Stability Analysis of Integral Delay Systems with Multiple Delays

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Abstract

This note is concerned with stability analysis of integral delay systems with multiple delays. To study this problem, the well-known Jensen inequality is generalized to the case of multiple terms by introducing an individual slack weighting matrix for each term, which can be optimized to reduce the conservatism. With the help of the multiple Jensen inequalities and by developing a novel linearizing technique, two novel Lyapunov functional based approaches are established to obtain sufficient stability conditions expressed by linear matrix inequalities (LMIs). It is shown that these new conditions are always less conservative than the existing ones. Moreover, by the positive operator theory, a single LMI based condition and a spectral radius based condition are obtained based on an existing sufficient stability condition expressed by coupled LMIs. A numerical example illustrates the effectiveness of the proposed approaches.

Keywords: Stability of integral delay systems, Multiple Jensen inequality, Linearization technique, Positive operator theory, Spectral radius

1 Introduction

An integral delay system (IDS) in the form of $x(t) = \int_{-\tau}^{0} F(s) x(t+s) ds$, where F(s) is a matrix function with bounded variation, has many important applications in the study of time-delay systems (see, for example, [3], [8, 20] and [21]). In Hale's book [8] this class of IDSs were named as D operators, which were also treated as generalized difference equations there, and their stability is necessary for the stability of the associated neutral time-delay systems. This class of IDSs also come from the model reduction approach for stability analysis of time-delay systems, which are frequently named as the additional dynamics (see [6] and [10]) and their stability is necessary for the stability of the transformed time-delay systems. This class of IDSs are very closely related with the predictor feedback control of linear systems with input delays, for example,

- In [7], the author proved that the numerical implementation of the predictor feedback for linear systems with input delays is safe only if an IDS is exponentially stable.
- In [14] it is shown that the stability of this class of IDSs is necessary for the robust stability of linear systems with input delay by the well-known predictor feedback.
- In [19] we have shown that an input delayed linear system by the so-called pseudo-predictor feedback is exponentially stable if and only if an IDS is exponentially stable.

Stability analysis of IDSs can be traced back at least to Cruz and Hale [2], Henry [9] and Melvin [17], in the study of the stability of neutral time-delay systems [8]. When the right hand side of the IDS only contains terms at some time points, a general theory was build in [1]. This class of IDSs have received renewed interest in recent years. A general stability theorem was build in [15] and was later applied on different forms of IDSs (see [13], [16], [18], and the references therein). In the paper [4] stability conditions are derived for IDS with matrix discrete-continuous measures.

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In this note we also study stability analysis of this class of IDSs in which F(s) is a piecewise constant matrix function, namely, the system can be expressed as $x(t) = \sum_{i=1}^{N} A_i \int_{-\tau_i}^{0} x(t+s) ds$ where A_i and τ_i are constants (τ_i can be unknown, time-varying, yet bounded). This class of IDSs with multiple time delays have been investigated in [16] with the help of the well-known discrete-time and continuous-time Jensen inequalities. In this note, by recognizing that the jointed usage of the conventional discrete-time and continuous-time Jensen inequalities requires that all the integration terms $\int_{-\tau_i}^0 x^{\mathrm{T}} (t+s) A_i^{\mathrm{T}} Q A_i x (t+s) \mathrm{d}s$ share the same weighting matrix Q, we first establish a so-called multiple Jensen inequality, by which as well as some novel Lyapunov functionals and a new linearization technique, every individual integration term possesses a different weighting matrix, which can introduce more weighting matrices that will be optimized to reduce the conservatism of the corresponding sufficient stability conditions. Indeed, it is shown in both theory and by numerical examples that the stability conditions obtained by the multiple Jensen inequalities are always less conservative than that obtained by the jointed usage of the conventional discretetime and continuous-time Jensen inequalities. Another contribution of this note is that, with the help of the positive operator theory, we are able to establish an equivalent linear matrix inequality (LMI) based stability condition involving only one constraint and one decision matrix and an equivalent spectral radius based stability condition of some existing stability conditions that are expressed by a set of coupled LMIs. These two equivalent stability conditions are appealing in both theory and in computation.

The remaining of this note is organized as follows. The problem formulation and some preliminaries are given in Section 2. In Section 3, two kinds of LMIs based sufficient conditions are established with the help of the multiple Jensen inequalities, some novel Lyapunov functionals and a new linearization technique. A spectral radius based stability condition and a single LMI based stability condition are then established in Section 4 in which a comparison of the proposed approaches and the existing one will also be carried out. A numerical example is worked out in Section 5 to illustrate the effectiveness of the proposed approaches and finally Section 6 concludes the note.

2 Problem Formulation and Preliminaries

Consider the following integral delay system (IDS) with multiple delays

$$x(t) = \sum_{i=1}^{N} A_i \int_{-\tau_i}^{0} x(t+s) \,\mathrm{d}s,$$
(1)

where $A_i \in \mathbf{R}^{n \times n}$, $i \in \mathbf{I}[1, N] \triangleq \{1, 2, ..., N\}$, are given matrices and $\tau_i, i \in \mathbf{I}[1, N]$, are given scalars and are such that

$$0 < \tau_i \le \tau \triangleq \max_{i \in \mathbf{I}[1,N]} \{\tau_i\}, \ i \in \mathbf{I}[1,N].$$
⁽²⁾

Let $\varphi \in \mathscr{C}_{n,\tau}$ be an initial condition for (1) and $x(t) = x(t,\varphi), \forall t \geq 0$ be the corresponding solution of (1) satisfying $x(t) = \varphi(t), \forall t \in [-\tau, 0)$. Here $\mathscr{C}_{n,\tau}$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into \mathbf{R}^n with the topology of uniform convergence. We say that the IDS (1) is exponentially stable if there exist two positive constants α and β such that $||x(t)|| \leq \alpha \sup_{s \in [-\tau, 0]} ||\varphi(s)|| e^{-\beta t}, \forall t \geq 0$.

