Sensor Networks Localization: Extending Trilateration via Shadow Edges

Gabriele Oliva*‡, Stefano Panzieri†, Federica Pascucci† and Roberto Setola*

Abstract

Distance-based network localization is known to have solution, in general, if the network is globally rigid. In this paper we relax this condition with reference to unit disk graphs. To this end "shadow edges" are introduced to model the fact that selected nodes are not able to sense each other. We provide a localization algorithm based on such edges and a necessary and sufficient localizability condition. We also inspect the relation of the the proposed approach with trilateration, showing from both a theoretical and empirical point of view that shadow edge localization may solve the problem also when trilateration fails.

Index Terms

Wireless Sensor Networks Localization; Rigidity; Trilateration; Unit Disk Graphs; Delaunay Graphs; Gabriel Graphs.

I. Introduction

In the literature several sensor network localization approaches based on relative distance measurements have been proposed [1]–[3]. Among others, trilateration algorithms [1], [4] are widely adopted, but they fail in some cases, especially when low-range communication devices are used, or when the environment contains obstacles.

Typical distance-based localization algorithms require the network to be globally rigid [5], [6]. In the case of *unit disk graphs* [8], i.e., such that each pair of nodes within a given distance threshold ρ are connected by means of an edge, while some additional piece of information can be used [1], [7]. Specifically, in [1] unit disk graphs are used to characterize the probability of having a trilateration graph depending on the number of nodes and on the value of the threshold ρ, while in [7], assuming the graph is a unit disk graph, an NP-hard algorithm is devised that inspects the possible configurations and gets rid of those in conflict with the unit disk graph structure.

* University Campus Bio-Medico of Rome, Italy. † University "Roma TRE", Roma, Italy. ‡ corresponding author, email: g.oliva@unicampus.it

Even if the idea that unit disk graphs may help to get rid of some ambiguities is not new in the literature, to the best of our knowledge no theoretical result is given on the localizability of a sensor network over a unit disk graph, and the available algorithms are intrinsically centralized and complex.

In this paper, based on the preliminary results of [9]–[11], we provide a localization framework for unit disk graphs that models the a priori information as additional virtual links, namely *shadow edge*, to localize networks that are not globally rigid.

As a result we provide an efficient and ready to be distributed algorithm, namely *Shadow Edge Localization Algorithm* (SELA), and we give conditions that guarantee the localizability of the sensor network also when the graph is not globally rigid. We also prove that set of nodes localized by SELA always contains the one localized by trilateration when the anchors (i.e., a small set of already localized nodes) coincide, while the maximum number of nodes that can be localized by SELA (i.e., choosing the best seed) is always greater or equal than the maximum number of nodes that can be localized via trilateration. Hence SELA algorithm is able to succeed when algorithms based on classical trilateration fail.

The remainder of the paper is as follows: Section II provides some preliminary definitions; Section III reviews the network localization problem; the proposed approach is discussed in Section IV, while some simulation results are given in Section V; eventually, some conclusions are drawn in Section VI, while the Appendix contains the proofs.

II. Preliminaries

Given a set H, let $|H|$ be the number of elements in the set H.

Let a *graph* $G = \{V, E\}$, where the set V denotes the nodes $v_1, \ldots, v_n$ and E is the set of edges (v_i, v_j). A graph G is *connected* if there is a path composed of edges in E that connects each pair of nodes in G, while it is *complete* if for each pair of nodes v_i, v_j the edge $(v_i, v_j) \in E$. In the following we will assume that G is *undirected*, i.e., $(v_i, v_j) \in E$ whenever $(v_j, v_i) \in E$. A set of $V_m \subseteq V$ of vertices of a graph G are *fully connected* if the subgraph of G induced by V_m is a full graph. A graph G is *edge 2-connected* if for any two vertices v_i and v_j there are at least 2 paths that connect them and do not have edges in common.

A graph G is *chordal* [12] if each of its cycles of four or more vertices has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Let us define a graph G as *perfectly chordal* if it is chordal and each node belongs to at least a cycle of three nodes. It can be noted that a graph G is perfectly chordal if it is chordal and edge 2-connected.

