Sequential Randomized Algorithms for Convex Optimization in the Presence of Uncertainty

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Abstract

In this paper, we propose new sequential randomized algorithms for convex optimization problems in the presence of uncertainty. A rigorous analysis of the theoretical properties of the solutions obtained by these algorithms, for full constraint satisfaction and partial constraint satisfaction, respectively, is given. The proposed methods allow to enlarge the applicability of the existing randomized methods to real-world applications involving a large number of design variables. Since the proposed approach does not provide a priori bounds on the sample complexity, extensive numerical simulations, dealing with an application to hard-disk drive servo design, are provided. These simulations testify the goodness of the proposed solution.

I. INTRODUCTION

In recent years, research on randomized and probabilistic methods for control of uncertain systems has successfully evolved along various directions, see e.g. [20] for an overview of the state of the art on this topic. For convex control design, two main classes of algorithms, sequential and non-sequential, have been proposed in the literature, and their theoretical properties have been rigorously studied, see e.g. [10].

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Regarding non-sequential methods, the approach that has emerged is the so-called scenario approach, which has been introduced in [7], [8]. Taking random samples of the uncertainty $q \in \mathbb{Q}$, the main idea of this particular line of research is to reformulate a semi-infinite convex optimization problem as a sampled optimization problem subject to a finite number of random constraints. Then, a key problem is to determine the sample complexity, i.e. the number of random constraints that should be generated, so that the so-called probability of violation is smaller than a given accuracy $\varepsilon \in (0, 1)$, and this event holds with a suitably large confidence $1 - \delta \in (0, 1)$. On the other hand, if accuracy and confidence are very small, and the number of design parameters is large, then the sample complexity may be large, and the sampled convex optimization problem may be difficult to solve in practice.

Motivated by this discussion, in this paper we develop a sequential method specifically tailored to the solution of the scenario-based optimization problem. The proposed approach iteratively solves *reduced-size* scenario problems of increasing size, and it is particularly appealing for large-size problems. This line of research follows and improves upon the schemes previously developed for various control problems, which include linear quadratic regulators, linear matrix inequalities and switched systems discussed in [10], [20]. The main idea of these sequential methods is to introduce the concept of validation samples. That is, at step k of the sequential algorithm, a "temporary solution" is constructed and, using a suitably generated validation sample set, it is verified whether or not the probability of violation corresponding to the temporary solution is smaller than a given accuracy ε, and this event holds with confidence $1 - \delta$. Due to their sequential nature, these algorithms may have wider practical applications than non-sequential methods, in particular for real-world problems where fast computations are needed because of very stringent time requirements due to on-line implementations. However, we remark that the sequential methods proposed here, contrary to the scenario approach previously discussed, do not provide a priori bounds on the sample complexity.

Compared to the sequential approaches discussed above, the methods proposed in this paper have the following distinct main advantages: 1. the termination of the algorithm does not require the knowledge of some user-determined parameters, such as the center of a feasibility ball; 2. the methods can be immediately implemented using existing off-the-shelf convex optimization tools, and no ad-hoc implementation of specific update rules (such as stochastic gradient, ellipsoid or cutting plane) is needed. We also remark that the methods presented here directly apply to

optimization problems, whereas the sequential methods discussed in [10], [20] are limited to feasibility.

In this paper, which is an expanded version of [14], we study two new sequential algorithms for optimization, with full constraint satisfaction and partial constraint satisfaction, respectively, and we provide a rigorous analysis of their theoretical properties regarding the probability of violation of the returned solutions. These algorithms fall into the class of sequential probabilistic validation (SPV) algorithms introduced in [3].

In the second part of the paper, using a non-trivial example regarding the position control of read/write head in a commercial hard disk drive, we provide extensive numerical simulations to compare the sample complexity of the scenario approach with the number of iterations required in the two sequential algorithms previously introduced. We remark that the sample complexity of the scenario approach is computed a priori, while for sequential algorithms, the numerical results regarding the size of the validation sample set are random. For this reason, mean values, standard deviation and other related parameters are experimentally computed for both proposed algorithms by means of extensive Monte Carlo simulations. We also highlight that the worst case complexity of the proposed methods may be larger than that of the scenario approach.

II. Problem Formulation and Preliminaries

An uncertain convex problem has the form

$$\min_{\theta \in \Theta} \quad c^T \theta \tag{1}$$

$$\text{subject to } f(\theta, q) \leq 0 \text{ for all } q \in \mathbb{Q}$$

where $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the vector of optimization variables and $q \in \mathbb{Q}$ denotes random uncertainty acting on the system, $f(\theta, q) : \Theta \times \mathbb{Q} \to \mathbb{R}$ is convex in θ for any fixed value of $q \in \mathbb{Q}$ and Θ is a convex and closed set. We note that most uncertain convex problems can be reformulated as (1). In particular, multiple scalar-valued constraints $f_i(\theta, q) \leq 0$, $i = 1, \ldots, m$ can always be recast into the form (1) by defining $f(\theta, q) = \max_{i=1,\ldots,m} f_i(\theta, q)$.

In this paper, we study a probabilistic framework where the uncertainty vector q is assumed to be a random variable and the constraint in (1) is allowed to be violated for some $q \in \mathbb{Q}$, provided that the rate of violation is sufficiently small. This concept is formally expressed using the notion of "probability of violation".

