \{-1, 0, 1\}^n$, $\theta(0) = 0$. With this choice of initial conditions, we note that $\mathcal{S}(0, 0) = I$, that is, every agent undergoes a discrete update at the initial time: $t_0^i = 0$ for every $i \in I$. We also remark that inherent in the definition of the discrete evolution (3), (4) is the property that the period between two consecutive updates of agent i ’s controller is never smaller than $\varepsilon/(4d_i)$.

The model (2)–(3) describes the following protocol, which is implemented by each agent i to collect information and compute the control law:

Protocol A (for each $i \in I$)

- 1: **initialization:** set $u_i(0) \in \{-1, 0, +1\}$ and $\theta_i(0) = 0$;
- 2: **while** $\theta_i(t) > 0$ **do**
 - 3: i applies the control $u_i(t)$;
- 4: **end while**
- 5: **if** $\theta_i(t) = 0$ **then**
 - 6: **for all** $j \in \mathcal{N}_i$ **do**
 - 7: i polls j and collects the information $x_j(t) - x_i(t)$;
 - 8: **end for**
 - 9: i computes $\text{ave}_i(t)$;
 - 10: i computes $\theta_i(t^+) = f_i(x(t))$;
 - 11: i computes $u_i(t^+)$ by (3);
- 12: **end if**

After these remarks, we are ready to state our first convergence result:

Theorem 1 (Practical Consensus): For every initial condition $\bar{x}$, let $x(t)$ be the solution to (2)–(3) such that $x(0) = \bar{x}$. Then $x(t)$ converges in finite time to a point x^* belonging to the set

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n : \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right| < \varepsilon \forall i \in I \right\}. \quad (7)$$

This result can be seen as a *practical* consensus result: indeed the magnitude of the local “disagreement” can be made as small as desired by choosing ε . Moreover, the connectivity of the graph implies that, as ε goes to zero, the set $\mathcal{E}$ reduces to the usual consensus subspace.

Additionally, we can estimate the time cost of the consensus system, as follows:

Proposition 1 (Time Cost): Let $x(\cdot)$ be the solution to system (2)–(3). Define the *time cost* $T = \inf\{t \geq 0 : x(t) \in \mathcal{E}\}$.

Then,

$$T \leq \frac{2(d_{\max} + 1)}{\varepsilon} \sum_{\{i,j\} \in E} (\bar{x}_i - \bar{x}_j)^2$$

where $\bar{x} \in \mathbb{R}^n$ is the initial condition.

One can also define the *communication cost* $C = \max_{i \in I} \max\{k : t_k^i \leq T\}$. In view of (6), each agent polls its neighbors not more often than every $\varepsilon/(4d_{\max})$ units of time, thus implying that the number of communication events involving any agent i is not larger than

$$\begin{aligned} C &\leq \frac{T}{\frac{\varepsilon}{4d_{\max}}} \leq \frac{\frac{2(d_{\max}+1)}{\varepsilon} \sum_{\{i,j\} \in E} (\bar{x}_i - \bar{x}_j)^2}{\frac{\varepsilon}{4d_{\max}}} \\ &= \frac{8d_{\max}(d_{\max} + 1)}{\varepsilon^2} \sum_{\{i,j\} \in E} (\bar{x}_i - \bar{x}_j)^2. \end{aligned}$$

Since each polling action involves polling at most $d_{\max}$ neighbors, we also conclude that the total number of messages to be

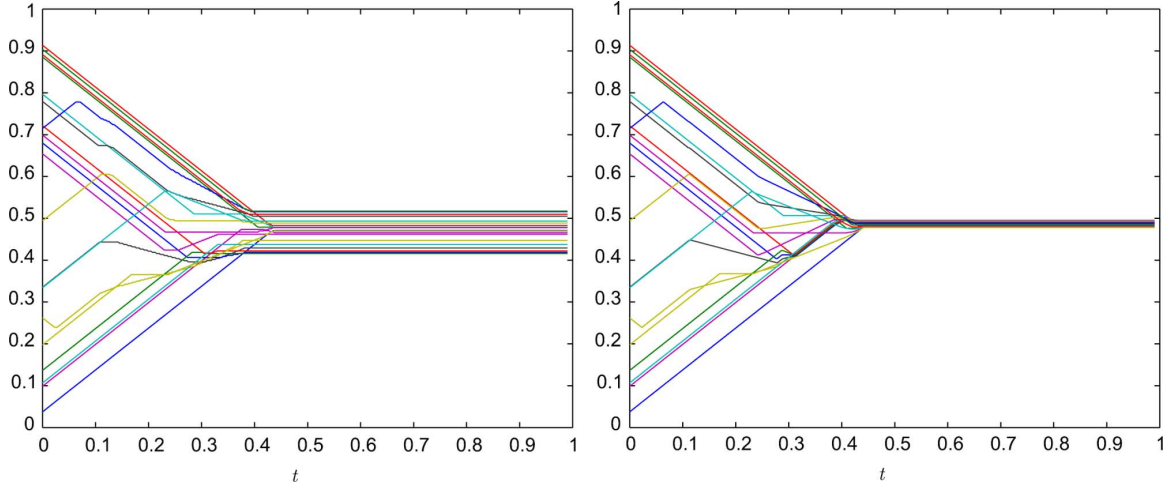


Fig. 1. Two sample evolutions of the states x in (2)–(3) starting from the same initial condition and on the same graph (a cycle graph with $n = 20$ nodes). Left plot assumes $\varepsilon = 0.01$, right plot assumes $\varepsilon = 0.001$.

exchanged in the whole network in order to achieve (practical) consensus is not larger than

$$\frac{8d_{\max}^2(d_{\max} + 1)n}{\varepsilon^2} \sum_{\{i,j\} \in E} (\bar{x}_i - \bar{x}_j)^2.$$

Our theoretical results suggest that, by choosing the sensitivity ε , we are trading between precision and cost, both in terms of time and of communication effort. However, simulations indicate that the role of ε in controlling the speed of convergence is limited, as long as $x(t)$ is far from $\mathcal{E}$. Before approaching the limit set, solutions are qualitatively similar to the solutions of consensus dynamics with (binary) controls in $\{-1, +1\}$: indeed the proposed control scheme may be seen as a self-triggered implementation of the binary controllers in [2]. This remark is confirmed if we compare Fig. 1 with Fig. 1 (rightmost) in [2]. Consistently, Fig. 2 demonstrates that the state trajectories “brake”, and the controls switch between zero and non-zero, as the states approach the region of convergence. Once this is reached (in finite time), the controls stop switching and remain constantly to zero, as the analysis in the next section shows.

The similarity with the dynamics in [2] also gives useful insights about the convergence values. Indeed, system (2)–(3) does not preserve the average of states, and simulations show that the states converge to values which are close to $(\min_i x_i(0) + \max_i x_i(0))/2$. This observation is again consistent with the results in [2], but we are not able so far to provide a tight formal statement about the distance between the consensus value and $(\min_i x_i(0) + \max_i x_i(0))/2$.

A. Convergence Analysis

This subsection is devoted to the proofs of Theorem 1 and Proposition 1.

Proof of Theorem 1: First of all, we recall that

$$t_{k+1}^i = t_k^i + \begin{cases} \frac{|\text{ave}_i(t_k^i)|}{4d_i} & \text{if } |\text{ave}_i(t_k^i)| \geq \varepsilon \\ \frac{\varepsilon}{4d_i} & \text{if } |\text{ave}_i(t_k^i)| < \varepsilon. \end{cases} \quad (8)$$

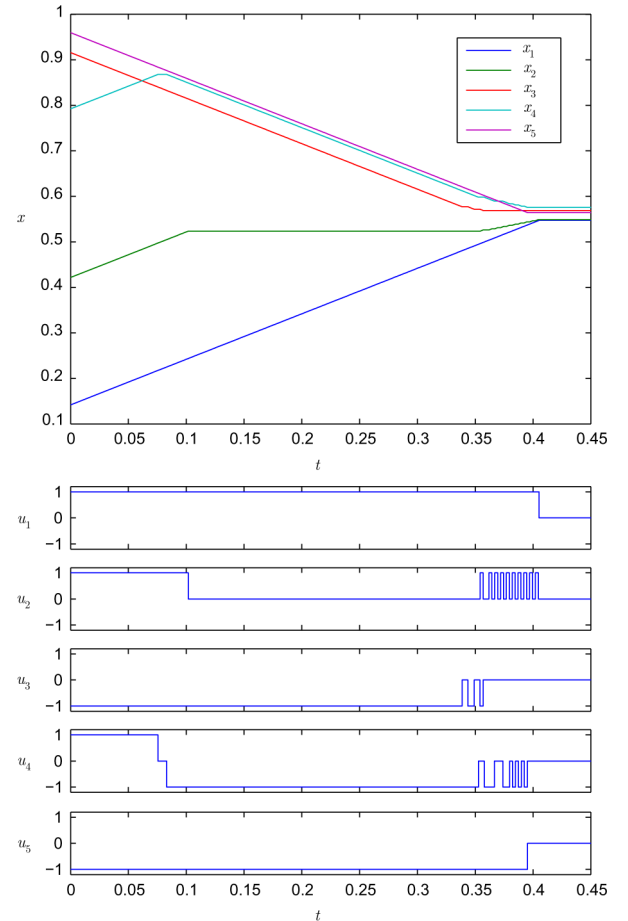


Fig. 2. Sample evolutions of states x and corresponding controls u in (2)–(3) on a cycle with $n = 5$ nodes, $\varepsilon = 0.02$.