Let us suppose that the nodes of a graph G are embedded in $\mathbb{R}^2$ and let p_i the position in $\mathbb{R}^2$ of the i-th node of G. A *framework* (G,P) is a graph $G = \{V,E\}$ together with a coordinate assignment $P : V \to \mathbb{R}^2$ for the vertices of the graph.

Let us define a *partial framework* (G,P_l) as a graph $G = \{V,E\}$ together with a coordinate assignment $P_l : V_l \to \mathbb{R}^2$ for a subset $V_l \subseteq V$ of the nodes of the graph.

Two frameworks (G,P) and (G,P^*) are *equivalent* if $||p_i - p_j|| = ||p_i^* - p_j^*||$ holds for all pairs v_i, v_j such that $(v_i, v_j) \in E$, i.e., for all the available edges. Two frameworks (G,P) and (G,P^*) are *congruent* if $||p_i - p_j|| = ||p_i^* - p_j^*||$ holds for all pairs $v_i, v_j \in V$, i.e., for any couple of vertices.

A framework (G,P) is *globally rigid* if every framework which is equivalent to (G,P) is also congruent to (G,P). This definition implies that the position of nodes can not be continuously deformed nor flipped without violating the distance constraints. If the position of at least 3 nodes is *generic* in $\mathbb{R}^2$, i.e., they do not lie on the same line, then the graph topology alone determines the global rigidity of the framework, which in this case is called *generic global rigidity* [5], [13]. Let a *trilateration* graph be a graph $G_T = \{V_T, E_T\}$ with an ordering of the vertices $v_1, \ldots, v_n$, such that the nodes $v_1, \ldots, v_3$ are fully connected and each node v_i for $i = 4, \ldots, n$ is connected to at least 3 of the vertices $v_1, \ldots, v_{i-1}$. A set of 3 fully connected nodes is often referred to as a *seed*.

Note that in $\mathbb{R}^2$, if a graph $G = \{V,E\}$ contains a trilateration graph such that $V_T = V$ and $E_T \subseteq E$, then it is generically globally rigid [1], [14].

A framework (G,P) is a *Delaunay framework* if G is composed of triangles (i.e., G is perfectly chordal) and for each 3 vertices v_i, v_j and v_k of G the circumference passing through the points

p_i, p_j and p_k does not contain any other point in (G,P). A well known property of a Delaunay framework is that the union of the triangles that compose it coincide with the convex hull of the nodes [15]. Let a framework (G,P); the *Gabriel circle* of an edge (v_i, v_j) of G is the circle with diameter equal to d_{ij} that intersects the points p_i and p_j. A framework (G,P) is a *Gabriel framework* [16] if for each edge (v_i, v_j) of G the corresponding Gabriel circle does not contain any other point of P. In [17] it is proved that an edge (v_i, v_j) belongs to a Gabriel graph if and only if the angle $\angle p_i p_k p_j$ is acute for every $v_k \in V$, $v_k \neq v_i, v_j$.

III. Network Localization Problem

Let a sensor network $N = \{\Sigma \cup \Sigma_a, D\}$ composed of n sensors $\sigma_i \in \Sigma \cup \Sigma_a$, each with a fixed position in $\mathbb{R}^2$. Suppose that only the position of a small set of *anchors* $\sigma_a \in \Sigma_a$ is known a priori while the location of the sensors in Σ is not known. Suppose further that the distances $d_{ij} \in D$ of some pairs of sensors (σ_i, σ_j) are known. We can represent the sensor network by means of a framework (G,P) where the points $p_i \in P$ are the coordinates of the sensors $\sigma_i \in \Sigma \cup \Sigma_a$ in $\mathbb{R}^2$ and the graph $G = \{V, E\}$ is obtained associating each sensor $\sigma_i \in \Sigma \cup \Sigma_a$ to a node $v_i \in V$ and each known distance d_{ij} to an edge $(v_i, v_j) \in E$.

The *(relative) network localization problem* consists in finding a coordinate assignment P for the non-anchor nodes which assigns coordinates $p_i = [x_i, y_i]^T \in \mathbb{R}^2$ (with respect to a relative framework of reference) to each sensor $\sigma_i \in \Sigma$ in a way such that $||p_i - p_j|| = d_{ij}$ holds for all sensor pairs (σ_i, σ_j) for which d_{ij} is given (we assume $d_{ij} = d_{ji}$).