Definition 1 (Probability of Violation): The probability of violation of θ for the function $f :$ $\Theta \times \mathbb{Q} \to \mathbb{R}$ is defined as

$$V(\theta) \doteq \Pr \left\{ q \in \mathbb{Q} \; : \; f(\theta, q) > 0 \right\}. \tag{2}$$

The exact computation of $V(\theta)$ is in general very difficult since it requires the computation of multiple integrals associated to the probability in (2). However, this probability can be estimated using randomization. To this end, assuming that a probability measure is given over the set $\mathbb{Q}$, we generate N independent identically distributed (i.i.d.) samples within the set $\mathbb{Q}$

$$\mathbf{q} = \{q^{(1)}, \ldots, q^{(N)}\} \in \mathbb{Q}^N,$$

where $\mathbb{Q}^N \doteq \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (N times). Next, a Monte Carlo approach is employed to obtain the so called "empirical violation" which is introduced in the following definition.

Definition 2 (Empirical Violation): For given $\theta \in \Theta$ the empirical violation of $f(\theta, q)$ with respect to the multisample $\mathbf{q} = \{q^{(1)}, \ldots, q^{(N)}\}$ is defined as

$$\widehat{V}(\theta, \mathbf{q}) \doteq \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_f(\theta, q^{(i)}) \tag{3}$$

where $\mathbb{I}_f(\theta, q^{(i)})$ is an indicator function defined as

$$\mathbb{I}_f(\theta, q^{(i)}) \doteq \begin{cases} 0 & \text{if } f(\theta, q^{(i)}) \leq 0 \\ 1 & \text{otherwise} \end{cases}.$$

A. The Scenario Approach

In this subsection, we briefly recall the so-called scenario approach, also known as random convex programs, which was first introduced in [7], [8], see also [11] for additional results. In this approach, a set of independent identically distributed random samples of cardinality N is extracted from the uncertainty set and the following scenario problem is formed

$$\min_{\theta \in \Theta} \quad c^T \theta \tag{4}$$

$$\text{subject to } f(\theta, q^{(i)}) \leq 0, \, i = 1, \ldots, N.$$

The function $f(\theta, q)$ is convex for fixed $q \in \mathbb{Q}$ and a further assumption is that the problem (4) is feasible for any finite number of samples and attains a unique solution $\widehat{\theta}_N$. These assumptions are now formally stated.

Assumption 1 (Convexity): $\Theta \subset \mathbb{R}^{n_\theta}$ is a convex and closed set and $f(\theta, q)$ is convex in θ for any fixed value of $q \in \mathbb{Q}$.

Assumption 2 (Feasibility and Uniqueness): The sampled optimization problem (4) is feasible for any multisample extraction and its feasibility domain has a nonempty interior. Furthermore, the solution of (4) exists and is unique.

We remark that the uniqueness assumption can be relaxed in most cases by introducing a tie-breaking rule (see Section 4.1 of [7]). The probabilistic property of the optimal solution obtained from (4) is stated in the next lemma taken from [11].

Lemma 1: Let Assumptions 1 and 2 hold and let $\delta, \varepsilon \in (0, 1)$ and N satisfy the following inequality

$$\sum_{i=0}^{n_\theta - 1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i} \leq \delta. \tag{5}$$

Then, with probability at least $1 - \delta$, the solution of the optimization problem (4) $\widehat{\theta}_N$ satisfies the inequality $V(\widehat{\theta}_N) \leq \varepsilon$.

We remark that Assumption 2, which guarantees that the sample problem is feasible, is rather common in the literature on random convex programs, and can be relaxed using the approach introduced in [6]. In particular, in this case $n_\theta - 1$ in (5) should be replaced by n_θ.

B. Scenario with Discarded Constraints

The idea of scenario with discarded constraints [6], [12] is to generate N i.i.d. samples and then purposely discard $r < N - n_\theta$ of them. In other words, we solve an optimization problem of the form

$$\min_{\theta \in \Theta} \quad c^T \theta \tag{6}$$

$$\text{subject to } f(\theta, q^{(i)}) \leq 0, \ i = 1, \ldots, N - r,$$

where, for notation ease, we assumed that the discarded constraints correspond to the last r ones[1].

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[1]In the more general case, the constraint in (6) should be written as follows $f(\theta, q^{(i_v)}) \leq 0$, $v = 1, \ldots, N - r$, where $i_v \in \{1, \ldots, N\}$ represent the not discarded constraints. Note that this assumption is made without loss of generality, since the two sets of constraints are equivalent up to a reordering.

The r discarded samples are chosen so that the largest improvement in the optimal objective value is achieved. We remark that the optimal strategy to select r discarded samples is a mixed-integer optimization problem, which may be hard to solve numerically. The following lemma [12] defines the probabilistic properties of the optimal solution obtained from (6).

Lemma 2: Let Assumptions 1 and 2 hold and let δ, $\varepsilon \in (0, 1)$, N and $r < N - n_\theta$ satisfy the following inequality

$$\binom{r + n_\theta - 1}{r} \sum_{i=0}^{r+n_\theta-1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i} \leq \delta. \tag{7}$$

Then, with probability at least $1 - \delta$, the optimal solution of the optimization problem (6) $\widehat{\theta}_N$ satisfies the inequality $V(\widehat{\theta}_N) \leq \varepsilon$.