The IDS (1) arises when some transformations are made on differential-difference systems [16]. In this note we are concerned with the stability analysis of the IDS (1). By choosing some suitable Lyapunov functionals and developing a new linearization technique for handling nonlinear matrix inequalities, we will establish two classes of LMIs based sufficient conditions guaranteeing the exponential stability of the IDS (1). Moreover, with the help of the positive operator theory, we will also provide a spectral radius based sufficient stability condition. The relationships among these different sufficient conditions are also revealed. Our results improve those in [16]. Both theoretical analysis and numerical examples will demonstrate that the obtained results are always less conservative and more efficient than the existing ones especially those in [16].

The following general Lyapunov stability theorem for the IDS (1) will be used later in this note.

Lemma 1 [16] The IDS (1) is exponentially stable if there exists a differentiable functional $V : \mathscr{C}_{n,\tau} \to \mathbf{R}$

and three positive constants α_i , i = 1, 2, 3, such that

$$\alpha_1 \int_{-\tau}^0 \|x(t+\theta)\|^2 \,\mathrm{d}\theta \le V(x_t) \le \alpha_2 \int_{-\tau}^0 \|x(t+\theta)\|^2 \,\mathrm{d}\theta,\tag{3}$$

$$\dot{V}(x_t) \le -\alpha_3 \int_{-\tau}^0 \|x(t+\theta)\|^2 \,\mathrm{d}\theta.$$
(4)

At the end of this section, we give the following technical lemma which is helpful for the linearization of nonlinear matrix inequalities in the sequel.

Lemma 2 Let $S \in \mathbf{R}^{n \times n}$ and $Q \in \mathbf{R}^{n \times n}$ be two positive definite matrices. Then $Q < S^{-1}$ if and only if there exists a matrix $R \in \mathbf{R}^{n \times n}$ such that

$$R^{\mathrm{T}}QR + S - (R + R^{\mathrm{T}}) < 0.$$
 (5)

The same statements hold true if "<" in the above two inequalities are replaced by " \leq ".

Proof. It follows from (5) that R is nonsingular. Since $(R-S)^T S^{-1} (R-S) \ge 0$, namely,

$$-R^{\mathrm{T}}S^{-1}R \le S - (R + R^{\mathrm{T}}),$$
 (6)

we have from (5) that

$$Q < -R^{-T}(S - (R + R^{T}))R^{-1} \le R^{-T}R^{T}S^{-1}RR^{-1} = S^{-1}.$$
(7)

On the other hand, if $Q < S^{-1}$ is satisfied, then (5) is satisfied by choosing R = S.

3 The Multiple Jensen Inequality Based Stability Conditions

3.1 The Multiple Jensen Inequality

We first recall the following well-known Jensen inequality.

Lemma 3 [5] For any positive definite matrix Q > 0, a positive number $\tau > 0$, and a vector valued function $\omega : [-\tau, 0] \to \mathbf{R}^n$ such that the integrals in the following are well-defined, then

$$\left(\int_{-\tau}^{0} \omega(s) \,\mathrm{d}s\right)^{\mathrm{T}} Q\left(\int_{-\tau}^{0} \omega(s) \,\mathrm{d}s\right) \le \tau \int_{-\tau}^{0} \omega^{\mathrm{T}}(s) \,Q\omega(s) \,\mathrm{d}s.$$
(8)

Moreover, for a series of vectors $\xi_i \in \mathbf{R}^n, i \in \mathbf{I}[1, N]$, there holds

$$\left(\sum_{i=1}^{N} \xi_{i}\right)^{\mathrm{T}} Q\left(\sum_{i=1}^{N} \xi_{i}\right) \leq N \sum_{i=1}^{N} \xi_{i}^{\mathrm{T}} Q \xi_{i}.$$
(9)

Inequalities (8) and (9) are respectively known as the continuous-time Jensen inequality and the discrete-time Jensen inequality, which have been widely used in the literature for the stability analysis and stabilization of time-delay systems (see [5] and the references that have cited it). By using these two inequalities jointly we get the following corollary.

Corollary 1 Let $\tau_i \ge 0, i \in \mathbf{I}[1, N]$, be N given nonnegative scalars. Assume that $\omega_i : [-\tau_i, 0] \to \mathbf{R}^n, i \in \mathbf{I}[1, N]$, are such that the integrals in the following are well-defined, then

$$\left(\sum_{i=1}^{N} x_{i}\right)^{\mathrm{T}} Q\left(\sum_{i=1}^{N} x_{i}\right) \leq N \sum_{i=1}^{N} \tau_{i} \int_{-\tau_{i}}^{0} \omega_{i}^{\mathrm{T}}(s) Q\omega_{i}(s) \,\mathrm{d}s,$$
(10)

where $x_i = \int_{-\tau_i}^0 \omega_i(s) \, \mathrm{d}s, i \in \mathbf{I}[1, N]$.

We notice that all the N integrations $\int_{-\tau_i}^{0} \omega_i^{\mathrm{T}}(s) Q\omega_i(s) ds, i \in \mathbf{I}[1, N]$, on the right hand side of (10) share the *same* weighting matrix Q, which is clearly very restrictive. To reduce the possible conservatism, we introduce the following multiple Jensen inequality.