Inequality (6) implies that there exists a positive dwell time between subsequent switches and this fact in turn implies that for each initial condition, (2) has a piecewise constant right-hand side. Hence the system has a unique solution $x(\cdot)$, which

is an absolutely continuous function of its time argument. Furthermore, solutions are bounded, since one can show that for all $t > 0$ it holds that $\min_i x_i(0) \leq \min_i x_i(t)$ and $\max_i x_i(0) \geq \max_i x_i(t)$. We are interested in studying the convergence properties of such solutions. For every $t \geq 0$, we let

$$V(t) = \frac{1}{2} x^T(t) L x(t).$$

We note that $V(t) \geq 0$ and we consider the evolution of $\dot{V}(t)$ along the solution. Since L is symmetric, and letting $t_k^i = \max\{t_h^i : t_h^i < t, h \in \mathbb{Z}_{\geq 0}\}$, we have

$$\begin{aligned} \dot{V}(t) &= x^T(t) L u(t) = u^T(t) L x(t) \\ &= - \sum_{i=1}^n \left(\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) \right) \text{sign}_\varepsilon(\text{ave}_i(t_k^i)) \\ &= - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \text{ave}_i(t) \text{sign}_\varepsilon(\text{ave}_i(t_k^i)). \end{aligned}$$

Using Equation (8) we observe that, for $t \in [t_k^i, t_{k+1}^i]$, if $\text{ave}_i(t_k^i) \geq \varepsilon$, then

$$\begin{aligned} \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) &\geq \sum_{j \in \mathcal{N}_i} (x_j(t_k^i) - x_i(t_k^i)) - 2d_i(t - t_k^i) \\ &\geq \frac{\text{ave}_i(t_k^i)}{2}. \end{aligned} \quad (9)$$

Similarly, if $\text{ave}_i(t_k^i) \leq -\varepsilon$, then

$$\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) \leq \frac{\text{ave}_i(t_k^i)}{2}.$$

These inequalities imply that, if $|\text{ave}_i(t_k^i)| \geq \varepsilon$, then $\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$ preserves the sign during continuous flow by continuity of $x(t)$, and consequently

$$\begin{aligned} \text{ave}_i(t) \text{sign}_\varepsilon(\text{ave}_i(t_k^i)) &= \text{ave}_i(t) \text{sign} \left(\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) \right) \\ &= |\text{ave}_i(t)|. \end{aligned} \quad (10)$$

Moreover

$$\begin{aligned} |\text{ave}_i(t)| &\geq \left| \sum_{j \in \mathcal{N}_i} (x_j(t_k^i) - x_i(t_k^i)) \right| - 2d_i(t - t_k^i) \\ &\geq \frac{|\text{ave}_i(t_k^i)|}{2}. \end{aligned} \quad (11)$$

Hence, using (10) and (11) we deduce

$$\begin{aligned} \dot{V}(t) &\leq - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \frac{|\text{ave}_i(t_k^i)|}{2} \\ &\leq - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \frac{\varepsilon}{2}. \end{aligned} \quad (12)$$

This inequality implies there exists a finite time $\bar{t}$ such that $|\text{ave}_i(t_k^i)| < \varepsilon$ for all $i \in I$ and all k such that $t_k^i \geq \bar{t}$. Indeed, otherwise there would be an infinite number of time intervals whose length is bounded away from zero and on which

$\dot{V}(t) \leq -\varepsilon/2$, contradicting the positivity of V . For all $i \in I$, let $\bar{k}_i = \min\{k \geq 0 : t_k^i \geq \bar{t}\}$ and define

$$\hat{t} = \inf\{t \geq 0 : t > t_{\bar{k}_i}^i \text{ for all } i \in I\}.$$

Note that $\hat{t} \geq \bar{t}$ and thus $|\text{ave}_i(t_k^i)| < \varepsilon$ if $t_k^i \geq \hat{t}$. Moreover, by definition of $\hat{t}$, for $t \geq \hat{t}$ and for all $i = 1, 2, \dots, n$, the controls $u_i(t)$ are zero and the states $x_i(t)$ are constant and such that $|\text{ave}_i(t)| < \varepsilon$ for all $i \in I$.

We conclude that there exists a point $x^* \in \mathbb{R}^n$ such that $x(t) = x^*$ for $t \geq \hat{t}$, and

$$x^* \in \left\{ x \in \mathbb{R}^n : \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right| < \varepsilon, \forall i \in I \right\}.$$

The above proof shows that convergence is reached in finite time: obtaining an estimate of this convergence time requires a deeper look into the dynamics of the system.

Proof of Proposition 1: In order to prove Proposition 1, we recall that Equation (12) implies that for every $t \geq 0$

$$\dot{V}(t) \leq - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \frac{\varepsilon}{2}.$$

We want to use this fact to estimate the time taken by $x(t)$ to reach the set $\mathcal{E}$. First of all, note that if $u(t) \neq 0$ for all $t < T$, then the set $\{i : |\text{ave}_i(t_k^i)| \geq \varepsilon\}$ is not empty and we can argue that the Lyapunov function decreases by at least $\varepsilon/2$ per time unit, until convergence is reached.

Let us then consider the more interesting case in which there exists $t' < T$ such that $u(t') = 0$. For all $i \in I$, define $k_i^* = \max\{h : t_h^i \leq t'\}$, and consider

$$t^* = \inf\{t \geq 0 : t > t_{k_i^*}^i \forall i \in I\}.$$

Clearly $u(t^*) = 0$ and $t_{k_i^*}^i \leq t^* \leq t' \leq t_{k_i^*+1}^i$ for all $i \in I$. Note that for $u(t')$ to be zero, necessarily $|\text{ave}_i(t_{k_i^*}^i)| < \varepsilon$, and then $t_{k_i^*+1}^i - t_{k_i^*}^i = \varepsilon/(4d_i)$ for all $i \in I$. If $|\text{ave}_i(t_{k_i^*+1}^i)| < \varepsilon$ for all $i \in I$ as well, then we can see that $u(t) = 0$ for all $t \geq t^*$, implying that convergence is reached and $T = t^* \leq t'$, contradiction. It must then exist² $j \in I$ such that $|\text{ave}_j(t_{k_j^*+1}^j)| \geq \varepsilon$.

Note that $t_{k_j^*+1}^j - t^* \leq \varepsilon/(4d_j) \leq \varepsilon/4$, whereas $\dot{V}(t) \leq -\varepsilon/2$ for $t \in (t_{k_j^*+1}^j, t_{k_j^*+1}^j + \varepsilon/(4d_j))$. The discussion above yields the following conclusion. Before convergence is reached, controls may possibly be zero and the set $\{i : |\text{ave}_i(\cdot)| \geq \varepsilon\}$ may be empty: however, this condition may only persist for a duration smaller than $\varepsilon/4$, after which the set $\{i : |\text{ave}_i(\cdot)| \geq \varepsilon\}$ is not empty for a time not shorter than $\varepsilon/(4d_{\max})$. Consequently, we argue that every $(\varepsilon/4)(1 + 1/d_{\max})$ units of time, $V(t)$ decreases by at least $(\varepsilon/2) \cdot \varepsilon/(4d_{\max})$. Hence, if

$$T > \frac{V(0)}{\frac{\varepsilon}{2} \cdot \frac{\varepsilon}{4d_{\max}}} \cdot \frac{\varepsilon}{4} \left(1 + \frac{1}{d_{\max}} \right) = \frac{V(0)}{\frac{\varepsilon}{2} \frac{1}{1+d_{\max}}}$$

²We remark that the existence of such j is permitted because, although $u(t) = 0$ when $t \in (t^*, \min_i t_{k_i^*+1}^i)$, actually $u(t)$ needs not to be zero for $t \in (\min_i t_{k_i^*}^i, t^*)$.

then the Lyapunov function would become negative, which is a contradiction. This implies that within $T \leq V(0)/((\varepsilon/2) \cdot (1/(1 + d_{\max})))$ units of time, the system must converge to the set of states (7) where $V(t)$ is constant.

The thesis follows if we recall that $\bar{x}$ is the initial condition and that $V(0) = \sum_{\{i,j\} \in E} (\bar{x}_i - \bar{x}_j)^2$. ■

We conclude this section with a significant remark about the versatility of Protocol A.

Remark 1 (Heterogeneous Sensitivities): It is apparent from the proof of Theorem 1 that convergence does not depend on the sensitivity ε to be the same for all the agents. We may thus think of allowing each agent the freedom to have its own sensitivity ε_i ; our arguments would thus ensure convergence to the set $\{x \in \mathbb{R}^n : |\sum_{j \in \mathcal{N}_i} (x_j - x_i)| < \varepsilon_i \ \forall i \in I\}$. □

This freedom may very well be understood as a robustness property against uncertain specification of the sensitivity threshold: this remark thus leads us to the topic of the next section.

III. ROBUSTNESS

In this section we discuss the robustness of Protocol A to some typical non-idealities which can occur in its implementation. We consider the issues of clock skew, delays, and limited precision of data: while these are not the only issues which can arise, we believe they are the most significant to our exposition, which regards networked problems. We do not study these three issues together, but we instead consider clock skews first in combination with delays, and then with quantization. This choice is made both for simplicity of presentation and in order to highlight the interest of robustness against clock skews. A model including all three issues can be studied using the same tools: a detailed analysis is left to future research.

The key idea to quantify the robustness properties, which are inherent to Protocol A, involves introducing a design parameter α which represents how conservative the agents are when planning their next sampling time. By proving convergence conditions for such extended model, we shall show that, provided the design parameters ε and α are properly chosen, our protocol can always be made robust to quantization errors, clock rate variabilities, and delays. The analysis reveals natural trade-offs between robustness and accuracy performance.