If a partial framework (G, P_l) that does not violate the distance constraints is found for a subset of the nodes $V_l \subseteq V$ then the network is (relatively) *partially localized*.

A. Trilateration

In [1], [14] it is shown that a sensor network is localizable provided that the graph G is generically globally rigid.

Although localizing a globally rigid framework (G,P) is in general hard, a computationally efficient algorithm has been provided using *trilateration*. Trilateration is the operation whereby a node v_i, knowing its relative distance from three generically positioned and localized nodes v_j, v_h and v_k, is able to derive its own position by intersecting the circumferences centered in v_j, v_h and v_k whose radius is d_{ij}, d_{ih} and d_{ik}, respectively.

An easy way to localize a sensor network whose graph contains a trilateration graph is thus to iteratively localize the nodes via trilateration, until no more node can be localized. Let us refer to this approach as the *Trilateration Localization Algorithm* (TLA) algorithm [1], [18].

IV. Shadow Edge Localization

Let us assume that each sensor σ_i is characterized by a *communication radius* $\rho > 0$, and is able to detect the presence of any sensor σ_j within such a communication radius (i.e., those sensors σ_j such that $d_{ij} \leq \rho$), obtaining also information about the distance $d_{ij} \leq \rho$ between them. The resulting structure is a *unit disk graph*, since every piece of distance information d_{ij} that can be obtained given the communication radius ρ is taken into account; such an assumption is typically verified in practical situations, in particular when circular antennas are adopted [8].

Under the above assumptions, the sensor network may be localizable even when the underlying graph is not globally rigid, exploiting the idea illustrated in Figure 1. In the Figure, the sensor σ_i can not be localized using trilateration. Although sensor σ_j is out of reach for sensor σ_i, however, the fact that σ_j is not sensed by σ_i may contribute to identify the correct position sensor σ_i.

In the following, we will refer the edges like the one reported in a blue dotted line in Figure 1 as *shadow edges*.

Definition 1 (Shadow Edge): Let a unit disk graph sensor network defined for some $\rho > 0$, represented by a partially localized framework (G, P_l) where $V_l \subseteq V$ contains the localized nodes (i.e., anchors and nodes localized via trilateration). Suppose that a node $v_i \notin V_l$ is connected to two nodes $v_h, v_k \in V_l$. Given the two distances d_{ih} and d_{ik} there are two admissible positions p_{i_1} and p_{i_2} for the location of node v_i (the intersections of the circumferences centered in v_h and v_k, of radius d_{ih} and d_{ik}, respectively). Let another node $v_j \in V_l$ such that $\{(v_j, v_h), (v_j, v_k)\} \in E$.

A *shadow edge* is an edge (v_i, v_j) such that:

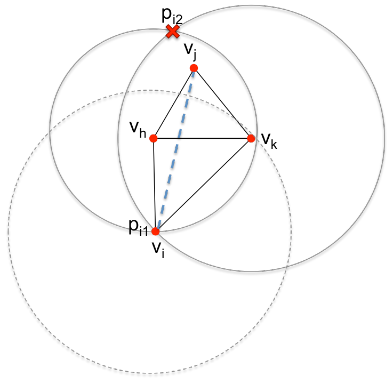

Figure 1. Example of localizable unit disk graph network whose associated graph G is not globally rigid. Sensors σ_j, σ_h and σ_k, represented by the nodes v_j, v_h and v_k are localized. Sensor σ_i perceives sensor σ_h and σ_k but it is unable to perceive sensor σ_j (the node v_j is outside the dashed circumference). On the base of the distances d_{ih} and d_{ik}, there are 2 distinct admissible positions p_{i_1} and p_{i_2} for the node v_i (i.e., the intersection of the solid circumferences centered in v_h and v_k of radius d_{ih} and d_{ik}, respectively). Sensor σ_i may exclude p_{i_2} (represented by the red cross), since σ_i does not perceive σ_j. The blue dotted line represents the *shadow edge* that may be created between v_i and v_j.

1) $(v_i, v_j) \notin E$;
2) either $||p_{i_1} - p_j|| < \rho$ or $||p_{i_2} - p_j|| < \rho$.

Hence a shadow edge is a "virtual" edge that does not exist in the original graph. Notice that a shadow edge does not always exist, since the conditions of Definition 1 may not be verified for particular frameworks.