Lemma 4 Let $Q_i \in \mathbf{R}^{n \times n}$, $i \in \mathbf{I}[1, N]$, be N given positive definite matrices and $\tau_i > 0, i \in \mathbf{I}[1, N]$, be N given scalars. Assume that the vector functions $\omega_i : [-\tau_i, 0] \to \mathbf{R}^n$, $i \in \mathbf{I}[1, N]$, are such that the integrals in the following are well-defined, then

$$\left(\sum_{i=1}^{N} x_i\right)^{\mathrm{T}} Q^{-1} \left(\sum_{i=1}^{N} x_i\right) \leq \sum_{i=1}^{N} \int_{-\tau_i}^{0} \omega_i^{\mathrm{T}}(s) \tau_i Q_i^{-1} \omega_i(s) \,\mathrm{d}s,\tag{11}$$

where $x_i = \int_{-\tau_i}^0 \omega_i(s) \, \mathrm{d}s, i \in \mathbf{I}[1, N]$ and $Q = \sum_{i=1}^N Q_i$. Moreover, for a series of vectors $\xi_i \in \mathbf{R}^n, i \in \mathbf{I}[1, N]$, there holds

$$\left(\sum_{i=1}^{N} \xi_{i}\right)^{\mathrm{T}} \left(\sum_{i=1}^{N} Q_{i}\right)^{-1} \left(\sum_{i=1}^{N} \xi_{i}\right) \leq \sum_{i=1}^{N} \xi_{i}^{\mathrm{T}} Q_{i}^{-1} \xi_{i}.$$
(12)

Proof. Notice that, for any $i \in \mathbf{I}[1, N]$, by a Schur complement, there holds

$$\begin{bmatrix} \omega_i^{\mathrm{T}}(s) \tau_i Q_i^{-1} \omega_i(s) & \omega_i^{\mathrm{T}}(s) \\ \omega_i(s) & \frac{1}{\tau_i} Q_i \end{bmatrix} \ge 0, \ i \in \mathbf{I}[1, N].$$

$$(13)$$

Taking integration on both sides of the above inequality gives

$$\begin{bmatrix} \int_{-\tau_i}^0 \omega_i^{\mathrm{T}}(s) \tau_i Q_i^{-1} \omega_i(s) \,\mathrm{d}s & \int_{-\tau_i}^0 \omega_i^{\mathrm{T}}(s) \,\mathrm{d}s \\ \int_{-\tau_i}^0 \omega_i(s) \,\mathrm{d}s & Q_i \end{bmatrix} \ge 0,$$
(14)

where $i \in \mathbf{I}[1, N]$, which implies

$$\begin{bmatrix} \sum_{i=1}^{N} \int_{-\tau_i}^{0} \omega_i^{\mathrm{T}}(s) \tau_i Q_i^{-1} \omega_i(s) \,\mathrm{d}s & \sum_{i=1}^{N} \int_{-\tau_i}^{0} \omega_i^{\mathrm{T}}(s) \,\mathrm{d}s \\ \sum_{i=1}^{N} \int_{-\tau_i}^{0} \omega_i(s) \,\mathrm{d}s & \sum_{i=1}^{N} Q_i \end{bmatrix} \ge 0.$$
(15)

By a Schur complement again, (15) is equivalent to (11). Finally, the inequality in (12) can be proven in a similar way. \blacksquare

Now every integration $\int_{-\tau_i}^0 \omega_i^{\mathrm{T}}(s) Q_i \omega_i(s) \mathrm{d}s, i \in \mathbf{I}[1, N]$, on the right hand side of (11) is weighted by an individual weighting matrix $Q_i, i \in \mathbf{I}[1, N]$, which can introduce more decision variables that can be optimized to reduce the conservatism of the resulting conditions. The multiple Jensen inequalities (11) and (12) are clearly less conservative than the inequalities in (10) and (9) since the later ones can be obtained immediately by setting $Q_i = Q^{-1}, i \in \mathbf{I}[1, N]$, in the former ones.

By applying the Jensen inequality (10) in Corollary 1 and choosing the following Lyapunov functional

$$V(x_{t}) = \int_{t-\tau}^{t} x^{\mathrm{T}}(s) Px(s) ds + \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} (s+\tau_{i}) x^{\mathrm{T}}(t+s) Q_{i}x(t+s) ds,$$
(16)

the following result was obtained in [16].

Lemma 5 [16] The IDS (1) is exponentially stable if there exist N + 1 positive definite matrices $P, Q_i \in \mathbb{R}^{n \times n}, i \in \mathbb{I}[1, N]$, such that the following coupled LMIs are satisfied

$$N\tau_i A_i^{\mathrm{T}} \left(P + \sum_{j=1}^N \tau_j Q_j \right) A_i - Q_i < 0, \ i \in \mathbf{I} [1, N].$$

$$(17)$$

In the next two subsections, we will show how to use the multiple Jensen inequality (11) to improve the above result.

3.2 The First Sufficient Stability Condition

In this subsection we present a new sufficient condition for the exponential stability of the IDS (1) by applying the multiple Jensen inequality (11) and choosing a similar Lyapunov functional as (16).