A. Clock Skews and Delays

In this section, we discuss the intrinsic robustness of Protocol A against model uncertainties in local clock specifications, combined with delays, which may occur during communication, computation, or actuation. To this goal, we extend the protocol to include a certain class of delays and clock rate variabilities. We thus generalize system (2)–(3) by considering the system $(x, u, \theta) \in \mathbb{R}^{3n}$ satisfying the continuous evolution

$$\begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -R_i \end{cases} \quad (13)$$

where $R_i > 0$ is the rate of the local clock at agent i , and the discrete evolution defined as follows. The set of switching agents is again defined as $\mathcal{S}(\theta, t) = \{i \in I : \theta_i(t) = 0\}$, and each agent

$i \in \mathcal{S}(\theta, t)$ polls its neighbors at time t : since implementing communication and actuation entails a nonnegative delay $\tau_i(t)$, each switching agent $i \in \mathcal{S}(\theta, t)$ undergoes the following update at time $\bar{t} = t + \tau_i(t)$:

$$\begin{cases} x_i(\bar{t}^+) = x_i(\bar{t}) & \forall i \in I \\ u_i(\bar{t}^+) = \begin{cases} \text{sign}_\varepsilon(\text{ave}_i(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ u_i(\bar{t}) & \text{otherwise} \end{cases} \\ \theta_i(\bar{t}^+) = \begin{cases} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(\bar{t}) & \text{otherwise} \end{cases} \end{cases} \quad (14)$$

where $\text{ave}_i(t)$ is defined as in (5), and for every $i \in I$ the map $f_i^\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ is defined by

$$f_i^\alpha(x) = \max \left\{ \frac{\alpha}{2d_i} \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right|, \frac{\alpha \varepsilon}{2d_i} \right\}$$

where $\alpha > 0$ is a design parameter. The initial conditions are chosen as before, namely $x(0) = \bar{x} \in \mathbb{R}^n$, $u(0) \in \{-1, 0, 1\}^n$, $\theta(0) = 0$. Note that system (2)–(3) is a special case of the above definition, assuming $\alpha = 1/2$, $R_i = 1$, and $\tau_i = 0$ for all $i \in I$.

Next, we move on to analyze the behavior of this system. In view of the presence of delays, for each $i \in I$ we define two sequences of time instants, namely the sequence $\{t_h^i : h \in \mathbb{Z}_{\geq 0}\}$ of times at which the controller i polls its neighbors, and the sequence $\{s_h^i : h \in \mathbb{Z}_{\geq 0}\}$ of times at which the variables u_i, θ_i are updated. More precisely, at time t_h^i node i polls its neighbors to receive the information needed to compute the quantity $f_i^\alpha(x(t_h^i))$. Next, a positive time $\tau_i(t_h^i)$ elapses between the time t_h^i when the node polls its neighbors and the time it updates the variables u_i, θ_i . Hence, the new control becomes effective only at time $s_h^i = t_h^i + \tau_i(t_h^i)$ and the control unit schedules the next sampling operation at time

$$s_h^i + f_i^\alpha(x(t_h^i)).$$

Due to the clock skew, however, the actual next sampling time t_{h+1}^i at which the controller polls the nodes is

$$t_{h+1}^i = s_h^i + \frac{1}{R_i} f_i^\alpha(x(t_h^i)).$$

This quantity satisfies

$$\begin{aligned} t_{h+1}^i &= s_h^i + \frac{1}{R_i} f_i^\alpha(x(t_h^i)) \\ &= t_h^i + \tau_i(t_h^i) + \frac{1}{R_i} f_i^\alpha(x(t_h^i)) \\ &\geq t_h^i + \frac{\alpha \varepsilon}{2d_i R_i}. \end{aligned}$$

Similarly

$$\begin{aligned} s_{h+1}^i &= t_{h+1}^i + \tau_i(t_{h+1}^i) \\ &= s_h^i + \frac{1}{R_i} f_i^\alpha(x(t_h^i)) + \tau_i(t_{h+1}^i) \\ &\geq s_h^i + \frac{\alpha \varepsilon}{2d_i R_i}. \end{aligned}$$

These bounds imply that solutions to (13)–(14) are well defined, and are useful to prove that, provided the system parameters α

and ε are chosen appropriately, the system ensures practical consensus according to the same definition as the ideal case studied before.

Proposition 2 (Clock Skew and Delay Robustness): Consider system (13)–(14) and assume that $R_i \geq R_{\min} > 0$ and $\tau_i(\cdot) \leq \tau_{\max}$ for all $i \in I$. If $\varepsilon > 4d_{\max}\tau_{\max}$ and

$$\alpha < \frac{\varepsilon - 4d_{\max}\tau_{\max}}{\varepsilon} R_{\min}$$

then $x(t)$ converges to a point in the set $\mathcal{E}$ defined in (7) in finite time.

Proof: Let t_k^i be a sampling time for node i , and s_k^i the corresponding update time. We observe that if at time $t \geq t_k^i$ the control $u_i(t) = \text{sign}_\varepsilon(\text{ave}_i(t_k^i))$ is being applied, then $s_k^i < t \leq s_{k+1}^i$ and

$$t - t_k^i \leq \tau_i(t_k^i) + \frac{1}{R_i} f_i^\alpha(x(t_k^i)) + \tau_i(t_{k+1}^i).$$

Using this inequality we observe that, if $\text{ave}_i(t_k^i) \geq \varepsilon$, then

$$\begin{aligned} \text{ave}_i(t) &\geq \text{ave}_i(t_k^i) - 2d_i(t - t_k^i) \\ &\geq \text{ave}_i(t_k^i) - 2d_i \left(\frac{\alpha |\text{ave}_i(t_k^i)|}{2d_i R_i} + \tau_i(t_k^i) + \tau_i(t_{k+1}^i) \right) \\ &\geq \text{ave}_i(t_k^i) \left(1 - \frac{\alpha}{R_{\min}} \right) - 4d_{\max}\tau_{\max} > 0. \end{aligned} \quad (15)$$

By symmetry, an analogous inequality holds if $\text{ave}_i(t_k^i) \leq -\varepsilon$. As in the proof of Theorem 1, we let $V(t) = 1/2x^T(t)Lx(t)$ for every $t \geq 0$, and we consider the evolution of $\dot{V}(t)$ along the solution.

We then have

$$\begin{aligned} \dot{V}(t) &= x^T(t)Lu(t) \\ &= - \sum_{i=1}^n \left(\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) \right) u_i(s_k^{i+}) \\ &= - \sum_{i=1}^n \text{ave}_i(t) \text{sign}_\varepsilon(\text{ave}_i(t_k^i)) \\ &= - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \text{ave}_i(t) \text{sign}_\varepsilon(\text{ave}_i(t_k^i)). \end{aligned} \quad (16)$$

From here on, the same reasoning as in the proof of Theorem 1 can be applied to show that

$$\dot{V}(t) \leq - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon} \left(\varepsilon \left(1 - \frac{\alpha}{R_{\min}} \right) - 4d_{\max}\tau_{\max} \right).$$

From this inequality, a similar Lyapunov argument as in the proof of Theorem 1 implies the desired convergence property. ■

We note that, according to Proposition 2, any (bounded) delay can be tolerated, but entails a proportionally large loss in the achievable precision.

Moreover, our result implies that convergence is guaranteed for clocks subject to an arbitrarily large but finite perturbation with respect to a perfect clock, provided the “time arrow” is preserved ($R_i > 0$) and as long as α is chosen to be small

enough. Specifically, there is no limitation on how large R_i can be. Indeed, a large R_i means a fast clock, which implies a faster sampling: while oversampling is certainly a drawback from the point of view of an efficient use of network resources, it does not prevent convergence. Instead, a low R_i means a slower clock, which implies a delayed sampling: this kind of error may disrupt the proper behavior of the system, and result in a loss of convergence.

Remark 2 (Local Robustness Condition): The robustness condition in Proposition 2 involves uniform bounds on node degrees, as well as clock skews and delays. We note however that this restriction is only made for simplicity: if we allow each node to choose its own robustness parameter α_i (and sensitivity), an analogous argument leads to the condition

$$\alpha_i < \frac{\varepsilon_i - 4d_i\tau_i^{\max}}{\varepsilon_i} R_i$$

where $\tau_i^{\max}$ is an upper bound on the delay affecting controller i . As a matter of fact, the inequality (15) can be replaced by

$$\begin{aligned} \text{ave}_i(t) &\geq \text{ave}_i(t_k^i) - 2d_i \\ &\quad \times \left(\frac{\alpha_i |\text{ave}_i(t_k^i)|}{2d_i R_i} + \tau_i(t_k^i) + \tau_i(t_{k+1}^i) \right) \\ &\geq \text{ave}_i(t_k^i) \left(1 - \frac{\alpha_i}{R_i} \right) - 4d_i\tau_i^{\max} > 0 \end{aligned}$$

and (16) can be replaced by

$$\dot{V}(t) = \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon_i} \text{ave}_i(t) \text{sign}_{\varepsilon_i}(\text{ave}_i(t_k^i)).$$

We then argue that

$$\dot{V}(t) \leq - \sum_{i: |\text{ave}_i(t_k^i)| \geq \varepsilon_i} \left(\varepsilon_i \left(1 - \frac{\alpha_i}{R_i} \right) - 4d_i\tau_i^{\max} \right)$$

which shows convergence in finite time to a set where $\text{ave}_i(x) < \varepsilon_i$ for all $i \in I$. □

B. Clock Skews and Quantized Information

A variation of the control scenario considered so far considers the possibility in which when an agent polls its neighbors it receives quantized information. This scenario can arise when the agent is endowed with a sensor which provides coarse (quantized) measurements of the neighbors’ relative states. We adopt for our analysis a standard uniform quantizer, defined as

$$q(x) = \Delta \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor$$

where $\Delta > 0$ is a parameter inversely proportional to the precision of the quantizer.

