DISTRIBUTED REINFORCEMENT LEARNING VIA GOSSIP

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Abstract: We consider the classical TD(0) algorithm implemented on a network of agents wherein the agents also incorporate the updates received from neighboring agents using a gossip-like mechanism. The combined scheme is shown to converge for both discounted and average cost problems.

Key words: reinforcement learning; gossip; stochastic approximation; TD(0); distributed algorithm

1 Introduction

Reinforcement learning with function approximation has been a popular framework for approximate policy evaluation and dynamic programming for Markov decision processes (Bertsekas 2012; Gosavi 2003; Lewis and Liu 2013; Powell 2007; Szepesvari 2010). In view of the growing interest in control across communication networks, there has been a growing need to consider distributed or multi-agent versions of these schemes. While there has been some early work in this direction, analysis of provably convergent schemes is lacking. (See, e.g., Lauer and Riedmiller (2000), Littman and Boyan (1993), Pendrith (2000), Weiss (1995), also Busoniu et al (2008) and Panait and Luke (2005) for surveys. The work closest to ours in spirit is Macua et al (2012).) The present article aims at filling up this lacuna. Specifically, we consider a distributed version of the celebrated TD(0) algorithm implemented across a network of processors or 'agents', who communicate with each other and incorporate, in addition to their own measurements, the estimates of their neighbors. For the latter aspect, we borrow a simple averaging scheme from gossip algorithms (Shah, 2008). We

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prove the convergence of this scheme. It may be noted that we *do not* prove consensus, in fact consensus is an unreasonable expectation here. This is so because each agent potentially has a different set of basis functions, even of a different cardinality. We do, however, justify the proposed scheme in terms of a certain performance measure.

Next section describes our convergence results for the infinite horizon discounted cost problem. Section 3 extends them to the average cost problem. Section 4 comments upon the results.

2 Discounted cost

Let $\{X_n\}$ denote an irreducible Markov chain on a finite state space $S := \{1, 2, \dots, m\}$ with transition matrix $P := [[p(i, j)]]_{i,j \in S}$, and an associated 'running' cost function $c : S \times S \times \mathcal{R}$. Thus c(i, j) denotes the cost associated with the transition from i to j. (While we have a controlled Markov chain in mind, we are interested in estimating the cost for a fixed policy, so we do not render explicit the policy dependence of P, c for sake of notational ease.) Consider the problem of estimating the infinite horizon cost

$$J(x) := E_x \left[\sum_{t=0}^{\infty} \alpha^m c(X_t, X_{t+1}) \right],$$

 $\alpha \in (0, 1)$ being the discount factor. The original TD(0) algorithm for approximate evaluation of J begins with the a priori approximation $J(\cdot) \approx \phi(\cdot)r = \sum_{i=1}^{K} r_i \phi_i(\cdot)$. Here $\phi(\cdot) = [\phi_1(\cdot) : \phi_2(\cdot) : \cdots : \phi_K(\cdot)]^T$ with $\phi_i :=$ the *i*th feature vector (these are kept fixed), and $r := [r_1, \cdots, r_K]^T$ are the weights that are to be learnt. The actual algorithm for doing so is as follows (Tsitsiklis and Van Roy, 1997):

$$r_{t+1} = r_t + \gamma_t \phi(X_t) [c(X_t, X_{t+1}) + \alpha \phi(X_{t+1})^T r_t - \phi(X_t)^T r_t],$$
(1)

where the step-sizes $\gamma_t > 0$ satisfy $\sum_t \gamma_t = \infty$, $\sum_t \gamma_t^2 < \infty$. A convergence proof and error estimates relative to the exact J may be found in Tsitsiklis and Van Roy (1997). We sketch an alternative convergence proof of independent interest, using the 'o.d.e.' (for ordinary differential equation) approach of Derevitskii and Fradkov (1974) and Llung (1977). For simplicity, we rely on the exposition of Borkar (2008). Let η denote the unique stationary probability vector for the chain and D the diagonal matrix whose *i*th diagonal entry is $\eta(i)$. By Corollary 8, p. 74, Borkar (2008), the 'limiting o.d.e.' for the above iteration is

$$\dot{r}(t) = \phi^T D\bar{c} + \alpha \phi^T DP \phi r(t) - \phi^T D\phi r(t) := h(r(t))$$
(2)

for $h(x) := \phi^T D \bar{c} + \alpha \phi^T D P \phi x - \phi^T D \phi x$, $\bar{c} := \sum_j p(i,j) c(i,j)$. Then

$$h_{\infty}(x) := \lim_{a \uparrow \infty} \frac{h(ax)}{a} = \alpha \phi^T D P \phi x - \phi^T D \phi x.$$

It is easy to see that $\frac{1}{a}h(ax) \to h_{\infty}(x)$ uniformly on \mathbb{R}^{K} .