Theorem 1 Consider the IDS (1). Then

1. It is exponentially stable if there exist 2N positive definite matrices $S_i, Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, such that the following nonlinear matrix inequalities are satisfied

$$\tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i - S_i < 0, \ i \in \mathbf{I}[1, N],$$
(18)

$$\sum_{i=1}^{N} S_i < \left(\sum_{i=1}^{N} Q_i\right)^{-1}.$$
(19)

2. The nonlinear matrix inequalities in (18)–(19) are solvable if and only if there exist 2N positive definite matrices $S_i, Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, and a matrix $R \in \mathbf{R}^{n \times n}$ such that the following LMIs are satisfied

$$\sum_{i=1}^{n} Q_i + \sum_{i=1}^{n} S_i - \left(R^{\mathrm{T}} + R\right) < 0,$$
(20)

$$\begin{bmatrix} -S_i & \tau_i A_i^{\mathrm{T}} R \\ \tau_i R^{\mathrm{T}} A_i & -Q_i \end{bmatrix} < 0, \ i \in \mathbf{I} [1, N].$$

$$(21)$$

Proof. Proof of Item 1. The inequality in (19) implies that there exists a positive definite matrix P > 0 such that $P \leq \varepsilon I_n$ for some sufficiently small number $\varepsilon > 0$ and such that

$$P + \sum_{i=1}^{N} S_i < \left(\sum_{i=1}^{N} Q_i\right)^{-1}, \ i \in \mathbf{I}[1, N].$$
(22)

Consider the following Lyapunov functional

$$V(x_{t}) = \int_{t-\tau}^{t} x^{\mathrm{T}}(s) Px(s) ds + \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} \left(\frac{s}{\tau_{i}} + 1\right) x^{\mathrm{T}}(t+s) S_{i}x(t+s) ds,$$
(23)

which is in the form of (16) that was used in [16]. The time-derivative of $V(x_t)$ satisfies

$$\dot{V}(x_{t}) \leq x^{\mathrm{T}}(t) \left(P + \sum_{i=1}^{N} S_{i}\right) x(t) - \sum_{i=1}^{N} y_{i}$$

$$\leq x^{\mathrm{T}}(t) \left(\sum_{i=1}^{N} Q_{i}\right)^{-1} x(t) - \sum_{i=1}^{N} y_{i}$$

$$\leq \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} x^{\mathrm{T}}(t+s) \tau_{i} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} x(t+s) \,\mathrm{d}s - \sum_{i=1}^{N} y_{i}$$

$$= \sum_{i=1}^{N} \frac{1}{\tau_{i}} \int_{-\tau_{i}}^{0} x^{\mathrm{T}}(t+s) \left(\tau_{i}^{2} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} - S_{i}\right) x(t+s) \,\mathrm{d}s$$

$$\leq -\gamma \int_{-\tau}^{0} \|x(t+s)\|^{2} \,\mathrm{d}s, \qquad (24)$$

where $y_i = \frac{1}{\tau_i} \int_{-\tau_i}^0 x^{\mathrm{T}} (t+s) S_i x (t+s) \mathrm{d}s$, $\gamma > 0$ is some constant and we have used the multiple Jensen inequality (11) and the nonlinear matrix inequalities (18). Hence, it follows from Lemma 1 that the IDS (1) is exponentially stable.

Proof of Item 2. Let $S = \sum_{i=1}^{N} S_j$ and $Q = \sum_{i=1}^{N} Q_j$. Then (19) is equivalent to $Q < S^{-1}$, which, by Lemma 2, is satisfied if and only if there exists an $R \in \mathbf{R}^{n \times n}$ such that

$$R^{\mathrm{T}}QR + S - (R^{\mathrm{T}} + R) < 0.$$
 (25)

On the other hand, by the Schur complement, the inequalities in (18) are satisfied if and only if

$$\begin{bmatrix} -S_i & \tau_i A_i^{\mathrm{T}} \\ \tau_i A_i & -Q_i \end{bmatrix} < 0, \ i \in \mathbf{I} [1, N],$$
(26)

which, by a congruence transformation, are equivalent to

$$\begin{bmatrix} -S_i & \tau_i A_i^{\mathrm{T}} R\\ \tau_i R^{\mathrm{T}} A_i & -R^{\mathrm{T}} Q_i R \end{bmatrix} < 0, \ i \in \mathbf{I} [1, N].$$

$$(27)$$

It is clear that (25) and (27) are respectively equivalent to (20) and (21) by a substitution $R^{T}Q_{i}R \rightarrow Q_{i}, i \in \mathbf{I}[1, N]$ (and thus $R^{T}QR \rightarrow Q$). The proof is finished.

3.3 The Second Sufficient Stability Condition

With the help of the multiple Jensen inequality (11), we further present in this subsection a new sufficient condition for the exponential stability of the IDS (1) with an alternative Lyapunov functional, which may possess some advantages over (16) and (23).

Theorem 2 Consider the IDS (1). Then

1. It is exponentially stable if there exist N positive definite matrices $Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, such that the following nonlinear matrix inequality is satisfied

$$\sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i - \left(\sum_{i=1}^{n} Q_i\right)^{-1} < 0.$$
(28)

2. The nonlinear matrix inequality (28) is solvable if and only if there exist N positive definite matrices $Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, such that the following LMI is satisfied

$$\sum_{i=1}^{N} \begin{bmatrix} \tau_1 A_1 \\ \vdots \\ \tau_N A_N \end{bmatrix} Q_i \begin{bmatrix} \tau_1 A_1 \\ \vdots \\ \tau_N A_N \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} Q_1 \\ \ddots \\ Q_N \end{bmatrix} < 0.$$
(29)

Proof. Proof of Item 1. Let $Q = \sum_{i=1}^{N} Q_j$. Then it follows from (28) that there exist two sufficiently small numbers $\delta > 0$ and $\varepsilon > 0$ such that

$$\sum_{i=1}^{N} \left(\tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i + \tau_i \delta I_n \right) - Q^{-1} \le -\varepsilon Q^{-1}.$$
(30)

