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The following publication Huang, J., Li, X., & Wang, T. (2015). Mean-field linear-quadratic-Gaussian (LQG) games for stochastic integral systems. IEEE Transactions on Automatic Control, 61(9), 2670-2675 is available at https://doi.org/10.1109/TAC.2015.2506620

Mean-Field Linear-Quadratic-Gaussian (LQG) Games for Stochastic Integral Systems

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***Abstract*—In this paper, we formulate and investigate a class of mean-field linear-quadratic-Gaussian (LQG) games for stochastic integral systems. Unlike other literature on mean-field games where the individual states follow the controlled stochastic differential equations (SDEs), the individual states in our large-population system are characterized by a class of stochastic Volterra-type integral equations. We obtain the Nash certainty equivalence (NCE) equation and hence derive the set of associated decentralized strategies. The ϵ-Nash equilibrium properties are also verified. Due to the intrinsic integral structure, the techniques and estimates applied here are significantly different from those existing results in mean-field LQG games for stochastic differential systems. For example, some Fredholm equation in the mean-field setup is introduced for the first time. As for applications, two types of stochastic delayed systems are formulated as the special cases of our stochastic integral system, and relevant mean-field LQG games are discussed.**

***Index Terms*—**ϵ-Nash Equilibrium, Fredholm Equation, Mean Field LQG Games, Controlled Stochastic Delay System, Stochastic Volterra Equation.

I. Introduction

Large-population (LP) systems have great importance in both theoretical analysis and practical applications. They emerge naturally from a variety of different fields, including but not limited to mathematical economics [14], multi-agent systems [16], [22], coupled oscillators [23], wireless communication [9], etc. One efficient and powerful methodology to analyze the controlled large-population system is the mean-field game (MFG) which leads to an HJB equation coupled with the Fokker-Planck (FP) equation. The mathematical framework of mean-field games and considerable literature in this direction can be found in [15]. In particular, [2] provides a comprehensive study of the linear-quadratic mean field games via the adjoint equation approach; [3] introduces a complete probabilistic analysis of a large class of stochastic differential games for which the interaction between the players is of mean-field type; [4] studies the mean-field game in the limit of large number of banks in the presence of a common noise; [7] further develops mean-field games with financial applications; [21] analyzes a class of risk-sensitive mean-field stochastic differential games. One important case is the controlled linear stochastic large-population system and related mean-field linear-quadratic-Gaussian (LQG) game. For this case, some relevant works include: the N-person linear differential mean-field games [1], mean-field LQG games with a major player and a large number of minor players [8], mean-field LQG games with nonuniform agents [11], mean-field LQG mixed games with continuum-parameterized minor players [18], etc.

It is remarkable that all the above works mainly focus on the stochastic large-population system driven by stochastic differential equations (SDEs). On the other hand, there exist various models arising from physics and economics which are beyond the framework of standard stochastic differential systems. For instance, the stochastic Volterra equations (SVIEs) are often introduced when studying the production-exchange models ([5]) or nanoscale biophysics ([13]), etc. In addition, the Volterra equation setup also includes the stochastic delay equations from the advertisement model ([6]) as its special case. Therefore, based on these models, it is natural to investigate the mean-field LQG games when studying the mass behavior of considerable negligible agents in terms of stochastic integral systems. For example, when discussing the dynamic optimization of large population wireless interaction knots, there always exist some delay effects in signal transmission, which motivates us to formulate some mean-field stochastic delay games. Inspired by the above discussions, we consider the stochastic large-population system in which the individual states satisfy the following controlled SVIE,

$$\begin{aligned} x_i(t) &= \varphi(t) + \int_0^t f(t,s)x^N(s)ds \\ &\quad + \int_0^t c(t,s)u_i(s)ds + \int_0^t \sigma(t,s)dW_i(s), \end{aligned} \tag{I.1}$$

where $i = 1, 2, \cdots, N$, $x^N = \frac{1}{N}\sum_{j=1}^N x_j$ is the state average term which characterizes the average interaction and mass effects of our population in the *spatial* variable, and φ, f, c, σ are deterministic functions.

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Like the adjoint equation approach [2], the Nash certainty equivalence principle [8], [10] and [12], stochastic controls [9], the ϵ-Nash equilibrium [11] and [18], we adopt the standard analysis route of mean-field games (MFGs) as most current literature does. However, it turns out that various novel features arise in our new integral system framework. We briefly point them out as follows. Firstly, instead of feedback control and Riccati equations in decoupling for differential systems, the optimal (decentralized) strategies are now represented by a new kind of stochastic Fredholm-Volterra equations (SFVEs) for integral systems, whose solvability is discussed under certain conditions. Secondly, in the procedure of deriving the Nash certainty equivalence (NCE) equation, it is necessary to introduce certain Fredholm integral equations (FIEs) instead of Riccati equations. Thirdly, in order to discuss the asymptotic equilibrium of decentralized strategies, some nontrivial extension of classical results is in need, and new techniques are thus developed for error estimates. Last but not least, our study enables us to formulate and discuss mean-field LQG games for two types of stochastic delayed systems, which are new in MFG studies, to our best knowledge.