Theorem 1: Under the above assumptions, $r_t \to \hat{r}$ a.s., where \hat{r} is the unique solution to $h(\hat{r}) = 0$.

Proof: The 'scaled o.d.e.' $\dot{r}(t) = h_{\infty}(r(t))$ is a linear system with the origin as its globally asymptotically stable equilibrium, in fact $V(r(t)) = ||r(t)||^2$ is a Liapunov function, as seen from Lemma 9 of Tsitsiklis and Van Roy (1997) with r^* therein replaced by the zero vector. By Theorem 9, p. 75, Borkar (2008), $\sup_t ||r_t|| < \infty$ a.s. In turn, (2) has \hat{r} as its globally asymptotically stable equilibrium, again $V(r(t)) = ||r(t) - \hat{r}||^2$ is a Liapunov function, as seen from Lemma 9 of Tsitsiklis and Van Roy (1997). The claim follows by Theorem 7 – Corollary 8, p. 74, Borkar (2008).

We now describe the distributed version of this scheme. Consider n agents sitting on the nodes of a connected graph, each with a different set of feature vectors. We denote by $\mathcal{N}(i)$ the set of neighbors of i. Let the feature vectors of the *i*th agent be denoted by $\phi_1^i, \phi_2^i, \dots, \phi_{n_i}^i$, with $\Phi^i := [\phi_1^i : \phi_2^i : \dots : \phi_{n_i}^i]^T$. Let q(i, j) denote the probability by which *i*th agent polls agent $j \in \mathcal{N}(i)$. The *i*th agent runs the following n_i -dimensional iteration:

$$r_{t+1}^{i} = r_{t}^{i} + \gamma_{t}\phi^{i}(X_{t})[c(X_{t}, X_{t+1}) + \alpha\phi^{Y_{t+1}^{i}}(X_{t+1})^{T}r_{t}^{Y_{t+1}^{i}} - \phi^{i}(X_{t})^{T}r_{t}^{i}].$$
 (3)

Here Y_t^i is a [1, 2...n] valued random variable taking value j with probability q(i, j). We further assume it is independent of $\{X_s, Y_s^j, j \neq i, s \leq t; Y_s^i, s < t\}$. We make the following key assumptions:

- (A1) $\phi_1^i, \phi_2^i, \dots, \phi_{n_i}^i$ are linearly independent for all *i*.
- (A2) The Markov chain $\{X_t\}$ is irreducible and aperiodic.
- (A3) The stochastic matrix Q := [[q(i, j)]] is irreducible, aperiodic and doubly stochastic.

Remark: The *i*th row $q(i, \cdot)$ of the matrix Q indicates the 'weights' node *i* assigns to its neighbors. Since it stands to reason that each node values its own opinions, q(i, i) > 0, which automatically ensures aperiodicity.

Rewrite above iteration as

$$r_{t+1}^{i} = r_{t}^{i} + \gamma_{t} [\phi^{i}(X_{t})\overline{c}(X_{t}) + \alpha \phi^{i}(X_{t})(\sum_{j=1}^{n} q(i,j) \sum_{s=1}^{m} p(X_{t},s)\phi^{j}(s)^{T}r_{t}^{j}) - \phi^{i}(X_{t})\phi^{i}(X_{t})^{T}r_{t}^{i} + M_{t+1}^{i}],$$

$$(4)$$

where $\overline{c}(i) = \sum_{j} p(i, j) c(i, j)$, $\overline{c} = [\overline{c}(1), \overline{c}(2), \dots, \overline{c}(m)]^T$, and $M_{t+1}^i, t \ge 0$, is a martingale difference sequence w.r.t. $\sigma(X_n, Y_n, n \le t)$, given by

$$M_{t+1}^{i} := (c(X_{t}, X_{t+1})\phi^{i}(X_{t}) - \overline{c}(X_{t})\phi^{i}(X_{t})) + \alpha \Big(\phi^{i}(X_{t})\phi^{Y_{t+1}^{i}}(X_{t+1})^{T}r_{t}^{Y_{t+1}^{i}} - \phi^{i}(X_{t})(\sum_{j=1}^{n}q(i,j)\sum_{s=1}^{m}p(X_{t},s)\phi^{j}(s)^{T}r_{t}^{j})\Big).$$