Let $R_i > 0, i \in \mathbf{I}[1, N]$, be such that

$$\sum_{i=1}^{N} R_i \triangleq R = Q^{-1} = \left(\sum_{i=1}^{N} Q_i\right)^{-1},$$
(31)

and consider an associated nonnegative functional

$$V_{1}(x_{t}) = \sum_{i=1}^{N} \int_{t-\tau_{i}}^{t} x^{\mathrm{T}}(s) R_{i}x(s) \,\mathrm{d}s, \qquad (32)$$

whose time-derivative is given by

$$\dot{V}_{1}(x_{t}) = x^{\mathrm{T}}(t) \left(\sum_{i=1}^{N} R_{i}\right) x(t) - \sum_{i=1}^{N} y_{i}$$

$$= x^{\mathrm{T}}(t) \left(\sum_{i=1}^{N} Q_{i}\right)^{-1} x(t) - \sum_{i=1}^{N} y_{i}$$

$$\leq \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} \tau_{i} x^{\mathrm{T}}(t+s) A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} x(t+s) \,\mathrm{d}s - \sum_{i=1}^{N} y_{i},$$
(33)

where $y_i = x (t - \tau_i)^{\mathrm{T}} R_i x (t - \tau_i)$, and we have used the IDS (1) and the multiple Jensen inequality (11). Choose another nonnegative functional

$$V_{2}(x_{t}) = \sum_{i=1}^{N} \int_{0}^{\tau_{i}} \int_{t-s}^{t} x^{\mathrm{T}}(l) \left(\tau_{i} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} + \delta I_{n}\right) x(l) \,\mathrm{d}l\mathrm{d}s,$$
(34)

whose time-derivative can be evaluated as

$$\dot{V}_{2}(x_{t}) = \sum_{i=1}^{N} x^{\mathrm{T}}(t) \left(\tau_{i}^{2} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} + \delta \tau_{i} I_{n}\right) x(t) - \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} x^{\mathrm{T}}(t+s) \left(\tau_{i} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} + \delta I_{n}\right) x(t+s) \,\mathrm{d}s.$$
(35)

Hence, it follows from (33) and (35) that

$$\dot{V}_{1}(x_{t}) + \dot{V}_{2}(x_{t}) \\
\leq x^{\mathrm{T}}(t) \sum_{i=1}^{N} \left(\tau_{i}^{2} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} + \delta \tau_{i} I_{n}\right) x(t) - \sum_{i=1}^{N} y_{i} - \mu \\
\leq (1 - \varepsilon) x^{\mathrm{T}}(t) Q^{-1} x(t) - \sum_{i=1}^{N} y_{i} - \mu \\
\leq (1 - \varepsilon) \sum_{i=1}^{N} \left(x^{\mathrm{T}}(t) R_{i} x(t) - x(t - \tau_{i})^{\mathrm{T}} R_{i} x(t - \tau_{i})\right) - \mu \\
= (1 - \varepsilon) \dot{V}_{1}(x_{t}) - \delta \sum_{i=1}^{N} \int_{-\tau_{i}}^{0} \|x(t + s)\|^{2} \mathrm{d}s,$$
(36)

where $\mu = \delta \sum_{i=1}^{N} \int_{-\tau_i}^{0} \|x(t+s)\|^2 ds$ and we have used (30). Therefore

$$\dot{V}(x_t) \triangleq \varepsilon \dot{V}_1(x_t) + \dot{V}_2(x_t) \le -\delta \int_{-\tau}^0 \|x(t+s)\|^2 \,\mathrm{d}s.$$
(37)

Finally, it is trivial to show that $V(x_t)$ satisfies (3). The conclusion then follows from Lemma 1. *Proof of Item 2.* By a Schur complement, the LMI in (29) is satisfied if and only if

$$\begin{bmatrix} -Q & \tau_1 Q A_1^{\mathrm{T}} \cdots \tau_N Q A_N^{\mathrm{T}} \\ \hline \tau_1 A_1 Q & -Q_1 \\ \vdots & \ddots \\ \tau_N A_N Q & -Q_N \end{bmatrix} < 0.$$
(38)

By using a Schur complement again, the inequality (38) can be equivalently transformed into

$$-Q + \sum_{i=1}^{N} \tau_i^2 Q A_i^{\mathrm{T}} Q_i^{-1} A_i Q < 0, \qquad (39)$$

which is further equivalent to (28). The proof is finished.

At the end of this section, we will show that the stability condition (29) has a very interesting relationship with the stability condition of the following IDS

$$x(t) = \sum_{i=1}^{N} A_i x(t - \tau_i), \qquad (40)$$

which was originally studied in [1] by using a Lyapunov functional approach, where $A_i \in \mathbf{R}^{n \times n}$, $i \in \mathbf{I}[1, N]$, are given matrices and $\tau_i, i \in \mathbf{I}[1, N]$, are given scalars and are such that $0 < \tau_1 < \cdots < \tau_N$.

Lemma 6 [1] The IDS (40) is exponentially stable independent of the delays $\tau_i, i \in \mathbf{I}[1, N]$, if there exist N positive definite matrix $Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, such that the following LMI is satisfied

$$\sum_{i=1}^{N} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} Q_i \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}^{1} - \begin{bmatrix} Q_1 \\ \ddots \\ Q_N \end{bmatrix} < 0.$$
(41)