To take into account the presence of quantized measurements, model (2)–(3) is modified as follows. The continuous evolution obeys the equations

$$\begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -R_i \end{cases} \quad (17)$$

where $R_i > 0$ are the local clock rates. At every t such that the set $\mathcal{S}(\theta, t) = \{i \in I : \theta_i = 0\}$ is non-empty, the system instead satisfies the following discrete evolution:

$$\begin{cases} x_i(t^+) = x_i(t) & \forall i \in I \\ u_i(t^+) = \begin{cases} \text{sign}_\varepsilon(\text{qave}_i(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ u_i(t) & \text{otherwise} \end{cases} \\ \theta_i(t^+) = \begin{cases} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \end{cases} \quad (18)$$

where for every $\alpha > 0$ and $i \in I$ the map $f_i^\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ is defined by

$$f_i^\alpha(x) = \begin{cases} \frac{\alpha}{2d_i} |\sum_{j \in \mathcal{N}_i} q(x_j - x_i)| & \text{if } |\sum_{j \in \mathcal{N}_i} q(x_j - x_i)| \geq \varepsilon \\ \frac{\alpha \varepsilon}{2d_i} & \text{otherwise} \end{cases}$$

and we have used the notation

$$\text{qave}_i(t) = \sum_{j \in \mathcal{N}_i} q(x_j(t) - x_i(t)).$$

We are now ready to state a second robustness result.

Proposition 3 (Clock Skew and Quantization Robustness): Consider system (17)–(18) and assume that $R_i \geq R_{\min} > 0$ for all $i \in I$. If $\varepsilon > (d_{\max} \Delta)/2$ and

$$\alpha < \frac{2\varepsilon - d_{\max} \Delta}{2\varepsilon} R_{\min} \quad (19)$$

then $x(t)$ converges in finite time to a point in

$$\mathcal{E}_2 = \{x \in \mathbb{R}^n : |\sum_{j \in \mathcal{N}_i} (x_j - x_i)| < 2\varepsilon\}.$$

Proof: Similarly to the previous protocols, this algorithm ensures a guaranteed minimum inter-sampling time given by $(\alpha\varepsilon)/(2d_{\max})$. Hence, the solutions to the system are well-defined and unique. Along these solutions, the Lyapunov function $V = (x^T L x)/2$ satisfies

$$\dot{V}(t) = - \sum_{i: |\text{qave}_i(t_k^i)| \geq \varepsilon} \text{ave}_i(t) \text{sign}_\varepsilon(\text{qave}_i(t_k^i))$$

where as earlier t_k^i denotes the largest time at which agent i polls its neighbors before time t .

Observe that for all t ,

$$\text{ave}_i(t) - \frac{\Delta}{2} d_i \leq \text{qave}_i(t) \leq \text{ave}_i(t) + \frac{\Delta}{2} d_i \quad (20)$$

and also

$$|\text{ave}_i(t)| - \frac{\Delta}{2} d_i \leq |\text{qave}_i(t)| \leq |\text{ave}_i(t)| + \frac{\Delta}{2} d_i. \quad (21)$$

For $t \in [t_k^i, t_{k+1}^i]$ and $\text{qave}_i(t_k^i) \geq \varepsilon$

$$t - t_k^i \leq \frac{\alpha}{2d_i} \text{qave}_i(t_k^i) \frac{1}{R_i}.$$

Using this fact and (20), we argue that, if $\text{qave}_i(t_k^i) \geq \varepsilon$, then

$$\begin{aligned} \text{ave}_i(t) &\geq \text{ave}_i(t_k^i) - 2d_i(t - t_k^i) \\ &\geq \text{ave}_i(t_k^i) - 2d_i \left(\frac{\alpha}{2d_i} \text{qave}_i(t_k^i) \frac{1}{R_i} \right) \\ &\geq \text{qave}_i(t_k^i) - \frac{1}{2} d_i \Delta - \frac{\alpha}{R_i} \text{qave}_i(t_k^i) \\ &\geq \left(1 - \frac{\alpha}{R_i} \right) \varepsilon - \frac{1}{2} d_i \Delta \\ &\geq \left(1 - \frac{\alpha}{R_{\min}} \right) \varepsilon - \frac{1}{2} d_{\max} \Delta. \end{aligned}$$

An analogous inequality holds in the case $\text{qave}_i(t_k^i) \leq -\varepsilon$. Using the inequalities above, arguments similar to those in the proof of Theorem 1 lead to

$$\dot{V}(t) \leq - \sum_{i: |\text{qave}_i(t_k^i)| \geq \varepsilon} \left(\left(1 - \frac{\alpha}{R_{\min}} \right) \varepsilon - \frac{1}{2} d_{\max} \Delta \right)$$

and ultimately to prove convergence in finite time to the set such that

$$|\sum_{j \in \mathcal{N}_i} q(x_j(t) - x_i(t))| < \varepsilon.$$

The result thus follows from (21) and the condition on ε . ■

We conclude from Proposition 3 that the system is robust to quantized communication, and the achievable precision is proportional to the precision of the quantizer.

We remark that similarly to the robustness condition for delays and clock skews, also condition (19) can be reformulated in terms of quantities that are available *locally* at the agents (cf. Remark 2).

IV. ASYMPTOTICAL CONSENSUS

In this section we propose a modification of system (2)–(3), which drives the system to asymptotical consensus.

The key idea involves decreasing the sensitivity threshold with time and concurrently introducing a *time-varying decreasing gain* in the control loop.

Let $\varepsilon : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ and $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ be non-increasing functions such that

$$\lim_{t \rightarrow +\infty} \varepsilon(t) = \lim_{t \rightarrow +\infty} \gamma(t) = 0.$$

We consider the system $(x, u, \tau) \in \mathbb{R}^{3n}$ which satisfies the following continuous evolution

$$\begin{cases} \dot{x}_i = \gamma u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -1 \end{cases} \quad (22)$$

except for every t such that the set $\mathcal{S}(\theta, t) = \{i \in I : \theta_i = 0\}$ is non-empty. At such time instants the system satisfies the following discrete evolution

$$\begin{cases} x_i(t^+) = x_i(t) & \forall i \in I \\ u_i(t^+) = \begin{cases} \text{sign}_{\varepsilon(t)}(\text{ave}_i(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ u_i(t) & \text{otherwise} \end{cases} \\ \theta_i(t^+) = \begin{cases} \frac{1}{\gamma(t)} f_i(x(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \end{cases} \quad (23)$$

where $f_i(x) = \max \{ |\sum_{j \in \mathcal{N}_i} (x_j - x_i)| / (4d_i), \varepsilon(t) / (4d_i) \}$ for every $i \in I$.

We also adopt the same initial conditions as before, namely $x(0) = \bar{x} \in \mathbb{R}^n$, $u(0) \in \{-1, 0, +1\}$, $\theta(0) = 0$. As a result $t_k^i = 0$ for every $i \in I$.

The corresponding protocol is the following:

Protocol B (for each $i \in I$)

- 1: **initialization** : set $u_i(0) \in \{-1, 0, +1\}$ and $\theta_i(0) = 0$;
- 2: **while** $\theta_i(t) > 0$ **do**
 - 3: i applies the control $u_i(t)$;
- 4: **end while**
- 5: **if** $\theta_i(t) = 0$ **then**
 - 6: **for all** $j \in \mathcal{N}_i$ **do**
 - 7: i polls j and collects the information $x_j(t) - x_i(t)$;
 - 8: **end for**
 - 9: i computes $\text{ave}_i(t)$;
 - 10: i computes $\theta_i(t^+) = f_i(x(t)) / \gamma(t)$;
 - 11: i computes $u_i(t^+)$ by (23);
- 12: **end if**

In this new protocol we let the parameter ε , which—as established in the previous sections—gives a measure of the size of the region of practical convergence, to be time-varying and converging to zero. The obvious underlying rationale is that if the size of the convergence region goes to zero as time elapses, one might be able to establish asymptotical convergence rather than practical. However, letting ε go to zero does not suffice and may induce agents to poll their neighbors infinitely often in a finite interval of time (Zeno phenomenon). To prevent this occurrence we slow down both the process of requesting information to the neighbors and the velocity of the system. The former is achieved via a factor $1/\gamma(t)$ multiplying the map $f_i(x)$, the latter via the factor $\gamma(t)$ which weights the control value $u_i(t)$. It is intuitive that to fulfill the purpose, the function $\gamma(t)$ must be “comparable” with $\varepsilon(t)$. This is achieved assuming that there exists $c > 0$ such that

$$\frac{\varepsilon(t)}{\gamma(t)} \geq c \quad \forall i \in I, \forall t \geq 0. \quad (24)$$

Before stating the main result of this section, we briefly comment on the assumptions of the new algorithm. We note that in order to implement Protocol B the agents need to evaluate γ and ε as functions of t . Hence absolute time is assumed to be known to the agents. For this reason, the robustness properties of (22)–(23) do not trivially follow from the analysis in Section III: while some results covering clocks errors are given below, a complete study including delays and quantized communication is left to future work. We also note that one of the features of the practical coordination algorithm of the previous sections is the use of *ternary* control inputs, namely controls taking values