Let us denote by E_s a set of shadow edges. Note that, for any $(v_i, v_j) \in E_s$, v_i and v_j have to be considered as virtual two-hop neighbors, because their distance is $d_{ij} \in (\rho, 2\rho]$.

Definition 2 (Shadow localizable framework): Let a framework (G, P) with n nodes. A framework (G', P) contained in (G, P) is a *shadow localizable framework* if:

- G' is connected and perfectly chordal;
- for each perfectly chordal subframework (G_{sub}, P_{sub}) of (G, P) with 4 nodes which is not complete, the corresponding subframework (G'_{sub}, P_{sub}) of (G', P) is Delaunay and Gabriel.

A. Main Result

Let us provide the following result, whose proof is given in the appendix.

Theorem 1: Let a unit disk graph sensor network with $n \geq 3$ sensors, represented by the graph $G = \{V, E\}$ and let E_s be the maximum set of shadow edges that can be obtained starting from a seed V_l. Let P be the position of the nodes in the sensor network. The extended graph $G_e = \{V, E \cup E_s\}$ contains a trilateration graph $G_T = \{V, E_T\}$, $E_T \subseteq E \cup E_s$ if and only if the framework (G, P) contains a shadow localizable framework (G', P) with n nodes.

Let us provide the following corollary.

Corollary 1: A unit disk graph sensor network with $n \geq 3$ sensors, represented by a graph G can be localized over the extended graph G_e if and only if the framework (G, P) contains a shadow localizable framework (G', P) with n nodes.

B. Shadow Edge Localization Algorithm

The above results yield an algorithm to localize a sensor network based on shadow edges, namely *Shadow Edge Localization Algorithm* (SELA). Starting with a seed of 3 generically positioned nodes (we can say, without loss of generality that $V_l = \{v_1, v_2, v_3\}$), the nodes are iteratively tested for localizability. Specifically, if each tested node v_i is connected to 3 already localized nodes, or to 2 localized nodes and there is a shadow edge, then it is localized. The algorithm terminates when all nodes have been localized or when the remaining nodes can no longer be localized.

The following result is a corollary of Definition 2.

Corollary 2: Let a unit disk graph sensor network with $n \geq 3$ sensors, represented by the graph $G = \{V, E\}$ and suppose that SELA and TLA algorithms are executed starting from the same seed $V_l \subseteq V$. Let V^S, V^T be the set of nodes localized via SELA and TLA, respectively; it holds $V^T \subseteq V^S$.

We can provide the following result, whose proof is given in the appendix.

Proposition 1: Let a unit disk graph sensor network with $n \geq 3$ sensors represented by the graph $G = \{V, E\}$ and let $V_l \subseteq V$ be a seed with $3 \leq m < n$ nodes. Let V_l^S, V_l^T be the set of nodes localized via SELA and TLA algorithms, respectively, starting from the seed V_l. It holds: $\max_{V_l \subseteq V} |V_l^S| \geq \max_{V_l \subseteq V} |V_l^T|$.

Remark 1: The SELA algorithm can be easily implemented in a distributed fashion: each node is able to calculate its own position knowing the position of 3 one-hop localized neighbors and its distance from them (just like trilateration) of knowing the position of a two-hop localized neighbor if one of the 3 one-hop localized neighbors can not be found.

V. Simulation Results

We compare the performances of SELA and TLA in terms of percentage ψ of localized nodes, i.e. $\psi = \frac{l}{n}$, where l is the number of localized nodes and n is the total number of nodes in the network.

To this end we consider 100 randomly generated networks (the nodes have uniformly randomly distributed positions in the unit square) for several choices of the number of nodes $n = 30, 50, 70$ and 100 and considering a communication radius ρ ranging from 0.1 to 0.4. We choose the same 3 anchors in fixed positions in the lower left corner of the unit square for each of the trials. Figure 2 shows some examples for different choices of the network size n and of the communication radius ρ. Looking at this figure, it is evident that in situations where TLA is unable to localize the network, SELA is able to almost completely localize all the nodes. Specifically, there is up to about a $+19\%$ of localized nodes (when $\rho = 0.35$) for $n = 30$, and up to about a $+23\%$ both for $n = 50$ and 100, in correspondence of $\rho = 0.3$ and $\rho = 0.25$, respectively. Notice that, according to Corollary 2, SELA algorithm localizes the maximal localizable subset, which contains the maximal globally rigid subset localized also by TLA algorithm. Figure 3 shows the percentage of localized nodes plotted against ρ; for each choice of n and ρ, the average of 100 runs is reported. In all the runs, SELA algorithm localizes more nodes than TLA. The proposed algorithm appears particularly effective for moderately dense networks, i.e., intermediate values of ρ. Indeed, for very dense networks (right extrema of Figure 3), each node is generally connected with 3 or more nodes, hence TLA can be successfully applied. On the other hand, for very sparse networks (left extrema of Figure 3), the connectivity of the graph is so low that no localization algorithm can succeed.