The rest of this paper is organized as follows. Section II formulates the mean-field game for the stochastic integral system. Section III discusses the NCE equation and consistency condition. The ϵ-Nash equilibrium property of the decentralized strategy is also discussed therein. In Section IV, we discuss some special but important cases. Section V concludes our work.

II. Problem formulation

In this paper the state equation is set to be one-dimensional for the sake of notation simplicity as there is no essential difficulty in extending our results to multi-dimensional cases. Suppose $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$ is a complete filtered probability space on which $W_i(\cdot)$, $1\le i\le N$ are independent scalar-valued Brownian motions and $\mathcal{F}_t=\sigma\{W_i(s),\ 1\le i\le N,\ 0\le s\le t\}$. The large population integral system consists of N individual but negligible agents $\mathcal{A}_i, i=1,2,\cdots,N$, the dynamics of which are given by (I.1). All individual players are coupled via their individual cost functional as follows:

(II.1)
$$\mathcal{J}_i(u_i,u_{-i})=\mathbb{E}\int_0^T\left[(x_i(t)-\gamma x^N(t)-\eta)^2+Ru_i^2(t)\right]dt,$$

where $u_{-i}(\cdot)=(u_1(\cdot),\cdots,u_{i-1}(\cdot),u_{i+1}(\cdot),\cdots,u_N(\cdot))$ is the set of strategies applied except $\mathcal{A}_i$. One of our main targets is to find suitable strategies satisfying the following ϵ-Nash equilibrium property. To this end, let us denote by $L^2_{\mathcal{F}}(0,T;\mathbb{R})$, $C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$, the set of $\mathbb{R}$-valued $\mathcal{F}$-progressively measurable process $X(\cdot)$ such that $\|X(\cdot)\|_1^2\doteq\mathbb{E}\int_0^T|X(s)|^2ds<\infty$ and $\|X(\cdot)\|_2^2\doteq\sup_t\mathbb{E}|X(t)|^2<\infty$, respectively.

Definition 2.1: Given $\widehat{u}_i(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ with $1\le i\le N$, if for any i, $\mathcal{J}_i(\widehat{u}_i,\widehat{u}_{-i})\le\mathcal{J}_i(u_i,\widehat{u}_{-i})+\epsilon$, where $\epsilon>0$, $u_i\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$, then $\{\widehat{u}_i,\ 1\le i\le N\}$ are said to satisfy an ϵ-Nash equilibrium with respect to the functional cost $\mathcal{J}_i$. Now, let

(II.2)
$$\begin{aligned}&L_0\doteq\sup_{t,s\in[0,T]}|f(t,s)|;\quad \sigma_i(\cdot)=\int_0^{\cdot}\sigma(\cdot,s)dW_i(s);\\&\mathcal{C}_1\doteq\frac{1}{R^2}\sup_t\int_0^T\int_0^{s\wedge t}|c(t,r)c(s,r)|^2drds;\\&\mathcal{C}_2\doteq 2T^2\big(1+L_0^2e^{2L_0T}T^2\big)\mathcal{C}_1;\quad \Psi(t,a(\cdot))\doteq\varphi(t)+\sigma_i(t)\\&\qquad+\frac{1}{R}\int_0^T\int_0^{s\wedge t}c(t,r)c(s,r)\big[\gamma a(s)+\eta\big]drds;\\&\mathcal{E}_t(F_1,F_2)\doteq-\frac{1}{R}\mathbb{E}_t\int_t^T c(s,t)\big[F_1(s)-\gamma F_2(s)-\eta\big]ds,\end{aligned}$$

and a generic constant K depends on T, f, c, γ, η, R. As for the coefficients, let us suppose,

(H1) $\varphi(\cdot)\in C(0,T;\mathbb{R})$, f, c, σ are bounded such that $t\mapsto f(t,\cdot)$, $c(t,\cdot)$, $\sigma(t,\cdot)$ are continuous, $R>0$, γ, η are constants. Moreover, $\max\{1,(\gamma-1)^2\}\cdot\mathcal{C}_2<\frac{1}{3}$.

Remark 2.1: Note that the continuity requirement for f, c and σ guarantees the solution of the NCE equation to be continuous as well. Moreover, the main reason of imposing the last inequality in (H1) is due to fixed point arguments in Lemma 2.1 below, and the well-posedness of the NCE equation, etc.