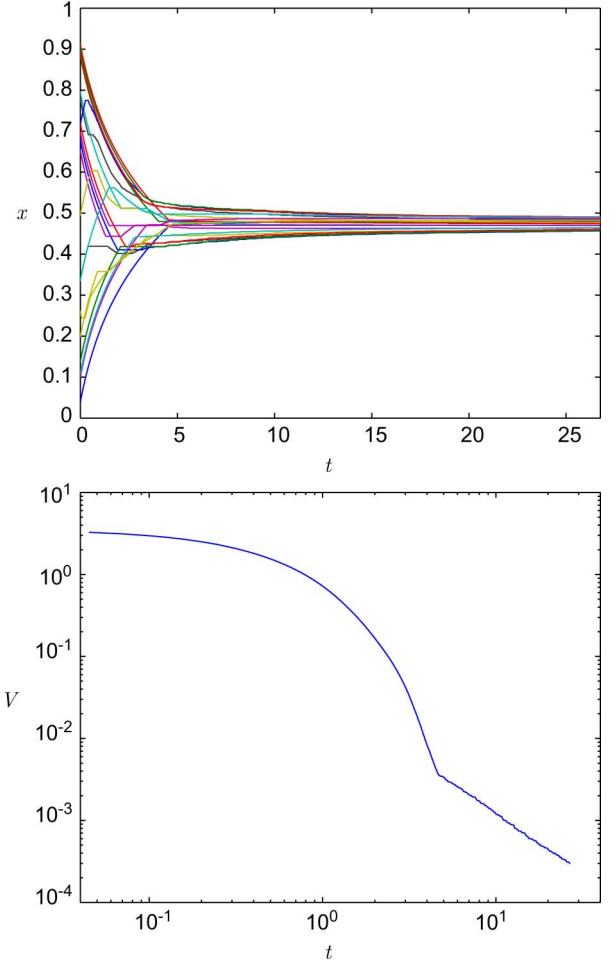


Fig. 3. Sample evolution of (22)–(23) starting from the same initial condition and on the same graph as Fig. 1. Top plot shows the state x , bottom plot shows the Lyapunov function V on a logarithmic scale. Simulation assumes $\varepsilon(t) = 0.05/(1+t)$, $\gamma(t) = 0.25/(1+t)$.

in the set $\{-1, 0, +1\}$. Instead, in the asymptotical coordination algorithm every control is weighted by the time-varying factor $\gamma(t)$, and thus belongs to $\{-\gamma(t), 0, +\gamma(t)\}$. One could think to modify the algorithm in the following way: each node i applies a constant ternary control input u_i during each time interval of continuous evolution $(t_k^i, t_{k+1}^i]$, where the control input u_i takes values in the set $\{-\gamma(t_k^i), 0, +\gamma(t_k^i)\}$. With this modification, the control laws are ternary on each inter-sampling time with a magnitude that asymptotically vanishes to zero. The analysis of such a modified algorithm does not follow from that given below and goes beyond the scope of this paper.

Theorem 2 (Asymptotical Consensus): Let $x(\cdot)$ be the solution to (22)–(23) under condition (24). Then, for every initial condition $\bar{x} \in \mathbb{R}^n$ there exists $\beta \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = \beta$ for all $i \in I$, if and only if $\int_0^{+\infty} \gamma(s) ds$ is divergent.

Before the proof, we discuss some illustrative simulation results, collected in Figs. 3 and 4. Simulations show that the evolution of (22)–(23) can be qualitatively divided into two phases. During a first phase of *fast* convergence, ε plays a little role and the behavior of x is reminiscent of Fig. 1. The first phase lasts until the states become close enough to consensus for $\varepsilon(t)$ to be comparable with the differences between the x_i 's. During the second phase, we can observe that the control actions (i.e., the

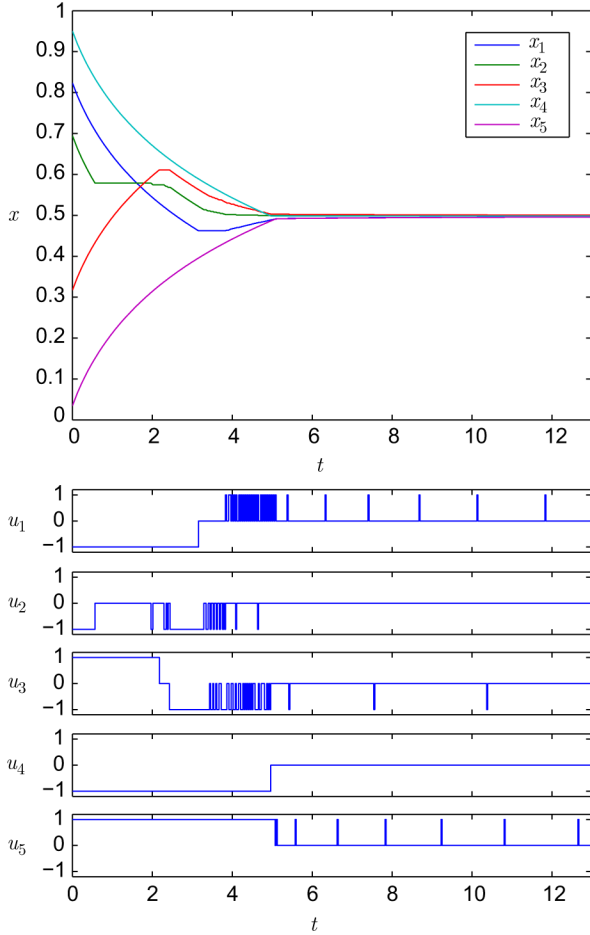


Fig. 4. Sample evolutions of states x and corresponding controls u in (22)–(23) on a ring with $n = 5$ nodes, $\varepsilon(t) = 0.05/(1+t)$, $\gamma(t) = 0.25/(1+t)$. The guaranteed minimal inter-switching time of the controllers is 0.025 units of time.

intervals of non-zero control) become sporadic and convergence depends on the decrease of ε : its speed is thus *slow*. Indeed, we are assuming that $\varepsilon(t)$ has divergent integral, so that it may not decrease faster than $1/t$. We want to stress that this technical condition is not due to a limitation of our analysis, but is inherent to the system. Indeed, assumption (24) relates ε and γ , so that $\gamma(t)$ may not be larger than (a constant times) $\varepsilon(t)$. In turn, as we shall see in the proof below, if $\gamma(t)$ had bounded integral, the control would not be large enough to stabilize the system to practical consensus from an arbitrary initial condition.

Proof of Theorem 2: First of all, we note that by (24), there exists a unique solution for every initial condition. In fact, observe that

$$t_{k+1}^i = t_k^i + \begin{cases} \frac{|\text{ave}_i(t_k^i)|}{4d_i\gamma(t_k^i)} & \text{if } |\text{ave}_i(t_k^i)| \geq \varepsilon(t_k^i) \\ \frac{\varepsilon(t_k^i)}{4d_i\gamma(t_k^i)} & \text{if } |\text{ave}_i(t_k^i)| < \varepsilon(t_k^i) \end{cases} \quad (25)$$

and hence $t_{k+1}^i - t_k^i \geq c/4d_i$ for all $i \in I$ and each $k \in \mathbb{Z}_{\geq 0}$. Next, we introduce the following notation. For every $t \geq 0$ and $i \in I$, we consider the sequence of switching times t_k^i and let

$$k_i^*(t) := \max\{k \geq 0 : t_k^i \leq t\}. \quad (26)$$

We now start a Lyapunov analysis to show that $\int_0^{+\infty} \gamma(s)ds = +\infty$ is sufficient for convergence. For every $t \geq 0$, we let $V(t) = (x^T(t)Lx(t))/2$, and we note that $V(t) \geq 0$ and

$$\begin{aligned} V(t) &\leq N \max_{i \in I} |x_i(t)| \max_{i \in I} |\text{ave}_i(t)| \\ &\leq N \max_{i \in I} |x_i(0)| \max_{i \in I} |\text{ave}_i(t)| \end{aligned}$$

since $\max_{i \in I} x_i(t)$ (resp. $\min_{i \in I} x_i(t)$) is non-increasing in time (respectively, non-decreasing). Indeed, let $m(t) = \max_{i \in I} x_i(t)$ and $\mu(t) = \text{argmax}_{i \in I} x_i(t)$ and note that also Protocol B ensures that at all times $\text{sign}(\text{ave}_i(t)) = \text{sign}(\text{ave}_i(t_{k_i^*}^i(t)))$ for every $i \in I$. In particular the latter is true for $i = \mu(t)$. Since $x_{\mu(t)}(t) = m(t)$, then $\text{ave}_{\mu(t)}(t) \leq 0$ and therefore $\text{ave}_{\mu(t)}(t_{k_{\mu(t)}^*}^{\mu(t)}(t)) \leq 0$. It follows by (23) that $u_{\mu(t)}(t) \leq 0$, which implies that $\dot{m}(t) = \dot{x}_{\mu(t)}(t) \leq 0$. Hence, during continuous evolution $m(t)$ cannot increase and during discrete transitions it remains constant. This shows the non-increasing monotonicity of $\max_{i \in I} x_i(t)$. Similarly, one proves the non-decreasing monotonicity of $\min_{i \in I} x_i(t)$.

After proving these bounds on $V(t)$, we study $\dot{V}(t)$ along the solution to the system. Similarly to the proof of Theorem 1, we have that if $\text{ave}_i(t_{k_i^*}^i(t)) \geq \varepsilon$, then

$$\begin{aligned} \text{ave}_i(t) &\geq \sum_{j \in \mathcal{N}_i} (x_j(t_{k_i^*}^i(t)) - x_i(t_{k_i^*}^i(t))) - 2d_i \int_{t_{k_i^*}^i(t)}^t \gamma(s)ds \\ &\geq \text{ave}_i(t_{k_i^*}^i(t)) - d_i \gamma(t_{k_i^*}^i(t))(t - t_{k_i^*}^i(t)) \\ &\geq \frac{\text{ave}_i(t_{k_i^*}^i(t))}{2}. \end{aligned} \quad (27)$$

Since $\dot{V}(s) = x^T(s)L\gamma(s)u(s)$, we deduce that for all $s \geq 0$,

$$\dot{V}(s) \leq -\gamma(s) \sum_{i: |\text{ave}_i(t_{k_i^*}^i(s))| \geq \varepsilon(t_{k_i^*}^i(s))} \frac{|\text{ave}_i(t_{k_i^*}^i(s))|}{2}$$

which in particular implies that $V(t)$ is non-increasing.