Note that in the literature there are different results regarding explicit sample complexity bounds N such that (5) or (7) are satisfied for given values of $\varepsilon, \delta \in (0, 1)$, see e.g. [3], [2], [6]. These bounds depend linearly on $1/\varepsilon$ and n_θ and logarithmically on $1/\delta$. However, in practice, the required number of samples can be very large even for problems with moderate number of decision variables. Therefore, the computational load of the random convex problems (4) and (6) might be beyond the capability of the available convex optimization solvers. Motivated by this observation, in the next section we propose two sequential randomized algorithms for optimization.

III. SEQUENTIAL RANDOMIZED ALGORITHMS

The main philosophy behind the proposed sequential randomized algorithms lies on the fact that it is easy from the computational point of view to evaluate a given "candidate solution" for a large number of random samples extracted from $\mathbb{Q}$. On the other hand, it is clearly more expensive to solve the optimization problems (4) or (6) when the sample bound N is large. The sequential randomized algorithms, which are presented next generate a sequence of "design" sample sets $\{q_d^{(1)}, \ldots, q_d^{(N_k)}\}$ with increasing cardinality N_k which are used in (4) and (6) for solving the optimization problem. In parallel, "validation" sample sets $\{q_v^{(1)}, \ldots, q_v^{(M_k)}\}$ of cardinality M_k are also generated by both algorithms in order to check whether the given candidate solution, obtained from solving (4) or (6), satisfies the desired violation probability.

The first algorithm is in line with those presented in [9] and [18], in the sense that it uses a similar strategy to validate the candidate solution. However, while these algorithms have been

designed for feasibility problems, the proposed algorithms deal with optimization problems.

A. Full Constraint Satisfaction

The first sequential randomized algorithm is presented in Algorithm 1, and its theoretical properties are stated in the following theorem.

Theorem 1: Let Assumptions 1 and 2 hold. Then with probability at least $1 - \delta$ the solution obtained from Algorithm 1 satisfies the inequality $V(\theta_{\text{sol}}) \leq \varepsilon$.

Proof: See Appendix A. ∎

We note that in steps 3 and 4, to preserve the i.i.d. assumptions, the design and validation samples need to be redrawn at each iteration, and sample-reuse techniques are not applicable.

Remark 1: It is important to observe that the probability $V(\theta_{\text{sol}}) \leq \varepsilon$ in the statement of Theorem 1 is the outcome probability of the algorithm. Hence, this probability is a measure on the whole collection of $\sum_k (N_k + M_k)$ samples that includes both design samples and validation samples. This measure is indeed different than the N-fold probability measure of the uncertain parameter q which appears in the scenario approach.

Remark 2: The proof of this result has similarities and differences compared to other results which appeared in the probabilistic design literature, see the survey paper [10]. Specifically, Theorem 1 in [9] studies the success of a probabilistic oracle, but it does not consider the validation sample techniques. The general framework of sequential algorithms with probabilistic validation is studied in [3], see Theorem 5 in particular. The contribution of the present paper is to exploit these methods for convex optimization problems in the context of the scenario approach.

Remark 3 (Optimal Value of α): The sample bound (10) has some similarities with the one derived in [10, Theorem 2], originally proven in [17], and also used in [3]. However, since we are using a finite sum[2], thanks to the finite scenario bound obtained solving (8), we can use the finite hyperharmonic series $\mathcal{S}_{k_t-1}(\alpha) = \sum_{j=1}^{k_t-1} j^{-\alpha}$ (also known as p-series) instead of the Riemann Zeta function $\sum_{j=1}^{\infty} j^{-\alpha}$. Indeed, the Riemann Zeta function does not converge when α is smaller than one, while in the presented bound (10), α may be smaller than one, which improves the overall sample complexity in particular for large values of k_t. The optimal value of

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[2]See in particular the summation (17) in the proof of Theorem 1.

Algorithm 1 SEQUENTIAL RANDOMIZED ALGORITHM: FULL CONSTRAINT SATISFACTION

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1) INITIALIZATION

   Set iteration counter to zero ($k = 0$). Choose probabilistic levels ε, δ and number of iterations $k_t > 1$.

2) UPDATE

   Set $k = k + 1$ and $N_k \geq N\frac{k}{k_t}$ where N is the smallest integer satisfying

   $$\sum_{i=0}^{n_\theta - 1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i} \leq \delta/2. \tag{8}$$

3) DESIGN

   - Draw N_k i.i.d. samples $\mathbf{q}_d = \{q_d^{(1)}, \ldots, q_d^{(N_k)}\} \in \mathbb{Q}$ based on the underlying distribution.

   - Solve the following *reduced-size scenario problem*

   $$\widehat{\theta}_{N_k} = \arg\min_{\theta \in \Theta} \quad c^T \theta \tag{9}$$

   $$\text{subject to} \quad f(\theta, q_d^{(i)}) \leq 0, \quad i = 1, \ldots, N_k.$$

   - **If** the last iteration is reached ($k = k_t$), set $\theta_{\text{sol}} = \widehat{\theta}_{N_k}$ and **Exit**.

   - **Else**, continue to the next step.