We have:

$$\begin{split} \sum_{l=1}^{m} d(l)\phi^{i}(l)\overline{c}(l) &= \Phi^{i^{T}}D\overline{c}, \\ \sum_{l=1}^{m} d(l)\phi^{i}(l)\phi^{i}(l)^{T}\overline{r}_{t}^{i} &= \Phi^{i^{T}}D\Phi^{i}\overline{r}_{t}^{i}, \\ \sum_{j=1}^{n} q(i,j)\sum_{l=1}^{m} d(l)\phi^{i}(l)\sum_{s=1}^{m} p(l,s)\phi^{j}(s)^{T}\overline{r}_{t}^{j} &= \Phi^{i^{T}}DP\sum_{j=1}^{n} q(i,j)\Phi^{j}\overline{r}_{t}^{j} \end{split}$$

By Corollary 8, p. 74, Borkar (2008), the o.d.e. corresponding to (3) is

$$\overline{r}^{i} = \Phi^{i^{T}} D\overline{c} + \alpha \Phi^{i^{T}} DP \sum_{j=1}^{n} q(i,j) \Phi^{j} \overline{r}^{j} - \Phi^{i^{T}} D\Phi^{i} \overline{r}^{i}.$$
(5)

$$\begin{bmatrix} \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} \overline{c} \\ \cdot \\ \vdots \\ \overline{c} \end{bmatrix}^{-} \begin{bmatrix} \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^{n} \end{bmatrix} \begin{bmatrix} \overline{r}^{i} \\ \cdot \\ \vdots \\ \overline{r}^{n} \end{bmatrix}$$
$$+ \alpha \begin{bmatrix} \Phi^{1^{T}}D & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} P\sum_{j=1}^{n}q(1,j)\Phi^{j}\overline{r}^{j} \\ \cdot \\ P\sum_{j=1}^{n}q(i,j)\Phi^{j}\overline{r}^{j} \\ \cdot \\ P\sum_{j=1}^{n}q(n,j)\Phi^{j}\overline{r}^{j} \end{bmatrix}$$

$$= \begin{bmatrix} \Phi^{1^{T}}D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} \overline{c} \\ \cdot \\ \overline{c} \\ \cdot \\ \overline{c} \end{bmatrix}$$
$$- \begin{bmatrix} \Phi^{1^{T}}D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} \Phi^{1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n} \end{bmatrix} \begin{bmatrix} \overline{r}^{1} \\ \cdot \\ \overline{r}^{i} \\ \cdot \\ \overline{r}^{n} \end{bmatrix}$$
$$+ \alpha \begin{bmatrix} \Phi^{1^{T}}D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}}D \end{bmatrix} \begin{bmatrix} q(1,1)P & \cdots & q(1,n)P \\ \vdots & \ddots & \vdots\\ q(n,1)P & \cdots & q(n,n)P \end{bmatrix} \begin{bmatrix} \Phi^{1}\overline{r}^{1} \\ \cdot \\ \Phi^{i}\overline{r}^{i} \\ \cdot \\ \Phi^{n}\overline{r}^{n} \end{bmatrix}.$$

Thus we get the following equation:

$$\begin{split} \dot{\bar{r}} = \begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} \bar{c}\\ \cdot\\ \bar{c}\\ \cdot\\ \bar{c} \end{bmatrix} + \\ & \alpha \begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} q(1,1)P & \cdots & q(1,n)P\\ \vdots & \ddots & \vdots\\ q(n,1)P & \cdots & q(n,n)P \end{bmatrix} \\ & & \times \begin{bmatrix} \Phi^{1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n} \end{bmatrix} \begin{bmatrix} \bar{r}^{1}\\ \cdot\\ \bar{r}^{i}\\ \cdot\\ \bar{r}^{n} \end{bmatrix} \\ & - \begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} \Phi^{1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n} \end{bmatrix} \begin{bmatrix} \bar{r}^{1}\\ \cdot\\ \bar{r}^{i}\\ \cdot\\ \bar{r}^{n} \end{bmatrix} \end{split}$$