It is also useful to notice that for all $t \in [t_{k_i^*}^i(t), t_{k_i^*}^i(t)+1)$, if $|\text{ave}_i(t_{k_i^*}^i(t))| < \varepsilon(t_{k_i^*}^i(t))$, then

$$\begin{aligned} |\text{ave}_i(t)| &\leq |\text{ave}_i(t_{k_i^*}^i(t))| + d_i \int_{t_{k_i^*}^i(t)}^t \gamma(s)ds \\ &\leq |\text{ave}_i(t_{k_i^*}^i(t))| + d_i \gamma(t_{k_i^*}^i(t))(t - t_{k_i^*}^i(t)) \\ &\leq \frac{5}{4} \varepsilon(t_{k_i^*}^i(t)). \end{aligned}$$

Similarly, if $|\text{ave}_i(t_{k_i^*}^i(t))| \geq \varepsilon(t_{k_i^*}^i(t))$, then

$$\begin{aligned} |\text{ave}_i(t)| &\leq |\text{ave}_i(t_{k_i^*}^i(t))| + 2d_i \int_{t_{k_i^*}^i(t)}^t \gamma(s)ds \\ &\leq \frac{3}{2} |\text{ave}_i(t_{k_i^*}^i(t))|. \end{aligned}$$

This inequality implies that for all $s \geq 0$

$$V(s) \leq N \max_{i \in I} |x_i(0)| \frac{3}{2} \max_{i \in I} \max\{|\text{ave}_i(t_{k_i^*}^i(s))|, \varepsilon(t_{k_i^*}^i(s))\}. \quad (28)$$

Next, we claim that for all $\delta > 0$ and $T > 0$, there exists $t \geq T$ such that $|\text{ave}_i(t_{k_i^*}^i(t))| < \delta$ for all $i \in I$. Indeed,

by contradiction there would exist $\delta > 0$ and $T > 0$ such that for all $t \geq T$, $|\text{ave}_i(t_{k^*}^i(t))| \geq \delta$ for some $i \in I$, implying $\dot{V}(t) \leq -(\delta/2)\gamma(t)$ and thus contradicting the positivity of $V(t)$ if $\int_0^{+\infty} \gamma(s)ds = +\infty$. Since the above claim holds true and $\varepsilon(t)$ converges to zero as t goes to infinity, we argue that for every $\delta' > 0$ it is possible to choose $T' \geq 0$ such that $\max_{i \in I} \max\{|\text{ave}_i(t_{k^*}^i(T'))|, \varepsilon(t_{k^*}^i(T'))\} < \delta'$. To complete the argument, we claim that for any $\lambda > 0$, there exists T_λ such that $V(t) < \lambda$ for all $t > T_\lambda$. To show the latter, choose δ' such that $N \max_{i \in I} |x_i(0)| (3/2)\delta' \leq \lambda$, and fix T' accordingly. Then by (8), $V(T') \leq \lambda$. As we have shown that $V(t)$ is monotone non-increasing, it is also true that $V(t) < \lambda$ for all $t > T'$, which proves the claim with $T_\lambda = T'$. Hence $V(t)$ goes to zero as t goes to infinity. This fact in turn implies that $|x_i - x_j| \rightarrow 0$ as $t \rightarrow +\infty$.

Finally, we need to show that each trajectory converges to one point in the subspace satisfying the above condition. To this goal, recall from the first part of the proof that $m(t) = \max_{i \in I} x_i(t)$ is monotonically non-increasing and that $m(t) \geq \min_{i \in I} x_i(0)$. Hence, $\lim_{t \rightarrow +\infty} m(t)$ exists finite, which together with the result above implies convergence of all x_i 's to a common limit point.

In order to conclude the proof, we still need to show that $\int_0^{+\infty} \gamma(s)ds = +\infty$ is necessary. As above, let $\mu(t) = \arg\max_{i \in I} x_i(t)$ and analogously $\nu(t) = \arg\min_{i \in I} x_i(t)$. Note that $\dot{x}_\mu(t) \leq \gamma(t)$ and $\dot{x}_\nu(t) \geq -\gamma(t)$. Then, for all $t \geq 0$

$$x_\mu(t) - x_\nu(t) \geq x_\mu(0) - x_\nu(0) - 2 \int_0^t \gamma(s) ds.$$

If we assume by contradiction that $\int_0^{+\infty} \gamma(s)ds = K > 0$, then $x_\mu(t) - x_\nu(t) \geq x_\mu(0) - x_\nu(0) - 2K$, which contradicts convergence for any initial condition such that $x_\mu(0) - x_\nu(0) \geq 2K + 1$. ■

We note that, while it is clear from the proof that for every $i \in I$ the quantity $\int_0^{+\infty} u_i(s)ds = \beta - x_i(0)$ is finite, our analysis is not able so far to understand whether $\int_0^{+\infty} |u_i(s)|ds$ (that can be viewed as an estimate of the "control effort") is always finite as well. This intriguing question is left as an open problem. In the rest of this section, we provide a preliminary analysis of the robustness of Protocol B.

A. Robustness Against Clock Errors

As we already remarked, Protocol B asks the agents to know the absolute time in order to evaluate the functions ε and γ . It is thus clear that inconsistent clocks may potentially disrupt the operation of the algorithm. In this section, we are going to quantify the robustness margin to this source of uncertainty and to propose suitable countermeasures. Other robustness issues, such as delays or quantized communications, are left out of the scope of this subsection.

Preliminary, let us note that an incorrect evaluation of the time t results in local time variables t_i 's, which are available at the agents and are different from t . Consequently, instead of the common weight functions ε and γ , some local copies ε_i and γ_i are available to the agents as weight functions, namely $\varepsilon_i(t) := \varepsilon(t_i(t))$ and $\gamma_i(t) := \gamma(t_i(t))$. Hence, a clock error is equivalent to an error in the protocol gains. In such a case, the

resulting implementation of Protocol B requires the control law in the first equation of (22) to be replaced by

$$\dot{x}_i = \gamma_i u_i \quad (29)$$

and the update law for u_i to be replaced by

$$u_i(t^+) = \begin{cases} \text{sign}_{\varepsilon_i(t)}(\text{ave}_i(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ u_i(t) & \text{otherwise.} \end{cases} \quad (30)$$

Consistently and in view of Section III, the update law for θ_i is replaced by

$$\theta_i(t^+) = \begin{cases} \frac{1}{\gamma_i(t)} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \quad (31)$$

where

$$f_i^\alpha(x) = \max \left\{ \frac{\alpha}{2d_i} \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right|, \frac{\alpha \varepsilon_i(t)}{2d_i} \right\}$$

for every $i \in I$ and $\alpha > 0$ is the conservativeness parameter.

This implementation may still be effective, provided certain conditions are met. We state this fact as the following result:

Proposition 4 (Clocks Error Robustness): Assume that non-increasing functions ε_i and γ_i are used as above by each node $i \in I$ to implement (22)–(23), according to (29), (30), and (31), and assume that for every $i \in I$, (i) $\lim_{t \rightarrow +\infty} \varepsilon_i(t) = 0$; (ii) $\int_0^{+\infty} \gamma_i(s)ds = +\infty$; and (iii) $\varepsilon_i(t)/\gamma_i(t) \geq c > 0$. If there exists $c' > 0$ such that

$$\frac{|\gamma_j(t) - \gamma_i(t)|}{\gamma_i(t)} \leq c' \quad \forall (i, j) \in E, \quad t \geq 0 \quad (32)$$

and α is chosen to satisfy

$$\alpha < \frac{2}{c' + 2} \quad (33)$$

then, for every initial condition $\bar{x} \in \mathbb{R}^n$ there exists $\beta \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = \beta$ for all $i \in I$.

Proof: The proof goes as the proof of Theorem 2, *mutatis mutandis*. Here we only explicitly verify the following key inequality [cf. (27)], where to make the notation compact we are using t_k^i instead of $t_{k^*}^i(t)$ as defined in (26). If $\text{ave}_i(t_k^i) \geq \varepsilon_i(t_k^i)$, then

$$\begin{aligned} \text{ave}_i(t) &\geq \sum_{j \in \mathcal{N}_i} (x_j(t_k^i) - x_i(t_k^i)) - \sum_{j \in \mathcal{N}_i} (\gamma_i(t_k^i) + \gamma_j(t_k^i))(t - t_k^i) \\ &\geq \text{ave}_i(t_k^i) - \frac{\alpha \sum_{j \in \mathcal{N}_i} (\gamma_i(t_k^i) + \gamma_j(t_k^i))}{d_i \gamma_i(t_k^i)} \text{ave}_i(t_k^i) \\ &= \left(1 - \alpha - \frac{\alpha \sum_{j \in \mathcal{N}_i} (-\gamma_i(t_k^i) + \gamma_j(t_k^i))}{d_i \gamma_i(t_k^i)} \right) \text{ave}_i(t_k^i) \\ &\geq \left(1 - \alpha - \frac{\alpha \sum_{j \in \mathcal{N}_i} |\gamma_j(t_k^i) - \gamma_i(t_k^i)|}{d_i \gamma_i(t_k^i)} \right) \text{ave}_i(t_k^i) \\ &\geq (1 - \alpha - \frac{\alpha}{2} c') \text{ave}_i(t_k^i). \end{aligned}$$

Then, the Lyapunov function $V(t) = (x^T(t)Lx(t))/2$ computed along the solutions of the system satisfies

$$\begin{aligned} \dot{V}(t) &= x^T(t)L \operatorname{diag}(\gamma_1(t), \dots, \gamma_n(t)) u(t) \\ &= - \sum_{i: |\operatorname{ave}_i(t_k^i)| \geq \varepsilon_i(t_k^i)} \gamma_i(t) \operatorname{ave}_i(t) \operatorname{sign}_{\varepsilon_i(t_k^i)}(\operatorname{ave}_i(t_k^i)) \\ &\leq - \sum_{i: |\operatorname{ave}_i(t_k^i)| \geq \varepsilon_i(t_k^i)} \gamma_i(t) \frac{|\operatorname{ave}_i(t_k^i)|}{2} \\ &\quad \times (2 - 2\alpha - \alpha c') \end{aligned}$$

and the result follows thanks to (33). $\blacksquare$

In some applications, assumption (32) can be satisfied by controlling the clock errors $|t_i - t_j|$ by means of some clock-synchronization protocol. Indeed, many clock-synchronization algorithms have been proposed in the literature (cf. [26]), but their implementation with self-triggered communication is not trivial. For all cases in which (32) can not be guaranteed, we propose a scheme which is intrinsically able to compensate for the errors induced by the clock skews. This scheme involves a different update rule for the variable θ , as follows:

$$\theta_i(t^+) = \begin{cases} \frac{\bar{f}_i(x(t))}{\sum_{j \in \mathcal{N}_i} (\gamma_j(t) + \gamma_i(t))} & \text{if } i \in \mathcal{S}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \quad (34)$$

where

$$\bar{f}_i(x) = \max \left\{ \frac{1}{2} \left| \sum_{j \in \mathcal{N}_i} (x_j - x_i) \right|, \frac{\varepsilon_i(t)}{2} \right\}. \quad (35)$$

It is clear that this update law requires the sampling node to receive from its neighbors the values of their gain functions at the time of sampling ($\gamma_j(t_k^i)$). The following result shows that this communication overhead is sufficient to ensure convergence.