Lemma 2.1: Suppose that (H1) holds true, and let

(II.3)
$$\begin{aligned}Y(t)=\alpha(t)&+\int_0^t f(t,s)Y(s)ds\\&-\frac{1}{R}\int_0^t c(t,s)\mathbb{E}_s\int_s^T c(r,s)Y(r)drds,\end{aligned}$$

where $\alpha(\cdot)\in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$. Then there exists a unique $Y(\cdot)\in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$ satisfying (II.3) such that,

(II.4)
$$\sup_t\mathbb{E}|Y(t)|^2\le 12e^{L_0T}\sup_t\mathbb{E}|\alpha(t)|^2.$$

Proof. By Lemma 1.1 in [19], as well as Fubini theorem, we can transform (II.3) into

(II.5)
$$Y(t)=\widetilde{\alpha}(t)-\frac{1}{R}\int_0^t\widetilde{c}(t,s)\mathbb{E}_s\int_s^T c(r,s)Y(r)drds,$$

where for $t,s\in[0,T]$, $|\widetilde{P}(t,s)|\le L_0e^{L_0(t-s)}$ and

(II.6)
$$\begin{aligned}&\widetilde{c}(t,s)=c(t,s)+\int_s^t\widetilde{P}(t,r)c(r,s)dr,\\&\widetilde{\alpha}(t)=\alpha(t)+\int_0^t\widetilde{P}(t,r)\alpha(r)dr,\\&\widetilde{P}(t,s)=\sum_{k=1}^{\infty}\widetilde{\Lambda}^k(t,s),\quad \widetilde{\Lambda}^1(t,s)=f(t,s),\\&\widetilde{\Lambda}^{k+1}(t,s)=\int_s^t f(t,r)\widetilde{\Lambda}^k(r,s)dr.\end{aligned}$$

By (H1), we can obtain the continuity of $t\mapsto f(t,\cdot)$, $\widetilde{c}(t,\cdot)$ and $\widetilde{\alpha}(\cdot)\in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$. Next we will use fixed point arguments to study equation (II.5). Given $\widehat{y}(\cdot) \doteq y_1(\cdot) - y_2(\cdot)$ with $y_i(\cdot) \in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$, we has $\widehat{Y}(\cdot) \doteq Y_1(\cdot) - Y_2(\cdot)$ with $Y_i(\cdot) \in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$ and

$$\widehat{Y}(t) = -\frac{1}{R}\int_0^t \widetilde{c}(t,s)\mathbb{E}_s \int_s^T c(r,s)\widehat{y}(r)drds.$$

Since $|\widetilde{P}(t,s)|$ is bounded by $L_0 e^{L_0(t-s)}$, by (H1) we has

$$\begin{aligned}\mathbb{E}|\widehat{Y}(t)|^2 &\leq \frac{T^2}{R^2}\int_0^T \int_0^{s\wedge t} |\widetilde{c}(t,r)c(s,r)|^2 drds \sup_t \mathbb{E}|\widehat{y}(t)|^2 \\ &\leq 2T^2\big(1 + L_0^2 e^{2L_0 T}T^2\big)\mathcal{C}_1 \cdot \sup_t \mathbb{E}|\widehat{y}(t)|^2.\end{aligned}$$

Thus, the existence and uniqueness of $Y(\cdot)$ of (II.5) follows from (H1). As for estimate (II.4), it follows from (H1) and,

$$\sup_t \mathbb{E}|Y(t)|^2 \leq 2\sup_t \mathbb{E}|\widetilde{\alpha}(t)|^2 + 2\mathcal{C}_2 \sup_t \mathbb{E}|Y(t)|^2.$$

III. NCE EQUATION SYSTEM AND ASYMPTOTIC EQUILIBRIUM ANALYSIS

This section aims to derive the Nash certainty equivalence principle and verify the relevant asymptotic equilibrium property of decentralized strategies.

A. NCE equation

Given $a(\cdot) \in C(0,T;\mathbb{R})$, for $\mathcal{A}_i$, let us formulate a control problem with state equation and cost functional

(III.1)
$$\begin{cases} X_i(t) = \varphi(t) + \displaystyle\int_0^t \big[f(t,s)a(s) + c(t,s)u_i(s)\big]ds + \sigma_i(t), \\ \bar{J}_i(u_i) = \mathbb{E}\displaystyle\int_0^T \big[(X_i(t) - \gamma a(t) - \eta)^2 + R|u_i(t)|^2\big]dt. \end{cases}$$