4) VALIDATION

   - Draw

   $$M_k \geq \frac{\alpha \ln k + \ln\left(\mathcal{S}_{k_t - 1}(\alpha)\right) + \ln \frac{2}{\delta}}{\ln\left(\frac{1}{1-\varepsilon}\right)} \tag{10}$$

   i.i.d. samples $\mathbf{q}_v = \{q_v^{(1)}, \ldots, q_v^{(M_k)}\} \in \mathbb{Q}$ based on the underlying distribution, and $\mathcal{S}_{k_t - 1}(\alpha) = \sum_{j=1}^{k_t - 1} j^{-\alpha}$, where $\alpha > 0$ is a tuning parameter.

   - **If** $\mathbb{I}_f(\widehat{\theta}_{N_k}, q_v^{(i)}) = 0$ for $i = 1, \ldots, M_k$; set $\theta_{\text{sol}} = \widehat{\theta}_{N_k}$ and **Exit**.

   - **Else**, goto step (2).

α which minimizes the sample bound (10) has been computed using numerical simulations for different values of the termination parameter k_t. The "almost" optimal value of α minimizing (10) for a wide range of k_t is $\alpha = 0.1$. The bound (10) (for $\alpha = 0.1$) improves upon the bound (17) in [10], by 5% to 15% depending on the termination parameter k_t. It also improves upon the bound in [18], which uses finite sum but in a less effective way.

Finally, we note that the dependence of M_k upon the parameters ε and δ is logarithmic in $1/\delta$ and substantially linear in $1/\varepsilon$. This is a key difference with an approach based on a straightforward (a posteriori) Monte Carlo analysis, which indeed requires $1/\epsilon^2$ validation samples, see e.g. [20].

B. Partial Constraint Satisfaction

In the "design" and "validation" steps of Algorithm 1, *all* elements of the design and validation sample sets are required to satisfy the constraint in (1). However, it is sometimes impossible to find a solution satisfying the constraint in (1) for the entire set of uncertainty. For this reason, in Algorithm 2, we consider the scenario design with discarded constraints where we allow a limited number of design and validation samples to violate the constraint in (1). We now provide a theorem stating the theoretical properties of Algorithm 2.

Theorem 2: Let Assumptions 1 and 2 hold. Then with probability at least $1 - \delta$ the solution obtained from Algorithm 2 satisfies the inequality $V(\theta_{\mathrm{sol}}) \leq \varepsilon$.

Proof: See Appendix B. ■

Algorithm 2 is different from the algorithm presented in [1], which was derived for non-convex problems, in a number of aspects. That is, the cardinality of the sequence of sample sets used for design and validation increases linearly with iteration counter k, while it increases exponentially in [1]. Furthermore, the cardinality of the validation sample set at the last iteration M_{k_t} in [1] is chosen to be equal to the cardinality of the sample set used for design at the last iteration N_{k_t} while, in the presented algorithm M_{k_t} and hence β_w are chosen based on the additive Chernoff bound which is less conservative.

We also note that both Algorithms 1 and 2 fall within the class of SPV algorithms in which the "design" and "validation" steps are independent, see [3]. As a result, in principle we could use the same strategy as Algorithm 1 to tackle discarded constraints problems. Nevertheless, Algorithm 2 appears to be more suitable for discarded constraints problems, since (13) forces the solution to violate some constraints.

Algorithm 2 SEQUENTIAL RANDOMIZED ALGORITHM: PARTIAL CONSTRAINT SATISFAC-
TION

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1) INITIALIZATION

   Set the iteration counter to zero ($k = 0$). Choose probabilistic levels ε, δ, number of iterations $k_t > 1$, number of discarded constraints r and define the following parameters:

   $$\beta_v \doteq \max\left\{1, \beta_w\left(k_t \ln \frac{2k_t}{\delta}\right)^{-1}\right\}, \quad \beta_w \doteq \frac{1}{4\varepsilon}\ln\frac{1}{\delta}. \tag{11}$$

2) UPDATE

   Set $k = k + 1$, $N_k \geq N\frac{k}{k_t}$ and $N_{k,r} \geq \frac{(N-r)k}{k_t}$ where N is the smallest integer satisfying

   $$\binom{r + n_\theta - 1}{r} \sum_{i=0}^{r+n_\theta-1} \binom{N}{i}\varepsilon^i(1 - \varepsilon)^{N-i} \leq \delta/2. \tag{12}$$

3) DESIGN

   - Draw N_k i.i.d. samples $\mathbf{q}_d = \{q_d^{(1)}, \ldots, q_d^{(N_k)}\} \in \mathbb{Q}$ based on the underlying distribution.

   - Solve the following *reduced-size scenario problem*

   $$\widehat{\theta}_{N_{k,r}} = \arg\min_{\theta \in \Theta} \quad c^T\theta \tag{13}$$

   $$\text{subject to} \quad f(\theta, q_d^{(i)}) \leq 0, \quad i = 1, \ldots, N_{k,r}.^3$$

   - **If** the last iteration is reached ($k = k_t$), set $\theta_{\text{sol}} = \widehat{\theta}_{N_k,r}$ and **Exit**.

   - **Else**, continue to the next step.

4) VALIDATION

   - Draw

   $$M_k \geq 2k\beta_v\frac{1}{\varepsilon}\ln\frac{2k_t}{\delta} \tag{14}$$

   i.i.d. samples $\mathbf{q}_v = \{q_v^{(1)}, \ldots, q_v^{(M_k)}\} \in \mathbb{Q}$ based on the underlying distribution.