Proposition 5 (Compensating Clock Errors): Let $x(\cdot)$ be the solution to (22)–(23), modified according to (29), (30), (34), and (35). Assume that for every $i \in I$, (i) $\lim_{t \rightarrow +\infty} \varepsilon_i(t) = 0$; (ii) $\int_0^{+\infty} \gamma_i(s) ds = +\infty$; and (iii) $\varepsilon_i(t)/\gamma_i(t) \geq c > 0$. Then, there exists $\beta \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = \beta$ for all $i \in I$.

Proof: The arguments follow the lines of the proof of Theorem 2: hence, most details are omitted and can be found in [27]. As in the proof of the previous result, we only explicitly verify the following key inequality [cf. (27)]. If $\operatorname{ave}_i(t_k^i) \geq \varepsilon$, then

$$\begin{aligned} \operatorname{ave}_i(t) &\geq \sum_{j \in \mathcal{N}_i} (x_j(t_k^i) - x_i(t_k^i)) - \sum_{j \in \mathcal{N}_i} (\gamma_i(t_k^i) + \gamma_j(t_k^i))(t - t_k^i) \\ &\geq \operatorname{ave}_i(t_k^i) - \frac{\sum_{j \in \mathcal{N}_i} (\gamma_i(t_k^i) + \gamma_j(t_k^i))}{2 \sum_{j \in \mathcal{N}_i} (\gamma_i(t_k^i) + \gamma_j(t_k^i))} \operatorname{ave}_i(t_k^i) \\ &\geq \frac{1}{2} \operatorname{ave}_i(t_k^i). \end{aligned}$$

V. INDEPENDENT POLLING OF NEIGHBORS

In Protocol A, each time an agent polls its neighbors, it polls all of them simultaneously. However, it is possible to design a similar protocol so that each agent collects information from

a neighbor independently of its other neighbors. This modification leads to similar convergence results, as we shall see in what follows.

Let us adopt a new set of state variables (x, u, θ) , which take value in the state space $\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d$, where d is the sum of the neighbors of all the agents, namely $d = \sum_{i=1}^n d_i$. The continuous evolution of the system obeys the equations

$$\begin{cases} \dot{x}_i = \sum_{j \in \mathcal{N}_i} u_i^j \\ \dot{u}_i^j = 0 \\ \dot{\theta}_i^j = -1 \end{cases} \quad (36)$$

where $i \in I$ and $j \in \mathcal{N}_i$. The system satisfies the differential equation above for all t except for those values of the time at which the set

$$\mathcal{J}(\theta, t) = \{\{i, j\} \in I \times I : j \in \mathcal{N}_i \text{ and } \theta_i^j(t) = 0\}$$

is non-empty. We denote the k th time t at which $\{i, j\} \in \mathcal{J}(\theta, t)$ by t_k^{ij} . At these times a discrete transition occurs, which is governed by the following discrete update:

$$\begin{cases} x_i(t^+) = x_i(t) & \forall i \in I \\ u_i^j(t^+) = \begin{cases} \operatorname{sign}_\varepsilon(x_j(t) - x_i(t)) & \text{if } \{i, j\} \in \mathcal{J}(\theta, t) \\ u_i^j(t) & \text{otherwise} \end{cases} \\ \theta_i^j(t^+) = \begin{cases} f_i^j(x(t)) & \text{if } \{i, j\} \in \mathcal{J}(\theta, t) \\ \theta_i^j(t) & \text{otherwise} \end{cases} \end{cases} \quad (37)$$

where for every $i \in I$ and $j \in \mathcal{N}_i$, the map $f_i^j : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ is defined by

$$f_i^j(x) = \max \left\{ \frac{|x_j - x_i|}{2(d_i + d_j)}, \frac{\varepsilon}{2(d_i + d_j)} \right\}. \quad (38)$$

The new protocol can be described as follows:

Protocol C (for each $i \in I$)

1: **initialization:** For all $j \in \mathcal{N}_i$, set $\theta_i^j = 0$, $u_i^j(0) \in \{-1, 0, +1\}$, and $u_i(0) = \sum_{j \in \mathcal{N}_i} u_i^j(0)$;

2: **for all** $j \in \mathcal{N}_i$ **do**

3: **while** $\theta_i^j(t) > 0$ **do**

4: i applies the control $u_i(t) = \sum_{j \in \mathcal{N}_i} u_i^j(t)$;

5: **end while**

6: **if** $\theta_i^j(t) = 0$ **then**

7: i polls j and collects the information $x_j(t) - x_i(t)$;

8: i updates $\theta_i^j(t^+) = f_i^j(x(t))$;

9: i updates $u_i^j(t^+) = \operatorname{sign}_\varepsilon(x_j(t) - x_i(t))$;

10: **end if**

11: **end for**

In contrast with Protocol A, we note that in this case the control applied by each agent is a sum of ternary controls.

Remark 3 (Self-Triggered Gossip): Note that for all $\{i, j\} \in E$ that $\theta_i^j(t) = \theta_j^i(t)$ and $u_i^j(t) = -u_j^i(t)$ for all $t \geq 0$. This edge synchrony is essential to the analysis which follows, and

points to the fact that (36)–(37) is actually an *edge-based* algorithm, although the active entities Protocol C are the agents, i.e., the nodes of the network. An equivalent implementation, in which the active entities are the edges, is to be presented in the forthcoming paper [28]. This feature is reminiscent of several pairwise *gossip* approaches which have recently appeared in the literature: references include randomized [29] and deterministic [30] protocols, with applications ranging from signal processing [31] to optimal deployment of robotic networks [32]. In view of these works, we may term Protocol C as a *self-triggered gossip algorithm*.

The following convergence result holds:

Theorem 3 (Practical Consensus): For every initial condition $\bar{x}$, let $x(t)$ be the solution to (36)–(37) such that $x(0) = \bar{x}$. Then, $x(t)$ converges in finite time to a point x^* in the set

$$\mathcal{E}' = \{x \in \mathbb{R}^n : |x_j - x_i| < \varepsilon \forall \{i, j\} \in E\}.$$

Moreover, if we define $T' = \inf\{t \geq 0 : x(t) \in \mathcal{E}'\}$, then

$$T' \leq \frac{d_{\max} + 1}{\varepsilon} \sum_{i \in I} \bar{x}_i^2.$$

Proof: In this proof we adopt the Lyapunov function $V(x) = (x^T x)/2$. For a given t , let $t_k^{ij} = \max\{t_\ell^{ij} : t_\ell^{ij} < t, \ell \in \mathbb{N}\}$. Along the solution of (36), the function satisfies

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N x_i(t) \dot{x}_i(t) \\ &= \sum_{i=1}^N x_i(t) \sum_{j \in \mathcal{N}_i} u_i^j(t) \\ &= \sum_{i=1}^N x_i(t) \sum_{j \in \mathcal{N}_i} \text{sign}_\varepsilon(x_j(t_k^{ij}) - x_i(t_k^{ij})) \\ &= - \sum_{\{i,j\} \in E} (x_j(t) - x_i(t)) \text{sign}_\varepsilon(x_j(t_k^{ij}) - x_i(t_k^{ij})). \end{aligned}$$

During the continuous evolution $|\dot{x}_j(t) - \dot{x}_i(t)| \leq d_i + d_j$, and at the jumps $x_i(t) - x_j(t)$ does not change its value. This implies that $x_i(t) - x_j(t)$ cannot differ from $x_i(t_k^{ij}) - x_j(t_k^{ij})$ in absolute value for more than $(d_i + d_j)(t - t_k^{ij})$. Exploiting this fact, if $|x_i(t_k^{ij}) - x_j(t_k^{ij})| \geq \varepsilon$, then by (38) for all $t \in [t_k^{ij}, t_{k+1}^{ij}]$, we have

$$|x_i(t) - x_j(t)| \geq \frac{|x_i(t_k^{ij}) - x_j(t_k^{ij})|}{2}$$

and $\text{sign}_\varepsilon(x_i(t) - x_j(t)) = \text{sign}_\varepsilon(x_i(t_k^{ij}) - x_j(t_k^{ij}))$. Hence

$$\begin{aligned} \dot{V}(t) &= - \sum_{\{i,j\} \in E: |x_i(t_k^{ij}) - x_j(t_k^{ij})| \geq \varepsilon} |x_i(t) - x_j(t)| \\ &\leq - \sum_{\{i,j\} \in E: |x_i(t_k^{ij}) - x_j(t_k^{ij})| \geq \varepsilon} \frac{|x_i(t_k^{ij}) - x_j(t_k^{ij})|}{2}. \end{aligned}$$

This implies that there exists a finite time T such that, for all $t \geq T$, $|x_i(t_k^{ij}) - x_j(t_k^{ij})| < \varepsilon$ for all $\{i, j\} \in E$, because if this were not true then there would exist $\{i, j\} \in E$ and an infinite subsequence $t_{k'}^{ij}$ of the sequence of switching times t_k^{ij} such that

$|x_i(t_{k'}^{ij}) - x_j(t_{k'}^{ij})| \geq \varepsilon$, which would contradict the positiveness of $V(t)$.