The corresponding optimal control is $\bar{u}_i(\cdot) = \mathcal{E}_t(\bar{X}_i, a)$ (see [20] or [24]), with $\mathcal{E}_t$ in (II.2) and

$$\begin{aligned}\bar{X}_i(t) &= \Psi(t,a(\cdot)) + \int_0^t f(t,s)a(s)ds \\ &\quad -\frac{1}{R}\int_0^T \int_0^{s\wedge t} c(t,r)c(s,r)\mathbb{E}^{\mathcal{F}_r}\bar{X}_i(s)drds,\end{aligned} \tag{III.2}$$

with $t \in [0,T]$. Note that for given process $X(\cdot)$,

$$\begin{aligned}&\int_0^t c(t,s)\mathbb{E}_s \int_s^T c(r,s)X(r)drds \\ &= \int_0^T \int_0^{r\wedge t} c(t,s)c(r,s)\mathbb{E}_s X(r)dsdr,\end{aligned} \tag{III.3}$$

which is due to Fubini theorem. So equation (III.2) is a special case of (II.3). Hence from (H1) and Lemma 2.1 it admits a unique solution $\bar{X}_i(\cdot) \in C_{\mathcal{F}}(0,T;L^2(\Omega;\mathbb{R}))$ such that

(III.4)
$$\|\bar{X}_i(\cdot)\|_2^2 \leq C\|\varphi(\cdot)\|_2^2 + C\|a(\cdot)\|^2 + C + C\|\sigma_i(\cdot)\|_2^2.$$

As a result, by taking expectation on both sides of (III.2),

(III.5)
$$\begin{aligned}\mathbb{E}\bar{X}_i(t) &= \varphi(t) + \int_0^t f(t,s)a(s)ds \\ &\quad +\frac{1}{R}\int_0^T \int_0^{s\wedge t} c(t,r)c(s,r)dr\big[\gamma a(s) + \eta - \mathbb{E}\bar{X}_i(s)\big]ds.\end{aligned}$$

Therefore, given $a(\cdot)$ and (H1), we can define $\big[\Gamma a\big](\cdot) \doteq \mathbb{E}\bar{X}_i(\cdot)$ and obtain NCE equation as (see [8], [18])

(III.6)
$$\begin{aligned}\widehat{a}(t) &= \varphi(t) + \int_0^t f(t,s)\widehat{a}(s)ds \\ &\quad +\frac{1}{R}\int_0^T \int_0^{s\wedge t} c(t,r)c(s,r)dr[(\gamma-1)\widehat{a}(s) + \eta]ds.\end{aligned}$$

By (III.3), we know that the arguments in Lemma 2.1 can be used to treat (III.6) as

Theorem 3.1: Suppose that (H1) holds true. Then NCE equation (III.6) admits a unique solution $\widehat{a}(\cdot) \in C(0,T;\mathbb{R})$ satisfying

$$\|\widehat{a}(\cdot)\|^2 \leq K + K\|\varphi(\cdot)\|^2. \tag{III.7}$$

Remark 3.1: The above arguments show that the obtained NCE equation relies on certain stochastic (or deterministic) equations of Volterra-Fredholm type (say, (III.2) and (III.6)) whose methodology is quite different from the Riccati equation approach in [8]-[12], [18].

B. Asymptotic equilibrium analysis

Next $\bar{X}_i(\cdot)$, $\bar{X}^N(\cdot) \doteq \frac{1}{N}\sum_{i=1}^N \bar{X}_i(\cdot)$ are associated with $\widehat{a}(\cdot)$ satisfying (III.6). Now let us define the decentralized strategy $\widehat{u}_i(\cdot) = \mathcal{E}_t(\widehat{x}_i, \widehat{a})$ and associated state equation

$$\begin{aligned}\widehat{x}_i(t) &= \Psi(t,\widehat{a}(\cdot)) + \int_0^t f(t,s)\widehat{x}^N(s)ds \\ &\quad -\frac{1}{R}\int_0^t c(t,s)\mathbb{E}^{\mathcal{F}_s}\int_s^T c(r,s)\widehat{x}_i(r)drds,\end{aligned} \tag{III.8}$$

where $\widehat{x}^N(t) = \frac{1}{N}\sum\limits_{i=1}^N \widehat{x}_i(t)$ and $\Psi(t,\widehat{a}(\cdot))$ is in (II.2). Note that $\widehat{x}^N(\cdot)$ satisfies equation (II.3) with $\alpha(\cdot) = \Psi(\cdot,\widehat{a})$. Under (H1) it then follows from Lemma 2.1 and Theorem 3.1 that $\|\widehat{x}^N(\cdot)\|_2 \leq K$. Plugging $\widehat{x}^N(\cdot)$ back to (III.8), under (H1) we then immediately obtain $\big[\|\widehat{x}_i(\cdot)\|_2 + \|\widehat{u}_i(\cdot)\|_1\big] \leq K$ as well. Hence $\mathcal{J}(\widehat{u}_i, \widehat{u}_{-i})$ is bounded.