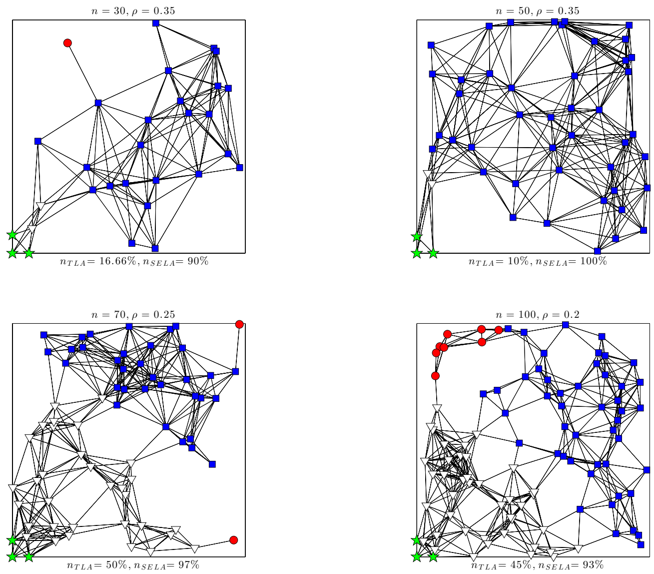

Figure 2. Comparison between TLA and SELA on some random sample graphs: green stars are the anchor nodes, white triangles are nodes localized by both TLA and SELA, while blue squares are nodes localized exclusively by SELA; the red circles are not localized nodes.

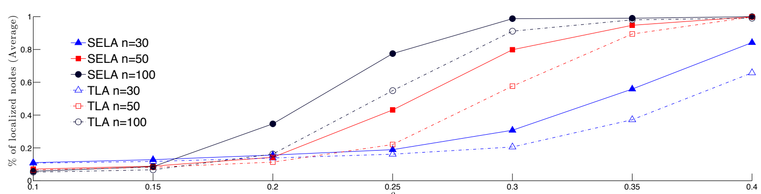

Figure 3. Comparison between TLA and SELA on the percentage of localized nodes plotted against the communication radius ρ: the figure represents the average percentage of localized nodes by SELA and TLA over 100 runs. Dotted lines and empty markers represent the results for TLA while the results for SELA are plotted with solid lines and filled markers; triangles, squares and circles represent networks with $n = 30, 50$ and 100 nodes, respectively.

VI. Conclusions

In this paper we extend trilateration over unit disk graphs by exploiting the information about not being connected, modeled as a link, namely shadow edge. Moreover we provide conditions for the localizability of the network and we prove that the proposed approach has better results than trilateration. The proposed algorithm, in fact is able to localize the sensor network also when trilateration fails. Future work will be devoted to extend the methodology to a 3D setting, so as to provide a partial localization of the sensor network in the case where only a single link

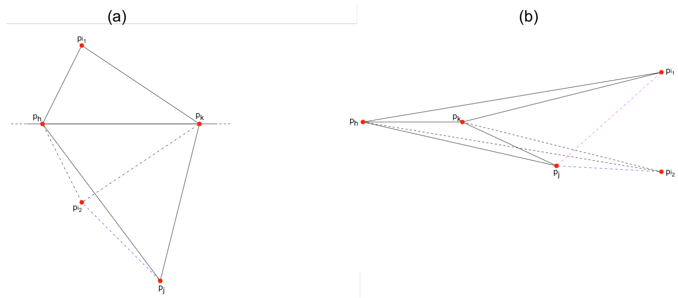

Figure 4. Situations considered in Lemma 1: (a) a framework which is both Gabriel and Delaunay is such that point p_{i_2} lies in the lower halfplane with respect to the line that intersects p_h and p_k; moreover, the angles $\angle p_{i_2} p_h p_j$ and $\angle p_{i_2} p_k p_j$ are acute, hence $d_{i_2 j} \leq \rho$; (b) only possible case where (G,P) is not both Delaunay and Gabriel and $d_{i_2 j} \leq \rho$; in this case $\angle p_{i_1} p_h p_j$ is acute, hence $d_{i_1 j}$ is smaller than ρ yielding to a contradiction.