Proof. By recognizing the characteristic polynomial of the IDS (40) (see Eq. (3.7) in [1]), it is not hard to see that it is exponentially stable if and only if the following IDS

$$x(t) = \sum_{i=1}^{N} A_i^{\mathrm{T}} x(t - \tau_i), \qquad (42)$$

is. Then by the results in Section 4 in [1], the IDS (42) is exponentially stable independent of the delays $\tau_i, i \in \mathbf{I}[1, N]$, if there exist N positive definite matrices $X_i, i \in \mathbf{I}[1, N]$, such that the following LMI is satisfied (see inequality (4.6) in [1])

$$\begin{bmatrix} \Pi_{1} & -A_{1}X_{1}A_{2}^{\mathrm{T}} & \cdots & -A_{1}X_{1}A_{N}^{\mathrm{T}} \\ -A_{2}X_{1}A_{1}^{\mathrm{T}} & \Pi_{2} & \cdots & -A_{2}X_{1}A_{N}^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{N}X_{1}A_{1}^{\mathrm{T}} & -A_{N}X_{1}A_{2}^{\mathrm{T}} & \cdots & X_{N} - A_{N}X_{1}A_{N}^{\mathrm{T}} \end{bmatrix} > 0,$$

$$(43)$$

where $\Pi_i = X_i - A_i X_1 A_i^{\mathrm{T}} - X_{i+1}, i \in \mathbf{I} [1, N-1]$. As the above LMI implies $X_i > X_{i+1}, i \in \mathbf{I} [1, N-1]$, we can let $Q_N = X_N$ and $Q_i = X_i - X_{i+1} > 0$, $i \in \mathbf{I} [1, N-1]$. Then the LMI in (43) can be exactly rewritten as (41).

It is very interesting to notice that the LMI (29) and the LMI (41) possess the very similar structures and the only difference is that the former LMI is delay dependent and the later one is not. This similarity may help us to understand the stability of these two classes of IDSs (1) and (40)

4 A Spectral Radius Based Condition and A Comparison

4.1 A Spectral Radius Based Stability Condition

In this subsection, we will present a spectral radius based sufficient condition for the exponential stability of the IDS (1) based on Lemma 5, as indicated by the following theorem.

Theorem 3 The following statements are equivalent.

A. There exist N + 1 positive definite matrices $P, Q_i \in \mathbf{R}^{n \times n}, i \in \mathbf{I}[1, N]$, such that the coupled LMIs in (17) are satisfied.

B. There exist N positive definite matrices $Q_i \in \mathbf{R}^{n \times n}$, $i \in \mathbf{I}[1, N]$, such that the following coupled LMIs are satisfied

$$N\tau_i^2 A_i^{\mathrm{T}}\left(\sum_{j=1}^N Q_j\right) A_i - Q_i < 0, \ i \in \mathbf{I}[1, N].$$

$$(44)$$

C. The following condition is met, where $\rho(A)$ denotes the spectral radius of a square matrix A:

$$\rho\left(\sum_{i=1}^{N}\tau_i^2 A_i \otimes A_i\right) < \frac{1}{N}.$$
(45)

D. There exists a positive definite matrix $Q \in \mathbf{R}^{n \times n}$ such that the following LMI is satisfied

$$\sum_{i=1}^{N} N\tau_i^2 A_i^{\mathrm{T}} Q A_i - Q < 0.$$
(46)

Proof. Proof of $A \Leftrightarrow B$. It is clear that the LMIs in (17) are satisfied if we choose $P = \varepsilon I_n$ with ε being sufficiently small and the following coupled LMIs

$$N\tau_i A_i^{\mathrm{T}}\left(\sum_{j=1}^N \tau_j Q_j\right) A_i - Q_i < 0, \ i \in \mathbf{I}\left[1, N\right],\tag{47}$$

are satisfied. The converse is obvious. Finally, the LMIs (47) are equivalent to (44) by the substitution $\tau_i Q_i \to Q_i, i \in \mathbf{I}[1, N]$.

Proof of $B \Leftrightarrow C$. Let $\mathcal{S}^{n \times n}_+ = (S_1, S_2, \cdots, S_N)$ where $S_i \in \mathbf{S}^{n \times n}_+ \triangleq \{S : S = S^T > 0\}$. Then

$$\mathscr{L}(\mathcal{Q}) = \left(N\tau_1^2 A_1^{\mathrm{T}} \sum_{j=1}^N Q_j A_1, \cdots, N\tau_N^2 A_N^{\mathrm{T}} \sum_{j=1}^N Q_j A_N \right),$$
(48)

where $Q = (Q_1, \dots, Q_N) \in S^{n \times n}_+$, is a linear positive operator (see Definition 1 in [11]). Consequently, the inequalities in (44) are satisfied if and only if there exists a $Q \in S^{n \times n}_+$ such that

$$\mathscr{L}(\mathcal{Q}) - \mathcal{Q} < 0,\tag{49}$$

where $\mathcal{P} < 0$ means $-\mathcal{P} \in \mathcal{S}^{n \times n}_+$. Then by Lemma 1 in [12], the inequality in (49) has a solution $\mathcal{Q} \in \mathcal{S}^{n \times n}_+$ if and only if $\rho(\mathscr{L}) < 1$. However, similar to (10)-(11) in [11], we can show that $\rho(\mathscr{L}) = \rho(NA^T) = \rho(NA)$ with

$$A = \begin{bmatrix} \tau_1^2 A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}} & \tau_1^2 A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}} & \cdots & \tau_1^2 A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}} \\ \tau_2^2 A_2^{\mathrm{T}} \otimes A_2^{\mathrm{T}} & \tau_2^2 A_2^{\mathrm{T}} \otimes A_2^{\mathrm{T}} & \cdots & \tau_2^2 A_2^{\mathrm{T}} \otimes A_2^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_N^2 A_N^{\mathrm{T}} \otimes A_N^{\mathrm{T}} & \tau_N^2 A_N^{\mathrm{T}} \otimes A_N^{\mathrm{T}} & \cdots & \tau_N^2 A_N^{\mathrm{T}} \otimes A_N^{\mathrm{T}} \end{bmatrix}$$

Hence the LMIs in (44) are solvable if and only if

$$\rho\left(A\right) < \frac{1}{N}.\tag{50}$$

Now notice that we can write A = BC where

$$B = \begin{bmatrix} \tau_1^2 A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}} \\ \vdots \\ \tau_N^2 A_N^{\mathrm{T}} \otimes A_N^{\mathrm{T}} \end{bmatrix}, \ C = \begin{bmatrix} I_{n^2} & I_{n^2} & \cdots & I_{n^2} \end{bmatrix}.$$
(51)

On the other hand, for any two matrices X and Y with appropriate dimensions, we have $\rho(XY) = \rho(YX)$. Hence

$$\rho(A) = \rho(BC) = \rho(CB) = \rho\left(\sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} \otimes A_i^{\mathrm{T}}\right),$$
(52)

by which the inequality in (50) is exactly the one in (45).