Hence for $t \geq T$, $|x_i(t_k^{ij}) - x_j(t_k^{ij})| < \varepsilon$ for all $\{i, j\} \in E$. Moreover, if $t \geq \max_{\{i,j\} \in E} t_{k+1}^{ij}$, $u_i^j(t) = 0$, the state stops evolving and satisfies $|x_i(t) - x_j(t)| < \varepsilon$, that is the first part of the thesis.

As far as the second part of thesis is concerned, similarly to the proof of Proposition 1, we observe that if for some t , $u(t) = 0$, then either $u_i^j(t_{k+1}^{ij}) \neq 0$ for some $i \in I$ and some $j \in \mathcal{N}_i$ (where t_{k+1}^{ij} denotes the smallest switching time larger than t at which agents i, j update their variables), or $u(t') = 0$ for all $t' \geq 0$. In the latter case, the state has already reached the set $\mathcal{E}'$. Since we are interested in characterizing the time T' by which convergence is achieved, we focus on the former case. Then we see that $\dot{V}(t) = 0$ for at most $\varepsilon/4$ units of time (the maximal length of an interval of time over which $u = 0$ before the state has reached $\mathcal{E}'$) and that the interval must be followed by an interval of at least $\varepsilon/(4d_{\max})$ units of time over which $\dot{V}(t) \leq -\varepsilon/2$. These estimates imply for T' the bound given in the statement. ■

A bound for C' , the communication cost, can be easily obtained starting from T' following a similar argument as the one after Proposition 1.

Remark 4 (Comparing Convergence Sets): We note that Theorem 3 proves convergence to a set $\mathcal{E}'$ which is different from Theorem 1. Writing the dependence on ε explicitly, we may observe that

$$\mathcal{E}'\left(\frac{\varepsilon}{d_{\max}}\right) \subset \mathcal{E}''(\varepsilon) \subset \mathcal{E}(\varepsilon)$$

where $\mathcal{E}'' = \{x \in \mathbb{R}^n : \sum_{j \in \mathcal{N}_i} |x_j - x_i| < \varepsilon \forall i \in I\}$. Furthermore, if $x \in \mathcal{E}'$, then for each pair of agents i, j , the distance $|x_i - x_j|$ is smaller than ε times the diameter of the network. In these respects, we may argue that the practical consensus condition of Protocol C is more precise than that of Protocol A. This performance is achieved by employing d_i time variables θ_i^j and controls u_i^j per agent, instead of a single one as in Protocol A.

A. Asymptotical Consensus

In the rest of this section, we present a modification of Protocol C leading to asymptotical consensus. While its design is largely inspired by Section IV, its analysis is partly different: hence we include a proof of convergence.

In order to yield asymptotical consensus, the protocol is modified as follows. The continuous evolution (36) is replaced by

$$\begin{cases} \dot{x}_i = \gamma(t) \sum_{j \in \mathcal{N}_i} u_i^j \\ \dot{u}_i^j = 0 \\ \dot{\theta}_i^j = -1 \end{cases} \quad (39)$$

whereas the discrete evolution (37) is replaced by

$$\begin{cases} x_i(t^+) = x_i(t) & \forall i \in I \\ u_i^j(t^+) = \begin{cases} \text{sign}_{\varepsilon(t)}(x_j(t) - x_i(t)) & \text{if } (i, j) \in \mathcal{J}(\theta, t) \\ u_i^j(t) & \text{otherwise} \end{cases} \\ \theta_i^j(t^+) = \begin{cases} \frac{1}{\gamma(t)} f_i^j(x(t)) & \text{if } (i, j) \in \mathcal{J}(\theta, t) \\ \theta_i^j(t) & \text{otherwise} \end{cases} \end{cases} \quad (40)$$

where $f_i^j(x)$ is defined in (38) and the functions $\varepsilon(t), \gamma(t)$ are as in Section IV. The protocol just introduced leads to the following result:

Theorem 4 (Asymptotical Consensus): Let $x(\cdot)$ be the solution to (39)–(40) under condition (24). Then for every initial condition $\bar{x} \in \mathbb{R}^n$ there exists $\beta \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = \beta$ for all $i \in I$, provided that $\int_0^{+\infty} \gamma(s)ds$ is divergent.

Proof: As in the proof of Theorem 2, one shows the equality

$$\dot{V}(t) = -\gamma(t) \sum_{\{i,j\} \in E: |x_j(t_k^{ij}) - x_i(t_k^{ij})| \geq \varepsilon(t_k^{ij})} \frac{|x_j(t_k^{ij}) - x_i(t_k^{ij})|}{2}.$$

From the latter and the properties $\varepsilon(t) \rightarrow 0$ and $\int_0^{+\infty} \gamma(s)ds = +\infty$, it follows that for each $\delta > 0$, for each $T_\delta > 0$, there exists $t \geq T_\delta$ such that $|x_j(t_{k_{ij}^*}(t)) - x_i(t_{k_{ij}^*}(t))| < \delta$ for all $(i, j) \in E$, where $k_{ij}^*(t) := \max\{k \geq 0 : t_k^{ij} \leq t\}$.

Consider now the function $W(x) = \max_{i \in I} x_i - \min_{i \in I} x_i$. The function $W(x(t))$ is non-increasing along the solutions to (39), (40). Indeed, Protocol C guarantees that, for all $\{i, j\} \in E$, the sign of $x_j(t_{k_{ij}^*}(t)) - x_i(t_{k_{ij}^*}(t))$ and the sign of $x_j(t) - x_i(t)$ are the same for all $t \in [t_{k_{ij}^*}(t), t_{k_{ij}^*}(t)+1]$. Furthermore we notice that $W(x) \leq \text{diam}(G) \cdot \max_{\{i,j\} \in E} |x_i - x_j|$. Bearing in mind the arguments above, we now prove that $\lim_{t \rightarrow +\infty} W(x(t)) = 0$. Indeed, for each $\varepsilon' > 0$, fix $\delta \leq \varepsilon'/\text{diam}G$ and choose t sufficiently large that $|x_j(t_{k_{ij}^*}(t)) - x_i(t_{k_{ij}^*}(t))| < \delta$ for all $\{i, j\} \in E$ —the existence of such t has been discussed in the first part of the proof. Then $W(x(t_{k_{ij}^*}(t))) \leq \text{diam}(G) \max_{\{i,j\} \in E} |x_j(t_{k_{ij}^*}(t)) - x_i(t_{k_{ij}^*}(t))| < \varepsilon'$. Since $W(x(t))$ is non-increasing then $W(x(t)) \leq W(x(t_{k_{ij}^*}(t))) < \varepsilon'$ for all $t \geq t_{k_{ij}^*}(t)$. Hence we have shown that for any $\varepsilon' > 0$, there exists a time $T_{\varepsilon'} := t_{k_{ij}^*}(t)$ such that $W(x(t)) < \varepsilon'$ for all $t \geq T_{\varepsilon'}$, which proves $\lim_{t \rightarrow +\infty} W(x(t)) = 0$. By definition of W the thesis follows. ■

The two protocols presented in this section can also be studied in terms of robustness: quantized communication, delays, and clock errors can be managed, as we have done in Sections III and IV-A, as long as synchrony and symmetry are preserved at the edge level: indeed we recall that these assumptions are crucial to the protocol. A detailed robustness analysis, however, is left to future research.

VI. CONCLUSIONS

In this paper we have addressed the problem of achieving consensus when agents collect information from the neighbors only at times which are designed iteratively and independently by each agent on the basis of its current local measurements. Following existing literature, this process can be termed self-triggered information collection. Compared with existing results, our approach presents a number of remarkable features. Our self triggered control policy, based on the use of relative measurements and ternary controls, achieves practical consensus with a guaranteed minimal inter-sampling time which can be freely tuned by the designer. Remarkably, no global information on the graph topology is required for either designing or running the algorithm, and the nodes need not even to agree on the sensitivity parameter. The approach lends

itself to an expressive characterization of the tradeoff between controller accuracy (sensitivity) and communications costs. We have also shown that our algorithm is inherently robust to uncertainties commonly found in networked systems, such as delays, quantized communication, and clock errors: the margin of robustness is adjustable via appropriate tuning of certain design parameters. Additionally, we have presented a modification of our basic self-triggered control scheme, which achieves asymptotical consensus: we have characterized its robustness margin against clock errors, and proposed a countermeasure for those cases in which this margin is exceeded. Finally, we have identified a self-triggered gossiping communication protocol, in which agents communicate in a pairwise fashion at times which are designed iteratively. From the methodological point of view, most of our results descend from Lyapunov-like analysis of the class of hybrid systems used to model our distributed self-triggered control schemes.

We believe that this paper can provide inspiration for future research in self-triggered algorithms for distributed and networked control. Besides some collateral open questions, which have been mentioned in the text, we envisage three main avenues for new research stemming from this work. First, we recall that the ternary nature of controllers has a key role in our approach, as it provides implicit information on the dynamics: this information is exploited in the computation of the sampling times. Thus, a natural extension would be to consider constrained controllers taking values in larger sets, for instance quantized or saturated controllers. Second, further investigation and extensions of the self-triggered gossip algorithm introduced in this paper may contribute to the rich literature on gossip algorithms, which has focused so far on time-triggered protocols. Third, it is worth to explore how similar approaches can be applied to coordination of higher dimensional systems and to more complex coordination tasks, such as formation control and collaborative tracking.

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