and several shadow edges are available and address the noisy case.

Appendix: Proofs

In order to prove Theorem 1 we need the following lemma, for which a visual explanation is given in Figure 4.

Lemma 1: Let a sensor unit disk graph network for some $\rho > 0$ be represented by framework (G,P) with 4 nodes and $G = \{V,E\}$ such that $V = \{v_i, v_j, v_h, v_k\}$ and

$$E = \{(v_i, v_h), (v_i, v_k), (v_h, v_k), (v_j, v_h), (v_j, v_k)\}.$$

Suppose the nodes v_j, v_h, v_k are localized and let p_{i_1} and p_{i_2} be the two options for the position of node v_i given the position of nodes v_h, v_k and the distances d_{ih}, d_{ik}. Suppose further that p_{i_1} is the true position of node v_1. It holds that $d_{i_2 j} \leq \rho$ if and only if the framework (G,P) is Delaunay and Gabriel.

Proof 1: $\Leftarrow$: Suppose (G,P) is a Delaunay and Gabriel framework. Being (G,P) a unit disk graph network, since $(v_i, v_j) \notin E$ it follows that $d_{i_1 j} > \rho$.

Since (G,P) is a Delaunay framework, the union of the triangles must coincide with the convex hull of the vertices [15]. In this case p_{i_2}, obtained by mirroring the triangle $\triangle_{h i_1 k}$ with respect to the line overlapping with segment $\overline{p_h p_k}$, lies in the same halfplane where p_j lies, while p_{i_1} lies in the opposite halfplane; as a consequence $d_{i_2 j} < d_{i_1 j}$.

The framework is Gabriel, hence both angles $\angle p_{i_1} p_h p_j$ and $\angle p_{i_1} p_k p_j$ must be acute [17]; as a consequence, also $\angle p_{i_2} p_h p_j$ and $\angle p_{i_2} p_k p_j$ are acute. Since d_{ih}, d_{ik}, d_{hj} and d_{kj} are all smaller than ρ and $\angle p_{i_2} p_h p_j$, $\angle p_{i_2} p_k p_j$ are acute, we can conclude that $d_{i_2 j} \leq \rho$.

$\Rightarrow$: Suppose $d_{i_2 j} \leq \rho$ but the framework (G,P) is not both Delaunay and Gabriel.

Note that, with respect to the above halfplane decomposition of the space, since $d_{i_1 j} > \rho$ and $d_{i_2 j} \leq \rho$, then $p_{i_2 j}$ must lie in the same halfplane of p_j, while p_{i_1} must lie in the other halfplane. In this case, since $\angle p_{i_1} p_h p_j$ is acute and both $d_{i_1 h}$ and d_{jh} are smaller than ρ, it follows that $d_{i_1, j} \leq \rho$, a contradiction.

Proof 2 (Proof of Theorem 1): $\Rightarrow$: If G_e contains a trilateration graph with n nodes, then by Lemma 1 each perfectly chordal subgraph of (G,P) with 4 nodes is either complete or such that it is possible to find a shadow edge, hence (G,P) contains a shadow localizable framework (G',P) with n nodes.

$\Leftarrow$: The framework (G,P) contains a shadow localizable framework (G',P) with n nodes, which is perfectly chordal. This implies that (G',P) contains at least 3 fully connected nodes v_1, v_2 and v_3. By Lemma 1, for each non complete perfectly chordal subgraph of (G',P) with 4 nodes it is possible to obtain a shadow edge. It is therefore possible to label the remaining nodes so that v_i is connected to at least 3 nodes v_j with $j < i$, $i = 3, \ldots, n$, hence G_e contains a trilateration graph and is generically globally rigid.