Proof of $C \Leftrightarrow D$. The proof is quite similar to the proof of $B \Leftrightarrow C$ by utilizing the linear positive operator

$$\mathscr{F}(Q) = \sum_{i=1}^{N} N \tau_i^2 A_i^{\mathrm{T}} Q A_i,$$
(53)

where $Q \in \mathbf{S}^{n \times n}_+$. The proof is finished.

Item C of Theorem 3 implies an interesting spectral radius based sufficient condition for testing the stability of the IDS (1), as highlighted in the following corollary.

Corollary 2 The IDS (1) is exponentially stable if the spectral radius condition (45) is satisfied. Particularly, if N = 1, the IDS (1) is exponentially stable if $\rho(A_1) < \frac{1}{\tau_1}$.

Remark 1 If N = 1, by Proposition 2 in [16], the IDS (1) is exponentially stable if $||A_1|| < \frac{1}{\tau_1}$, which is more conservative than $\rho(A_1) < \frac{1}{\tau_1}$ since $\rho(A_1) \leq ||A_1||$ for any matrix A_1 .

Corollary 3 The IDS (1) is exponentially stable if there exist N scalars $\alpha_i \in (0, 1), i \in \mathbf{I}[1, N]$ such that $\sum_{i=1}^{N} \alpha_i = 1$ and

$$\rho\left(\sum_{i=1}^{N} \frac{\tau_i^2}{\alpha_i} A_i \otimes A_i\right) < 1.$$
(54)

Proof. According to the proof of Theorem 3, (54) is satisfied if and only if there exists a Q > 0 such that

$$\sum_{i=1}^{N} \frac{\tau_i^2}{\alpha_i} A_i^{\mathrm{T}} Q A_i - Q < 0.$$
(55)

Hence the inequality (28) is satisfied with $Q_i = \alpha_i Q$. The result then follows from Theorem 2. Clearly, the spectral condition (54) reduces to (45) if we set $\alpha_i = \frac{1}{N}, i \in \mathbf{I}[1, N]$.

4.2 A Comparison of Different Sufficient Conditions

With the help of Theorem 3, we are able to make a comparison among these different stability conditions in Lemma 5, Theorem 1 and Theorem 2.

Proposition 1 The following statements are true.

- 1. If the set of LMIs (17) (or (44)) are solvable, then the set of LMIs in (20)-(21) are also solvable, namely, Theorem 1 is always less conservative than Lemma 5.
- 2. The set of LMIs in (20)-(21) are solvable if and only if the LMI in (29) is solvable. Hence Theorem 2 is equivalent to Theorem 1 and both of them are thus less conservative than Lemma 5.

Proof. Proof of Item 1. By Theorem 3, the set of LMIs (17) are solvable if and only if the set of LMIs (44) are solvable, which implies that there exists a sufficiently small number $\varepsilon > 0$ such that $\varepsilon I_n < Q_i, i \in \mathbf{I}[1, N]$, and

$$N\tau_i^2 A_i^{\mathrm{T}}\left(\sum_{j=1}^N Q_j\right) A_i - Q_i + \varepsilon I_n < 0, \ i \in \mathbf{I}[1, N].$$
(56)

Now for every $i \in \mathbf{I}[1, N]$ we choose

$$P_i^{-1} = N \sum_{i=1}^{N} Q_i \triangleq NQ, \ R_i = Q_i - \varepsilon I_n > 0,$$
(57)

which implies

$$\sum_{i=1}^{N} P_i = \sum_{i=1}^{N} \frac{1}{N} Q^{-1} = Q^{-1}.$$
(58)

Then it follows from (56) that, for all $i \in \mathbf{I}[1, N]$,

$$\tau_i^2 A_i^{\mathrm{T}} P_i^{-1} A_i - R_i = N \tau_i^2 A_i^{\mathrm{T}} Q A_i - Q_i + \varepsilon I_n < 0,$$
(59)

and it follows from (58) that

$$\sum_{i=1}^{N} R_{i} = \sum_{i=1}^{N} (Q_{i} - \varepsilon I_{n}) = Q - N\varepsilon I_{n} < Q = \left(\sum_{i=1}^{N} P_{i}\right)^{-1}.$$
(60)

Clearly, (59) and (60) are respectively in the form of (18)–(19).

Proof of Item 2. Assume that (18)–(19) are feasible. Summing the N nonlinear matrix inequalities in (18) on both sides gives

$$\sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i - \sum_{i=1}^{N} S_i < 0,$$
(61)

which, by using the nonlinear matrix inequality (19), implies

$$\sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i < \left(\sum_{i=1}^{N} Q_i\right)^{-1},\tag{62}$$

which is just in the form of (28) and is further equivalent to (29).