Now let us apply $\widehat{u}_j(\cdot)$ in (III.8) to all agents except $\mathcal{A}_i$. Hence for $j \neq i$ the state equations become

(III.9)
$$\begin{aligned}\widehat{x}_j(t) &= \varphi(t) + \sigma_j(t) + \int_0^t \big[f(t,s)\widetilde{x}^N(s) + c(t,s)\widehat{u}_j(s)\big]ds \\ x_i(t) &= \varphi(t) + \sigma_i(t) + \int_0^t \big[f(t,s)\widetilde{x}^N(s) + c(t,s)u_i(s)\big]ds,\end{aligned}$$

where $\widetilde{x}^N(\cdot) = \frac{1}{N}\big[\sum_{l=1,\ l\neq i}^N \widehat{x}_l(\cdot) + x_i(\cdot)\big]$. Now let us state the main result in this subsection.

Theorem 3.2: Suppose (H1) holds true. Then for any $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that for any $N > N_0$, $1 \le i \le N$,

$$\mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i}) - \epsilon \le \inf_{u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathcal{J}_i(u_i, \widehat{u}_{-i}) \le \mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i}).$$

The second inequality above is trivial. Therefore, we only focus on the first one. Before going further, let us introduce

Lemma 3.1: Suppose (H1) holds true. Then for any $\varepsilon > 0$, there exists $N_1(\varepsilon)$, N_2 such that for any $N > N_3 \doteq \max\{N_1, N_2\}$, $\|\widetilde{x}^N(\cdot) - \widehat{a}(\cdot)\|_2^2 < \frac{K}{N} < \varepsilon$.

Proof. For $x_i(\cdot)$ in (III.9), define $\widetilde{u}_i(\cdot) = \mathcal{E}_t(x_i, \widehat{a})$ with $\mathcal{E}_t$ in (II.2). Firstly we need the boundedness of $\|x_i(\cdot)\|_2$ from which one can obtain a similar result for $\widetilde{u}_i(\cdot)$. To this end, let us look at $\widetilde{x}^N(\cdot)$ which satisfies (II.3) with

$$\alpha(t) = \varphi(t) + \frac{1}{R}\int_0^T \int_0^{s\wedge t} c(t,r)c(s,r)dr[\gamma\widehat{a}(s) + \eta]ds + \frac{1}{N}\sum_{i=1}^N \sigma_i(t) + \frac{1}{N}\int_0^t c(t,s)\big[u_i(s) - \widetilde{u}_i(s)\big]ds.$$

From estimate (II.4) and Theorem 3.1, we then have $\|\widetilde{x}^N(\cdot)\|_2^2 \le K + \frac{K}{N^2}\|u_i(\cdot) - \widetilde{u}_i(\cdot)\|_1^2$.

On the other hand, the previous boundedness of $\mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i})$ indicates that it is sufficient to choose $u_i(\cdot)$ satisfying $\mathcal{J}_i(u_i, \widehat{u}_{-i}) \le \mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i})$ and $\|u_i(\cdot)\|_1 \le K$. By the notation of $\mathcal{E}_t(x_i, \widehat{a})$ in (II.2) we have $\|u_i(\cdot) - \widetilde{u}_i(\cdot)\|_1^2 \le K + K\|x_i(\cdot)\|_2^2$. As a result, $\|\widetilde{x}^N(\cdot)\|_2^2 \le K + \frac{K}{N^2}\big[1 + \|x_i(\cdot)\|_2^2\big]$.

Note that from (III.9) one has $\|x_i(\cdot)\|_2^2 \le K + K\|\widetilde{x}^N(\cdot)\|_2^2 + K\|u_i(\cdot)\|_1^2$. Hence to sum up one has $\|x_i(\cdot)\|_2^2 \le K + \frac{K}{N^2}\big[1 + \|x_i(\cdot)\|_2^2\big]$. Therefore, there exists N_2, for any $N > N_2$, $\frac{K}{N} < \frac{1}{2}$ from which we have the boundedness of $\|x_i(\cdot)\|_2$.

From the definitions of $\widetilde{x}^N(\cdot)$ and $\bar{X}^N(\cdot)$, we have $\widetilde{x}^N(\cdot) - \widehat{a}(\cdot)$ satisfying equation (II.3) with

$$\alpha(t) = \bar{X}^N(t) - \widehat{a}(t) + \frac{1}{N}\int_0^t c(t,s)[u_i(s) - \widetilde{u}_i(s)]ds + \frac{1}{R}\int_0^t c(t,s)\mathbb{E}^{\mathcal{F}_s}\int_s^T c(r,s)[\bar{X}^N(r) - \widehat{a}(r)]drds.$$

Therefore, under (H1), recalling the boundedness of $\|x_i(\cdot)\|_2$ obtained above for $N > N_2$, we have

$$\|\widetilde{x}^N(\cdot) - \widehat{a}(\cdot)\|_2^2 \le \frac{K}{N^2} + K\|\bar{X}^N(\cdot) - \widehat{a}(\cdot)\|_2^2. \tag{III.10}$$