   - **If**

   $$\frac{1}{M_k}\sum_{i=1}^{M_k}\mathbb{I}_f(\widehat{\theta}_{N_k,r}, q_v^{(i)}) \leq \left(1 - (k\beta_v)^{-1/2}\right)\varepsilon \tag{15}$$

   set $\theta_{\text{sol}} = \widehat{\theta}_{N_k,r}$ and **Exit**.

   - **Else**, goto step (2).

C. Algorithms Termination and Overall Complexity

Note that the maximum number of iterations of both Algorithms 1 and 2 is chosen by the user by selecting the termination parameter k_t. This choice affects directly the cardinality of the sample sets used for design N_k and validation M_k at each iteration, although they converge to fixed values (independent of k_t) at the last iteration. In problems for which the original scenario sample complexity is large, we suggest to use larger k_t. In this way, the sequence of sample bounds N_k starts from a smaller number and does not increase significantly with the iteration counter k. We also remark that, in Algorithm 2, the right hand side of the inequality (15) cannot be negative, which in turn requires β_v to be greater than one. This condition is taken into account in defining β_v in (11). However, we can avoid generating $\beta_v < 1$ by the appropriate choice of k_t. To this end, we solve the inequality $\beta_v \geq 1$ for k_t as follows:

$$\beta_v \doteq \beta_w \left(k_t \ln \frac{2k_t}{\delta} \right)^{-1} \geq 1 \Rightarrow k_t \ln \frac{2k_t}{\delta} \leq \beta_w \Rightarrow \frac{2k_t}{\delta} \ln \frac{2k_t}{\delta} \leq \frac{2\beta_w}{\delta}.$$

For implementation purposes, it is useful to use the function "LambertW" also known as "Omega function" or "product logarithm"[4] $k_t \leq \frac{\beta_w}{\text{LambertW}\left(\frac{2\beta_w}{\delta} \right)}$.

Furthermore, note that the overall complexity of Algorithm 1 and 2 is a random variable, because the number of iterations is random. Indeed, the number of iterations when the algorithm terminates (N_k and M_k) is only known *a posteriori*, while in the scenario approach we can establish *a priori* sample bounds. We remark that the computational cost of solving convex optimization problems does not increase linearly with the number of constraints. Hence, we conclude that, if the algorithms terminate with a smaller number of design samples than the original sample complexity of the scenario problem, the reduction in the number of design samples may significantly improve the overall computational cost. This was the case in all the extensive numerical simulations we have performed.

In the particular case when the constraints are linear matrix inequalities (LMIs), then the reduced-size scenario problem (9) can be reformulated as a semidefinite program by combining N_k LMIs into a single LMI with block-diagonal structure. It is known, see [5], that the

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[3]See footnote 1.

[4]This function is the inverse function of $f(W) = We^W$. In other words, $W = \text{LambertW}[f(W)]$; see e.g. [16] for more details.

computational cost of this problem with respect to the number of diagonal blocks N_k is of the order of $N_k^{3/2}$. Similar discussions hold for Algorithm 2. We conclude that a decrease in N_k can significantly reduce the computational complexity.

Finally, note that the computational cost of validation steps in both presented algorithms is not significant, since they just require *analysis* of a candidate solution for a number of i.i.d. samples extracted from the uncertainty set. For instance, consider the case when $\mathcal{H}_\infty$ performance of an n-dimensional system is of concern. This is generally expressed in terms of an LMI arising from a Riccati inequality. In this case, the number of floating point operations required to solve this LMI inequality is of the order of n^6. On the other hand, checking if a Riccati inequality is satisfied requires checking positive definiteness of a symmetric matrix, which is of complexity n^3, see further discussions in [19, page 1327].

IV. APPLICATION TO HARD DISK DRIVE SERVO DESIGN

In this section, we employ the developed algorithms to solve a non-trivial application. The problem under consideration is the design of a robust track following controller for a hard disk drive (HDD) servo system affected by parametric uncertainty. Servo system in HDD plays a crucial role in increasing the storage capacity by providing a more accurate positioning algorithm. The goal in this application is to achieve the storage density of 10 Tera bit per square inch $(10Tb/in^2)$. It requires the variance of the deviation of read/write head from the center of a data track to be less than 1.16 nanometer. Such a high performance has to be achieved in a robust manner, that is, for all drives produced in a mass production line. On the other hand, some imperfections in the production line such as manufacturing tolerances and slightly different materials or environmental conditions lead to slightly different dynamics over a batch of products.

A voice coil motor (VCM) actuator in a disk drive system can be modeled in the form
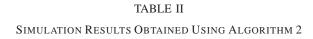
$$P_{VCM} = \sum_{i=1}^{3} \frac{A_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{16}$$

where ζ_i, ω_i and A_i are damping ratio, natural frequency and modal constant for each resonance mode, see [15] for their nominal values. We assume each natural frequency, damping ratio and modal constant to vary by 5%, 5% and 10% from their nominal values respectively. Hence, there are nine uncertain parameters in the plant. The objective is to design a full order dynamic output feedback controller which minimizes the worst case $\mathcal{H}_\infty$ norm of the transfer function from