Consider an augmented state space $S' := \{1, 2, ..n\} \times S$. Order it as

$$\{(1,1),(1,2),...(1,m),(2,1),...(2,m),....,(n,1),...,(n,m)\}.$$

Define

$$\begin{split} \tilde{p}((i,x),(j,y)) &:= q(i,j) \times p(x,y), \\ \tilde{c}((i,x),(j,y)) &:= c(x,y), \\ \psi_{jk}((i,x)) &:= \Phi_k^i(x) \text{ if } j = i, \text{ else } 0, \\ \Psi &:= [\psi_{11},\psi_{12},...\psi_{1n_1},....\psi_{n1},\psi_{n2},...\psi_{nn_n}] \end{split}$$

$$\begin{split} \rho &:= \left[\left[\tilde{P}((i,x),(j,y)) \right] \right] = \begin{bmatrix} q(1,1)P & \cdots & q(1,n)P \\ \vdots & \ddots & \vdots \\ q(n,1)P & \cdots & q(n,n)P \end{bmatrix}, \\ \Psi &:= \begin{bmatrix} \Phi^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^n \end{bmatrix}, \\ \Psi^T &:= \begin{bmatrix} \Phi^{1^T} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi^{n^T} \end{bmatrix}, \\ \nu &:= \frac{1}{n} \begin{bmatrix} D & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D \end{bmatrix}, \\ \tilde{c} &:= \begin{bmatrix} & \ddots \\ & \vdots \\ & E[\tilde{c}((i,x),(j,y))|(i,x)] \\ & \vdots \end{bmatrix} = \begin{bmatrix} \overline{c} \\ \vdots \\ \overline{c} \\ \vdots \\ \overline{c} \end{bmatrix}. \end{split}$$

Then the ODE for $r(\cdot)$ is

$$\dot{r} = n \left(\Psi^T \nu \tilde{c} + \alpha \Psi^T \nu \rho \Psi r - \Psi^T \nu \Psi r \right).$$
(6)

Lemma 1: Ψ is a full rank matrix.

Proof: This is immediate from (A1).

Lemma 2: ρ is irreducible (hence positively recurrent) and aperiodic under (A2)-(A3).

Proof: Let $p^{(n)}(i, j), q^{(n)}(k, \ell), \tilde{p}^{(n)}((k, i), (\ell, j))$ denote the *n*-step probabilities of going from *i* to *j*, *k* to ℓ , (k, i) to (ℓ, j) resp. for $n \ge 1$. Since P, Q are

irreducible aperiodic, there exist n_0, n'_0 such that $p^{(n)}(i, j) > 0, q^{(n')}(k, \ell) > 0$ for $n \ge n_0, n' \ge n'_0$ resp. So for $n \ge n_0 \lor n'_0, \tilde{p}^{(n)}((k, i), (\ell, j)) > 0$. The claim follows.

Let $(Z_t, X_t), t \ge 0$, denote the augmented Markov chain with transition matrix ρ . Note that the diagonal entries of ν are > 0 and are the stationary probabilities under ρ , i.e., letting η denote the ordered vector thereof, η is a unique stationary distribution under ν .

Theorem 2 As $t \uparrow \infty$, $r_t, t \ge 0$, a.s. converges to an r^* given as the unique solution to

$$\Psi^T \nu \tilde{c} + \alpha \Psi^T \nu \rho \Psi r^* - \Psi^T \nu \Psi r^* = 0.$$

Proof: The scalar n on the right hand side of (6) does not affect its asymptotic behavior, so can be ignored. But then (6) is exactly of the same form as (2) with the same assumptions being satisfied. Hence the same analysis applies. \Box

Remark: As in Tsitsiklis and Van Roy (1997), this can be extended to a positive recurrent Markov chain $\{X_t\}$ on a countably infinite state space under additional square-integrability assumptions on $\{c(X_t, X_{t+1}), \phi^i(X_t)\}$.

3 Average cost

Consider the problem of estimating average cost and a differential cost function on a finite, irreducible and aperiodic Markov chain. The average cost μ^* is given by $E_s[c(X_t, X_{t+1})]$, where $E_s[\cdot]$ denotes the stationary distribution. Let $\overline{\mathbf{1}}$ denote a vector with all components equal to 1. A differential cost function is any function $J: S \to R$ that satisfies the Poisson equation, which takes the form

$$J = \bar{c} - \mu^* \bar{\mathbf{1}} + PJ.$$

It is known that for an irreducible Markov chain, differential cost functions exist and the set of all differential cost functions takes the form $\{J^* + c\bar{\mathbf{1}} | c \in R\}$, for some J^* satisfying $\eta^T J^* = 0$. Such a J^* is referred to as the basic differential cost function. The original TD(0) algorithm for approximate evaluation of Jbegins with the a priori approximation $J(\cdot) \approx \phi(\cdot)r = \sum_{i=1}^{K} r_i \phi_i(\cdot)$. Here