For $\bar{X}_i(\cdot)$ in (III.2), we can see that $\bar{X}^N(\cdot) - \widehat{a}(\cdot)$ satisfies equation (II.3) with $f = 0$ and $\alpha(t) = \frac{1}{N}\sum_{i=1}^N \sigma_i(t)$. Hence from estimate (II.4), one has $\|\bar{X}^N(\cdot) - \widehat{a}(\cdot)\|_2^2 \le \frac{K}{N}$. Putting it back to (III.10) we obtain the desired result. □

Remark 3.2: Recall that to obtain similar error estimate in SDEs framework, one can apply Itô formula and Gronwall inequality, see e.g. Lemma A.2 and Appendix B of [18]. This is mainly due to the special features for differential systems like a feedback form of the optimal strategy and related Riccati equations. However, such features are no longer available in our integral systems. Thus some alternative techniques are needed, for example, the presence of N_2 in Lemma 3.1. On the other hand, though N_2 is used to obtain the boundedness of $\|x_i(\cdot)\|_2$, this term is not needed when $f(\cdot,\cdot) = 0$ as in (III.9). Actually, we can estimate $\|x_i(\cdot)\|_2$ directly by $\|\varphi(\cdot)\|$, $\|\sigma_i(\cdot)\|$ and $\|u_i(\cdot)\|_1$ without $\|\widetilde{x}^N(\cdot)\|_2$. This suggests some technical difference when introducing coupling term $\widetilde{x}^N(\cdot)$ and it is a new interesting phenomenon in error estimate of mean-field games for integral systems.

Proof of Theorem 3.2. For convenience, let us denote by $\lambda_a(\cdot) \doteq \widetilde{x}^N(\cdot) - \widehat{a}(\cdot)$. By the proof of Lemma 3.1, for any $\varepsilon > 0$, there exists N_3, for $N > N_3$, $\|\lambda_a(\cdot)\|_2^2 < \frac{K}{N} < \varepsilon$, $\|x_i(\cdot)\|_2 + \|u_i(\cdot)\|_1 \le K$. Given the cost functionals in (II.1), (III.1), for some $N_4 > 0$, and any $N > N_4$ we need to prove,

$$\mathcal{J}_i(u_i, \widehat{u}_{-i}) \ge \bar{J}_i(\bar{u}_i) - \tfrac{K}{\sqrt{N}}. \tag{III.11}$$

To this end, introduce

(III.12)

$$\xi_i(t) = \varphi(t) + \sigma_i(t) + \int_0^t f(t,s)\widehat{a}(s)ds + \int_0^t c(t,s)\zeta_i(s)ds$$

$$\mathbb{J}_i(v_i) = \mathbb{E}\int_0^T \big[|\xi_i(t) - \gamma\widehat{a}(t) - \eta|^2 + R|\zeta_i(t)|^2\big]dt,$$

where $\zeta_i(\cdot) \doteq [v_i(\cdot) + \mathcal{E}_\cdot(\xi_i, \widehat{a})]$, $v_i(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$. By Lemma 2.1, the existence and uniqueness of $\xi_i(\cdot)$ can be ensured under (H1). Obviously $\mathbb{J}_i(v_i) \ge \bar{J}_i(\bar{u}_i)$. Choose $v_i(t) = u_i(t) - \mathcal{E}_t(x_i, \widehat{a})$ with $x_i(\cdot)$ in (III.9). Then both $\xi_i(\cdot) - \bar{X}_i(\cdot)$ and $x_i(\cdot) - \xi_i(\cdot)$ satisfy equation (II.3) with $f = 0$ and $\alpha(t) = \int_0^t c(t,s)v_i(s)ds$, $\alpha(t) = \int_0^t f(t,s)[\widetilde{x}^N(s) - \widehat{a}(s)]ds$ respectively. Hence from estimate (II.4),

$$\begin{aligned} &\|\xi_i(\cdot) - \bar{X}_i(\cdot)\|_2^2 \le K\|v_i(\cdot)\|_1^2 \le K, \\ &\|x_i(\cdot) - \xi_i(\cdot)\|_2^2 \le K\|\lambda_a(\cdot)\|_2^2. \end{aligned} \tag{III.13}$$

Note that by (III.4) and (III.7), $\|\bar{X}_i(\cdot)\|_2$ is bounded. Hence so is $\|\xi_i(\cdot)\|_2$. Recall that for some N_3 and any $N > N_3$, $\|\widetilde{x}^N(\cdot)\|_2$, $\|x_i(\cdot)\|_2$ are bounded. Hence it follows from the estimates in (III.13) that,

$$|\mathcal{J}_i(u_i, \widehat{u}_{-i}) - \mathbb{J}_i(v_i)| \le K\|\lambda_a(\cdot)\|_2, \tag{III.14}$$

where K does not depend on λ_a. Considering $\mathbb{J}_i(v_i) \ge \bar{J}_i(\bar{u}_i)$, we then get the desired result (III.11).