ε	δ	k_t	Design Samples			Validation Samples			Objective Value			Iteration Number			Computational Time (sec)		
			Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case
0.2	10^{-2}	20	219.4	93	496	37	0	37	0.6106	0.006	0.6241	3.54	1.5	8	271.8	230.5	1195
0.1	10^{-4}	20	561.3	229.8	1397	121.9	0.37	123	0.6178	0.005	0.6275	4.42	1.81	11	1019	874	6025
0.05	10^{-6}	30	1041	387.8	1747	347.5	0.96	349	0.6211	0.04	0.6281	5.96	2.21	10	2633	1809	6963

TABLE I

SIMULATION RESULTS OBTAINED USING ALGORITHM 1

ε	δ	k_t	Design Samples			Validation Samples			Objective Value			Iteration Number			Computational Time (sec)		
			Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case	Mean	Standard Deviation	Worst Case
0.2	10^{-2}	20	141.3	27.9	186	189.2	37.4	249	0.6084	0.005	0.6217	2.28	0.45	3	109.36	41.48	179.53
0.1	10^{-4}	20	276.8	49	381	562	99.6	774	0.6125	0.04	0.6226	2.18	0.36	3	253.3	90.8	456.3
0.05	10^{-6}	30	443.9	93.9	699	1820.2	386.9	2866	0.6169	0.04	0.6253	2.54	0.53	4	600	233.6	1419

TABLE II

SIMULATION RESULTS OBTAINED USING ALGORITHM 2

disturbance to output. This problem can be reformulated in terms of linear matrix inequalities. Uncertain parameters enter into the plant description in a non-affine fashion; therefore, classical robust techniques are unable to solve the problem without introducing conservatism.

The sequential algorithms of Section III are implemented in Matlab using the toolbox R-RoMulOC [13]. In the simulations, we assumed the probability density function of all uncertain parameters to be uniform. The choice of uniform distribution is chosen due to its worst case nature [4]. The number of discarded constraints r in Algorithm 2 is chosen to be zero. The resulting optimization problem is solved for different values of ε, δ and k_t. Furthermore, we run the simulation 100 times for each pair. The mean, standard deviation and worst case values of the number of design samples, validation samples, objective value, the iteration number in

ε	δ	The Scenario Bound	Computational Time (Sec)
0.2	1×10^{-2}	1238	538
0.1	1×10^{-4}	2548	1536
0.05	1×10^{-6}	5240	–

TABLE III

THE SCENARIO BOUND AND THE REQUIRED COMPUTATIONAL TIME FOR THE SAME PROBABILISTIC LEVELS AS TABLES. I AND II

which the algorithm exits and total computational time[5] are tabulated in Table I and Table II. We remark that "design samples" and "validation samples" in Table I and Table II reflect the number of design and validation samples at the iteration when the algorithm exits. Table III shows the scenario bound along with the computational time required for solving the random convex problem for the same probabilistic levels as Tables I and II; we highlight that the number of design parameters in the problem at hand is 153. The average computational time of Tables I and II is much smaller than Table III which further proves the effectiveness of the proposed sequential randomized algorithms. Nevertheless, there are very rare cases when the computational time of the proposed methodology is larger than that of scenario (last column of Table I). When the probabilistic levels become stringent (last row of Table III), we could not solve the scenario problem while, using the two proposed Algorithms 1 and 2 the problem was solved efficiently.

V. CONCLUSIONS

We proposed two new sequential methods for solving in a computational efficient way uncertain convex optimization problems. The main philosophy behind the proposed sequential randomized algorithms stems from the consideration that it is easy, from a computational viewpoint, to validate a given "candidate solution" for a large number of random samples. The algorithms have been tested on a numerical example, and extensive numerical simulations show how the total computational effort is "diluted" by applying the proposed sequential methodology. Future theoretical work will concentrate on handling unfeasible problems.

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[5]All the simulations are carried on a work station with $2.83\,GHz$ Core 2 Quad CPU and $8\,GB$ RAM.

APPENDIX A

PROOF OF THE THEOREM 1

The proof follows similar reasoning to those in [18]. Notice that Algorithm 1, as constructed, always returns a solution θ_{sol}. Our goal is to bound the probability of this solution being "bad", that is we want to bound the probability of the event

ExitBad $\doteq$ {Algorithm 1 returns a bad solution, i.e. $V(\theta_{\text{sol}}) > \varepsilon$}. To do this, we introduce the following events

$$\text{Iter}_k \doteq \{\text{the validation step of the } k\text{th iteration is reached}\},$$

$$\text{Feas}_k \doteq \{\widehat{\theta}_{N_k} \text{ is declared as feasible in the "validation" step}\},$$

$$\text{Bad}_k \doteq \{V(\widehat{\theta}_{N_k}) > \varepsilon\},$$

$$\text{ExitBad}_k \doteq \{\text{Algorithm 1 exits at iteration } k \cap \text{Bad}_k\}.$$

The goal is to bound the probability of the event "ExitBad". Since $\text{ExitBad}_i \cap \text{ExitBad}_j = \emptyset$ for $i \neq j$, the probability of the event "ExitBad" can be reformulated in terms of the event "ExitBad_k" as

$$\Pr\{\text{ExitBad}\} = \Pr\{\text{ExitBad}_1 \cup \text{ExitBad}_2 \cup \cdots \cup \text{ExitBad}_{k_t}\}$$
$$= \Pr\{\text{ExitBad}_1\} + \Pr\{\text{ExitBad}_2\} + \cdots + \Pr\{\text{ExitBad}_{k_t-1}\} + \Pr\{\text{ExitBad}_{k_t}\}.$$
$$(17)$$