Proof 3 (Proof of Proposition 1): Let V_s be the seed that maximizes the number of nodes localized by SELA, and let V_s^S be the set of such localized nodes. Similarly, let V_t be the seed that maximizes the number of nodes localized by TLA, and let V_t^T be the set of such localized nodes.

From Definition 2 and Corollary 2, if V_t is used as seed for the SELA algorithm, then it holds $V_t^T \subseteq V_t^S$. Hence $|V_s^S|$ is a fortiori greater or equal than $|V_t^T|$, proving the statement.

References

[1] T. Eren, O. Goldenberg, W. Whiteley, Y. R. Yang, A. S. Morse, B. D. Anderson, and P. N. Belhumeur, “Rigidity, computation, and randomization in network localization,” in *INFOCOM 2004. Twenty-third AnnualJoint Conference of the IEEE Computer and Communications Societies*, vol. 4. IEEE, 2004, pp. 2673–2684.

[2] D. Moore, J. Leonard, D. Rus, and S. Teller, "Robust distributed network localization with noisy range measurements," in *Proceedings of the 2nd international conference on Embedded networked sensor systems*. ACM, 2004, pp. 50–61.

[3] Y. Liu, Z. Yang, X. Wang, and L. Jian, "Location, localization, and localizability," *Journal of Computer Science and Technology*, vol. 25, no. 2, pp. 274–297, 2010.

[4] B. Hendrickson, "Conditions for unique graph realizations," *SIAM Journal on Computing*, vol. 21, no. 1, pp. 65–84, 1992.

[5] G. Laman, "On graphs and rigidity of plane skeletal structures," *Journal of Engineering Mathematics*, vol. 4, no. 331-340, 1970.

[6] B. Jackson and T. Jordán, "Connected rigidity matroids and unique realizations of graphs," *Journal of Combinatorial Theory, Series B*, vol. 94, no. 1, pp. 1–29, 2005.

[7] P. Kaewprapha, J. Li, and N. Puttarak, "Network localization on unit disk graphs," in *Global Telecommunications Conference (GLOBECOM 2011), 2011 IEEE*. IEEE, 2011, pp. 1–5.

[8] B. N. Clark, C. J. Colbourn, and D. S. Johnson, "Unit disk graphs," *Annals of Discrete Mathematics*, vol. 48, pp. 165–177, 1991.

[9] G. Oliva, S. Panzieri, F. Pascucci, and R. Setola, "Network localization by shadow edges," in *Control Conference (ECC), 2013 European*. IEEE, 2013, pp. 2263–2268.

[10] ——, "Exploiting routing information in wireless sensor networks localization," in *Network Science Workshop (NSW), 2013 IEEE 2nd*. IEEE, 2013, pp. 66–73.

[11] F. Pascucci, S. Panzieri, G. Oliva, and R. Setola, "Simultaneous localization and routing in sensor networks using shadow edges," in *Intelligent Autonomous Vehicles*, vol. 8, no. 1, 2013, pp. 199–204.

[12] G. A. Dirac, "On rigid circuit graphs," in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 25, no. 1. Springer, 1961, pp. 71–76.

[13] R. Connelly, "Generic global rigidity," *Discrete & Computational Geometry*, vol. 33, no. 4, pp. 549–563, 2005.

[14] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. Anderson, and P. N. Belhumeur, "A theory of network localization," *Mobile Computing, IEEE Transactions on*, vol. 5, no. 12, pp. 1663–1678, 2006.

[15] F. Aurenhammer, "Voronoi diagrams–a survey of a fundamental geometric data structure," *ACM Computing Surveys (CSUR)*, vol. 23, no. 3, pp. 345–405, 1991.

[16] K. R. Gabriel and R. R. Sokal, "A new statistical approach to geographic variation analysis," *Systematic Biology*, vol. 18, no. 3, pp. 259–278, 1969.

[17] D. W. Matula and R. R. Sokal, "Properties of gabriel graphs relevant to geographic variation research and the clustering of points in the plane," *Geographical analysis*, vol. 12, no. 3, pp. 205–222, 1980.

[18] J. B. Saxe, *Embeddability of weighted graphs in k-space is strongly NP-hard*. Carnegie-Mellon University, Department of Computer Science, 1980.