On the other hand, for $\widehat{x}_i$, $\bar{X}_i$ in (III.8) and (III.2) associated with $\widehat{a}$, we have $[\widehat{x}_i - \bar{X}_i](\cdot)$ satisfying equation (II.3) with $f = 0$ and $\alpha(t) = \int_0^t f(t,s)[\widehat{x}^N(s) - \widehat{a}(s)]ds$. Consequently under (H1) by estimate (II.4) and Lemma 3.1 with $u_i(\cdot) = \widehat{u}_i(\cdot)$, we then obtain $\|\widehat{x}_i(\cdot) - \bar{X}_i(\cdot)\|_2^2 \le \frac{K}{N} \le \varepsilon$ for some $N_5 > 0$, any $N > N_5$ and $1 \le i \le N$. Therefore, by definitions of $\mathcal{J}_i$ and $\bar{J}_i$, the boundedness of $\bar{u}_i(\cdot)$ and Lemma 3.1, for some $N_6 > 0$ and any $N > N_6$, we can obtain

$$\big|\mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i}) - \bar{J}_i(\bar{u}_i)\big| \le \frac{K}{\sqrt{N}}. \tag{III.15}$$

At last, for $N_0 = \max\{N_i,\ 3 \le i \le 6\}$ and any $N > N_0$, it follows from (III.11) and (III.15) that

$$\mathcal{J}_i(u_i, \widehat{u}_{-i}) \ge \bar{J}_i(\bar{u}_i) - \tfrac{K}{\sqrt{N}} \ge \mathcal{J}_i(\widehat{u}_i, \widehat{u}_{-i}) - \tfrac{K}{\sqrt{N}}.$$

The conclusion thus holds. $\square$

IV. Some special cases

The study of above SVIEs not only is of independent mathematical interest, but also enables us to apply the results to three special but important cases. For the sake of simplicity, we suppose $f(\cdot,\cdot)=0$ in the following analysis.

A. SDEs case

For $1\le i\le N$, and $x(0)=x$, consider

$$dx_i(s)=\big[B(s)x_i(s)+C(s)u_i(s)\big]ds+D(s)dW_i(s),$$

where the cost functional is defined in (II.1). The current literature on mean-field games mainly focuses on the above (linear) SDEs. In this setting, the NCE equation is (III.6) with

$$\begin{aligned}\varphi(t)&=e^{\int_0^t B(s)ds}x,\quad c(t,s)=C(s)e^{\int_s^t B(r)dr},\\ \sigma(t,s)&=D(s)e^{\int_s^t B(r)dr},\quad T\ge t\ge s\ge 0.\end{aligned}\tag{IV.1}$$

Define the decentralized strategy with $t\in[0,T]$,
(IV.2)

$$\widehat{u}_i(t)=-\frac{C(t)}{R}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\int_t^s B(r)dr}[\widehat{x}_i(s)-\gamma\widehat{a}(s)-\eta]ds,$$

where $\widehat{x}_i(\cdot)$ is the related state process.

Theorem 4.1: Suppose $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are bounded and deterministic such that (H1) holds true. Then with $\widehat{u}_i(\cdot)$ in (IV.2), the set of strategies $\big\{\widehat{u}_i(\cdot),\ 1\le i\le N\big\}$ for N agents is an ϵ-Nash equilibrium.

Remark 4.1: By the particular form of the state equation and optimal control (note that s and t are separable in (IV.2)), one can obtain the feedback form for $\widehat{u}_i(\cdot)$ via Riccati equations (see [8], [11] and [18]). Such procedure is different from ours here. Moreover, these skills are intractable to be applied in more general models like the state equations with delay, as we discussed below. In such sense, our approach applied here is more flexible and can fill this technical gap.

B. SDEs with delay in the state process

Suppose the cost functional is (II.1) and the state equation is given by

$$\begin{aligned}dx_i(t)=&\left[A(t)x_i(t-h)+\int_{t-h}^t B(t,s)x_i(s)ds\right]dt\\ &+C(t)u_i(t)dt+D(t)dW_i(t),\end{aligned}$$

where $h>0$, $x_i(t)=k(t)$, $t\in[-h,0]$, and $k(t)$ is bounded. Hence the delay term appears in the state process. Let us introduce function $\Phi_1(\cdot,\cdot)$ with $\Phi_1(0,0)=1$, $\Phi_1(t,s)=0$ for $t<0$,

$$\frac{\partial\Phi_1(t,s)}{\partial t}=A(t)\Phi_1(t-h,s)+\int_{t-h}^t B(t,r)\Phi_1(r,s)dr.$$