From the definition of the event "ExitBad_k" and by considering that to exit at iteration $k \leq k_{t-1}$, the algorithm needs i) to reach kth iteration and ii) to declare $\widehat{\theta}_{N_k}$ feasible in the validation step, for $k = 1, \ldots, k_{t-1}$, we have

$$\Pr\{\text{ExitBad}_k\} = \Pr\{\text{Feas}_k \cap \text{Bad}_k \cap \text{Iter}_k\}$$

$$= \Pr\{\text{Feas}_k \cap \text{Bad}_k \mid \text{Iter}_k\} \Pr\{\text{Iter}_k\} \leq \Pr\{\text{Feas}_k \cap \text{Bad}_k \mid \text{Iter}_k\}$$

$$= \Pr\{\text{Feas}_k \mid \text{Bad}_k \cap \text{Iter}_k\} \Pr\{\text{Bad}_k \mid \text{Iter}_k\} \leq \Pr\{\text{Feas}_k \mid \text{Bad}_k \cap \text{Iter}_k\}.$$

Using the result of Theorem 1 in [9], we can bound the right hand side of the last inequality

$$\Pr\{\text{Feas}_k \mid \text{Bad}_k \cap \text{Iter}_k\} < (1 - \varepsilon)^{M_k}. \tag{18}$$

Combining (17) and (18) results in

$$\Pr\{\text{ExitBad}\} < (1 - \varepsilon)^{M_1} + (1 - \varepsilon)^{M_2} + \cdots + (1 - \varepsilon)^{M_{k_t - 1}}$$

$$+ \Pr\{\text{ExitBad}_{k_t}\} = \sum_{k=1}^{k_t - 1} (1 - \varepsilon)^{M_k} + \Pr\{\text{ExitBad}_{k_t}\}. \qquad (19)$$

The summation in (19) can be made arbitrary small by an appropriate choice of M_k. In particular, by choosing

$$(1 - \varepsilon)^{M_k} = \frac{1}{k^\alpha} \frac{1}{\mathcal{S}_{k_t - 1}(\alpha)} \frac{\delta}{2}, \qquad (20)$$

we have

$$\sum_{k=1}^{k_t - 1} (1 - \varepsilon)^{M_k} = \sum_{k=1}^{k_t - 1} \frac{1}{k^\alpha} \frac{1}{\mathcal{S}_{k_t - 1}(\alpha)} \frac{\delta}{2} = \frac{1}{\mathcal{S}_{k_t - 1}(\alpha)} \frac{\delta}{2} \sum_{k=1}^{k_t - 1} \frac{1}{k^\alpha} = \frac{\delta}{2}. \qquad (21)$$

Note that the choice of the number of design samples in the last iteration guarantees that $\Pr\{\text{ExitBad}_{k_t}\} \leq \delta/2$. The statement follows, combining (19) with (21) and noting that the bound (10) is obtained solving (20) for M_k.

APPENDIX B

PROOF OF THE THEOREM 2

To prove the statement, define the events $\text{Iter}_k, \text{Feas}_k, \text{Bad}_k, \text{ExitBad}_k$, and ExitBad as in the proof of Theorem 1. Then, note that the event Feas_k can be written as

$$\text{Feas}_k = \left\{ \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v) \leq \left(1 - (k\beta_v)^{-1/2}\right) \varepsilon \right\},$$

that is, $\widehat{\theta}_{N_k}$ is declared feasible whenever the feasibility test (15) is passed. Again, the goal is to bound the probability of the event "ExitBad", which can be written as the summation of the events "ExitBad$_k$" as in (17). In turn, for $k \leq k_{t-1}$, we can write

$$\Pr\{\text{ExitBad}_k\} = \Pr\{\text{Feas}_k \cap \text{Bad}_k \cap \text{Iter}_k\} \leq \Pr\{\text{Feas}_k \cap \text{Bad}_k\} \doteq \Pr\{\text{MisClass}_k\},$$

where we denoted MisClass_k the event of misclassification at iteration k.

$$\text{MisClass}_k = \left\{ \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v) \leq \left(1 - (k\beta_v)^{-1/2}\right) \varepsilon \right\} \cap \left\{ V(\widehat{\theta}_{N_{k,r}}) > \varepsilon \right\}, \quad k = 1, \ldots, k_{t-1}.$$

By defining $\rho_k \doteq \left(1 - (k\beta_v)^{-1/2}\right) \varepsilon$ and $\varepsilon_k \doteq (k\beta_v)^{-1/2} \varepsilon$, this event can be rewritten as

$$\text{MisClass}_k \subseteq \left\{ \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v) \leq \rho_k \right\} \cap \left\{ V(\widehat{\theta}_{N_{k,r}}) - \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v) > \varepsilon_k \right\}, \quad k = 1, \ldots, k_{t-1}.$$

Applying the results of [1, Theorem 1], we can bound this event as follows

$$\Pr\{\text{MisClass}_k\} \leq \Pr\left\{\frac{V(\widehat{\theta}_{N_{k,r}}) - \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v)}{\sqrt{V(\widehat{\theta}_{N_{k,r}})}} > \frac{\varepsilon_k}{\sqrt{\varepsilon_k + \rho_k}}\right\}, \quad k = 1, \ldots, k_{t-1}. \tag{22}$$