Now assume that (28) is feasible. Denote

$$\Omega = \frac{1}{2N} \left(\left(\sum_{i=1}^{N} Q_i \right)^{-1} - \sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i \right),$$
(63)

and let

$$S_{i} = \tau_{i}^{2} A_{i}^{\mathrm{T}} Q_{i}^{-1} A_{i} + \Omega > 0, \ i \in \mathbf{I} [1, N] .$$
(64)

It follows that (18) is satisfied. Now, by (28), we have

$$\sum_{i=1}^{N} S_i = \frac{1}{2} \left(\left(\sum_{i=1}^{N} Q_i \right)^{-1} + \sum_{i=1}^{N} \tau_i^2 A_i^{\mathrm{T}} Q_i^{-1} A_i \right) < \left(\sum_{i=1}^{N} Q_i \right)^{-1}$$

which implies that (19) is satisfied. The proof is finished.

This proposition demonstrates in theory that the multiple Jensen inequality (11) used in the proofs of Theorems 1 and 2 can indeed reduce the conservatism of the resulting stability conditions.

Remark 2 Though Theorems 1 and 2 are equivalent by Proposition 1, Theorem 2 obtained by the novel Lyapunov functionals (32) and (34) possesses an advantage over Theorem 1. To see this, we notice that the total row size (denoted by Φ) and the total number of scalar decision variables (denoted by Ψ) in the LMIs of Theorems 1 and 2 are, respectively, given by

$$\begin{cases} \Phi_{\text{Th},1} = n + 2n, & \Psi_{\text{Th},1} = n (n+1) N + n^2, \\ \Phi_{\text{Th},2} = nN, & \Psi_{\text{Th},2} = \frac{n(n+1)}{2}N. \end{cases}$$
(65)

It is well known that the computational complexity of an LMI is bounded by $\mu\Phi\Psi^3$ where μ is a constant (see [22]). Hence the computation complexity of the LMIs in Theorem 2 is significantly lower than that in Theorem 1.

Combining Lemma 6, Proposition 1 and Theorem 3 gives the following corollary.

Corollary 4 The IDS (40) is exponentially stable independent of the delays $\tau_i, i \in \mathbf{I}[1, N]$, if $\rho\left(\sum_{i=1}^N A_i \otimes A_i\right) < \frac{1}{N}$.

	LMI (29)	LMIs (17)	LMI(46)	Condition (45)
$\tau_1 = 0.4$	0.0317	infeasible	infeasible	infeasible
$\tau_1 = 0.3$	0.1146	0.0474	0.0474	0.0474
$\tau_1 = 0.2$	0.2418	0.1527	0.1527	0.1527
$\tau_1 = 0.1$	0.4882	0.3414	0.3414	0.3414

Table 1: The maximal allowable τ_2 by using different approaches

5 A Numerical Example

We consider an IDS in the form of (1) with two delays, namely,

$$x(t) = A_1 \int_{-\tau_1}^0 x(t+s) \,\mathrm{d}s + A_2 \int_{-\tau_2}^0 x(t+s) \,\mathrm{d}s, \tag{66}$$

where

$$A_1 = \begin{bmatrix} -4 & 1\\ -13 & 2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix}.$$
(67)

If $A_2 = 0$, this system has been studied in [16]. It is shown there that the LMIs in Lemma 5 are feasible if and only if $0 \le \tau_1 \le 0.4473 = \tau_1^*$. On the other hand, direct computation gives $\rho(\tau_1^*A_1) = 0.9999$, which clearly validates Corollary 2.

For a fixed τ_1 , the maximal value of τ_2 such that the LMI in (29), the LMIs in (17), the LMI in (46), and the spectral condition (45) are feasible can be respectively computed by a bisection method. The results are recorded in Table 1. From this table we can observe that Theorem 2 is always less conservative than Lemma 5 established in [16], which indicates that our approach based on the multiple Jensen inequality can considerably reduce the conservatism in the stability analysis of this class of IDSs. Moreover, the LMIs (17), the LMI (46) and the spectral condition (45) lead to the same result, which validates Theorem 3. We mention that the results obtained by Theorem 1 are the same as that by Theorem 2. However, from the computational point of view, Theorem 3 is recommended to use in practice as it only involves one constraint and a single decision variable.

To illustrate Corollary 3, let $\tau_1 = 0.4$ and $\tau_2 = 0.02$. From Table 1 we can see that (45) is not satisfied. Since $||A_1|| \gg ||A_2||$, we may let the weighting factor $\frac{1}{\alpha_1}$ of $(\tau_1 A_1) \otimes (\tau_1 A_1)$ be small enough so that the spectral radius of the resulting matrix $\sum_{i=1}^{2} \frac{1}{\alpha_i} (\tau_i A_i) \otimes (\tau_i A_i)$ is less than one. Indeed, if we choose $\alpha_1 = 0.9$ and $\alpha_2 = 0.1$ we can compute $\rho(\sum_{i=1}^{2} \frac{1}{\alpha_i} (\tau_i A_i) \otimes (\tau_i A_i)) = 0.9783$, which implies the asymptotic stability of IDS (66) in this case.

We finally mention that the conclusions obtained in the above have been approved by many other randomly chosen numerical examples that are not included here to save spaces.

6 Conclusion

This note has studied the stability analysis of a class of integral delay systems (IDSs) with multiple delays, which have wide applications in the stability analysis of time-delay systems, especially for neutral timedelay systems. By generalizing the well-known Jensen inequality to the case with multiple terms through introducing multiple weighting matrices, two Lyapunov functional based approaches have been established to yield set of sufficient stability conditions. Moreover, it is shown by the positive operator theory that the obtained new conditions are always less conservative than the existing ones and a spectral radius based sufficient condition is obtained simultaneously. A numerical example has demonstrated the effectiveness of the established approaches.

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