By [17], we can transform the above stochastic delay equation into (I.1) with $f=0$,

$$\begin{aligned}\varphi(t)=&\Phi_1(t,0)k(0)+\int_{-h}^0\big[\Phi_1(t,s+h)A(s+h)\\ &+\int_0^h\Phi_1(t,u)B(u,s)du\big]k(s)ds,\\ c(t,s)=&\Phi_1(t,s)C(s),\quad\sigma(t,s)=\Phi_1(t,s)D(s).\end{aligned}$$

Moreover $\Phi_1(\cdot,\cdot)$ is bounded and continuous in t. In this situation, the NCE equation becomes

$$\begin{aligned}\widehat{a}(t)=&\varphi(t)+\frac{1}{R}\int_0^T\int_0^{s\wedge t}\Phi_1(t,r)\Phi_1(s,r)|C(r)|^2dr\\ &\cdot\big\{\gamma\widehat{a}(s)+\eta-\widehat{a}(s)\big\}ds,\qquad t\in[0,T].\end{aligned}$$

Given $\widehat{a}(\cdot)$ being a solution of the NCE equation, define the decentralized strategy with $t\in[0,T]$,
(IV.3)

$$\widehat{u}_i(t)=-\frac{C(t)}{R}\mathbb{E}^{\mathcal{F}_t}\int_t^T\Phi_1(s,t)[\widehat{x}_i(s)-\gamma\widehat{a}(s)-\eta]ds,$$

where $\widehat{x}_i(\cdot)$ is the corresponding state variable.

Theorem 4.2: Suppose that $A(\cdot)$, $B(\cdot,\cdot)$, $C(\cdot)$ and $D(\cdot)$ are bounded deterministic functions such that (H1) holds true. Then with $\widehat{u}_i(\cdot)$ in (IV.3), the set of strategies $\big\{\widehat{u}_i(\cdot),\ 1\le i\le N\big\}$ for N players satisfies the ϵ-Nash equilibrium.

C. SDEs with delay in the control

In this case, the cost functional is (II.1) and the state equation is given by

$$dx_i(t)=A(t)x_i(t)dt+C(t)u_i(t-h)dt+D(t)dW_i(t),$$

where $h>0$, $x_i(t)=k(t)$, $t\in[-h,0]$, $C(t)=0$ with $t<h$. The delayed term appears in the control variable. Similarly, by introducing function Φ_2 as

$$\frac{\partial\Phi_2(t,s)}{\partial t}=A(t)\Phi_2(t,s)\quad\Phi_2(s,s)=1,\quad t,s\in[0,T],$$

we can transform the above state equation into (I.1) with

$$\begin{aligned}&\varphi(t)=\Phi_2(t,0)k(0),\quad c(t,s)=\Phi_2(t,s+h)C(s+h),\\ &\sigma(t,s)=\Phi_2(t,s)D(s),\quad t,s\in[0,T].\end{aligned}$$

The NCE equation in this situation becomes

$$\begin{aligned}\widehat{a}(t)=&\varphi(t)+\frac{1}{R}\int_0^T\int_0^{s\wedge t}\Phi_2(t,r+h)\Phi_2(s,r+h)\\ &|C(r+h)|^2dr\cdot\big\{\gamma\widehat{a}(s)+\eta-\widehat{a}(s)\big\}ds.\end{aligned}$$

Given $\widehat{a}(\cdot)$ being the NCE solution, let us define the strategy $\widehat{u}_i(\cdot)$ with $i=1,2,\cdots,N$,

$$\begin{aligned}\widehat{u}_i(t)=&-\frac{1}{R}C(t+h)\mathbb{E}^{\mathcal{F}_t}\int_t^T\Phi_2(s,t+h)\\ &[\widehat{x}_i(s)-\gamma\widehat{a}(s)-\eta]ds,\end{aligned}\tag{IV.4}$$

where $\widehat{x}_i(\cdot)$ is the state process.

Theorem 4.3: Suppose that $A(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are bounded such that (H1) holds true. Then the set of strategies $\{\widehat{u}_i(\cdot),\ 1 \leq i \leq N\}$ associated with (IV.4) is an ϵ-Nash equilibrium.

V. Conclusion

Herein, we investigate a class of mean-field LQG games for stochastic Volterra integral systems. The NCE consistency condition is derived based on the Fredholm equations and the ϵ-Nash equilibrium property of decentralized controls is also established. Our work is the first attempt to the LQG games with stochastic integral systems and it suggests various future research directions. For example, more discussions to mean-field games of stochastic delayed systems with more general lag characteristics. It is anticipated that some new consistency conditions will be given which depend on the delay characteristics.

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