For any $\eta \in (0, 1)$, the one-sided multiplicative Chernoff inequality [20] guarantees that

$$\Pr\{V(\widehat{\theta}_{N_{k,r}}) - \widehat{V}(\widehat{\theta}_{N_{k,r}}, \mathbf{q}_v) \geq \eta V(\widehat{\theta}_{N_{k,r}})\} \leq e^{\frac{-V(\widehat{\theta}_{N_{k,r}})M_k\eta^2}{2}}. \tag{23}$$

Setting $\eta = \frac{\varepsilon_k}{\sqrt{\varepsilon_k + \rho_k}}\frac{1}{\sqrt{V(\widehat{\theta}_{N_{k,r}})}}$ in (23), combining with inequality (22), we obtain, for $k = 1, \ldots, k_{t-1}$: $\Pr\{\text{MisClass}_k\} \leq e^{\frac{-\varepsilon_k^2 M_k}{2(\varepsilon_k + \rho_k)}} \leq \frac{\delta}{2k_t}$, where the last inequality follows from the choice of M_k in (14). Notice also that the choice of the number of design samples at the last iteration N_{k_t} guarantees that the probability of misclassification at the last iteration ($k = k_t$) is at most $\delta/2$. Therefore, we can write

$$\Pr\{\text{ExitBad}\} \leq \sum_{k=1}^{k_t} \Pr\{\text{MisClass}_k\} \leq \sum_{k=1}^{k_t-1} \frac{\delta}{2k_t} + \Pr\{\text{MisClass}_{k_t}\} = \frac{\delta(k_t - 1)}{2k_t} + \frac{\delta}{2} \leq \delta,$$

which proves the statement.

REFERENCES

[1] T. Alamo, R. Tempo, and E.F. Camacho. Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems. *IEEE Transactions on Automatic Control*, 54:2545–2559, 2009.

[2] T. Alamo, R. Tempo, and A. Luque. On the sample complexity of probabilistic analysis and design methods. In J. Willems, S. Hara, Y. Ohta, and H. Fujioka, editors, *Perspectives in Mathematical System Theory, Control, and Signal Processing*, volume 398 of *Lecture Notes in Control and Information Sciences*, pages 39–50. Springer Berlin / Heidelberg, 2010.

[3] T. Alamo, R. Tempo, A. Luque, and D.R. Ramirez. Randomized methods for design of uncertain systems: Sample complexity and sequential algorithms. *Automatica*, 52:160–172, 2015.

[4] E. Bai, R. Tempo, and M. Fu. Worst-case properties of the uniform distribution and randomized algorithms for robustness analysis. *Mathematics of Control, Signals, and Systems*, 11:183–196, 1998.

[5] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. SIAM, Philadelphia, 2001.

[6] G.C. Calafiore. Random convex programs. *SIAM Journal on Optimization*, 20:3427–3464, 2010.

[7] G.C. Calafiore and M.C. Campi. Uncertain convex programs: randomized solutions and confidence levels. *Mathematical Programming*, 102:25–46, 2004.

[8] G.C. Calafiore and M.C. Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51:742–753, 2006.

[9] G.C. Calafiore and F. Dabbene. A probabilistic analytic center cutting plane method for feasibility of uncertain LMIs. *Automatica*, 43:2022–2033, 2007.

[10] G.C Calafiore, F. Dabbene, and R. Tempo. Research on probabilistic methods for control system design. *Automatica*, 47:1279–1293, 2011.

[11] M.C. Campi and S. Garatti. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM J. on Optimization*, 19:1211–1230, 2008.

[12] M.C. Campi and S. Garatti. A sampling-and-discarding approach to chance-constrained optimization: Feasibility and optimality. 148(2):257–280, 2011.

[13] M. Chamanbaz, F. Dabbene, D. Peaucelle, and R. Tempo. R-RoMulOC: A Unified Tool for Randomized and Robust Multiobjective Control. In *Proc. 8th IFAC Symposium on Robust Control Design*, Bratislava, Slovakia, 2015.

[14] M. Chamanbaz, F. Dabbene, R. Tempo, V. Venkataramanan, and Q-G. Wang. Sequential randomized algorithms for sampled convex optimization. In *Proc. IEEE Multi-Conference on Systems and Control (MSC)*, Hyderabad, India, 2013.

[15] M. Chamanbaz, E. Keikha, V. Venkataramanan, Q-G Wang, and A. Al Mamun. Probabilistic robust approach for discrete multi-objective control of track-following servo systems in hard disk drives. In *Proc. 7th IFAC Symposium on Robust Control Design*, Aalborg, Denmark, 2012.

[16] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the LambertW function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[17] F. Dabbene, P. S. Shcherbakov, and B. T. Polyak. A randomized cutting plane method with probabilistic geometric convergence. *SIAM Journal on Optimization*, 20:3185–3207, 2010.

[18] Y. Oishi. Polynomial-time algorithms for probabilistic solutions of parameter-dependent linear matrix inequalities. *Automatica*, 43:538–545, 2007.

[19] I. R. Petersen and R. Tempo. Robust control of uncertain systems: Classical results and recent developments. *Automatica*, 50:1315 – 1335, 2014.

[20] R. Tempo, G.C. Calafiore, and F. Dabbene. *Randomized Algorithms for Analysis and Control of Uncertain Systems: With Applications*. Springer, 2nd edition, 2013.