$$\phi(X_t) = [\phi_1(X_t), \phi_2(X_t), \dots, \phi_K(X_t)]^T,$$

with $\phi_i :=$ the *i*th feature vector (these are kept fixed), and $r := [r_1, \dots, r_K]^T$ are the weights that are to be learnt. The actual algorithm for doing so is as follows Tsitsiklis and Van Roy (1999):

$$r_{t+1} = r_t + \gamma_t \phi(X_t) [c(X_t, X_{t+1}) - \mu_t + \phi(X_{t+1})^T r_t - \phi(X_t)^T r_t],$$

$$\mu_{t+1} = \mu_t + k \gamma_t (c(X_t, X_{t+1}) - \mu_t),$$

where k is any arbitrary positive constant. A convergence proof and error estimates relative to the exact J may be found in Tsitsiklis and Van Roy (1999).

As before, we sketch an alternative argument using the 'o.d.e.' approach. Once again, by Corollary 8, p. 74, Borkar (2008), the limiting o.d.e for the above iteration is

$$\dot{r} = \Phi^T D \bar{c} - \mu \Phi^T \bar{\mathbf{1}} + \Phi^T D P \Phi r - \Phi^T D \Phi r,$$

$$\dot{\mu} = k(\mu^* - \mu).$$

Let $w_t = [\mu_t, r_t]^T$. In matrix notation, the o.d.e. can be written as

$$\dot{w} = \begin{bmatrix} \dot{\mu} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -k & 0 \cdots 0 \\ -\Phi^T D \bar{\mathbf{1}} & \Phi^T D P \Phi - \Phi^T D \Phi \end{bmatrix} \begin{bmatrix} \mu \\ r \end{bmatrix} + \begin{bmatrix} k \mu^* \\ \Phi^T D \bar{c} \end{bmatrix} =: h(w)$$
for $h(w) := \begin{bmatrix} -k & 0 \cdots 0 \\ -\Phi^T D \bar{\mathbf{1}} & \Phi^T D P \Phi - \Phi^T D \Phi \end{bmatrix} w + \begin{bmatrix} k \mu^* \\ \Phi^T D \bar{c} \end{bmatrix}$. Then
$$h_{\infty}(w) := \lim_{a \uparrow \infty} \frac{h(aw)}{a} = \begin{bmatrix} -k & 0 \cdots 0 \\ -\Phi^T D \bar{\mathbf{1}} & \Phi^T D P \Phi - \Phi^T D \Phi \end{bmatrix} w.$$

It is easy to see that $\frac{1}{a}h(aw) \to h_{\infty}(w)$ uniformly on \mathbb{R}^{K+1} .

Let

$$A := \begin{bmatrix} -k & 0 \cdots 0 \\ -\Phi^T D \bar{\mathbf{1}} & \Phi^T D P \Phi - \Phi^T D \Phi \end{bmatrix} \text{ and}$$
$$b := \begin{bmatrix} k\mu^* \\ \Phi^T D \bar{c} \end{bmatrix}.$$

Suppose we assume Φ has linearly independent columns and $\Phi r \neq e$ for any $r \in R^K$

Theorem 3: Under the above assumptions, $w_t \to \hat{w}$ a.s., where \hat{w} is the unique solution to h(w) = 0.

Proof: For sufficiently large k, the matrix A is negative definite as seen from Lemma 7 of Tsitsiklis and Van Roy (1999) (k corresponds to their l). Hence the 'scaled o.d.e.' $\dot{w}(t) = h_{\infty}(w(t))$ is a linear system with the origin as its globally asymptotically stable equilibrium, in fact $V(w(t)) = ||w(t)||^2$ is a Liapunov function. By Theorem 9, p. 75, Borkar (2008), $\sup_t ||w_t|| < \infty$ a.s. In turn, (2) has \hat{w} as its globally asymptotically stable equilibrium, again V(w(t)) = $||w(t) - \hat{w}||^2$ is a Liapunov function. This can be seen as follows

$$\frac{dV(w(t))}{dt} = (w(t) - \hat{w})^T (Aw(t) + b)
= (w(t) - \hat{w})^T (Aw(t) + b - A\hat{w} - b)
= (w(t) - \hat{w})^T A(w(t) - \hat{w})
\leq 0.$$

with equality iff $w(t) = \hat{w}$. The claim follows by Theorem 2, p. 15, Borkar (2008).

Consider a similar setting as section 2. The ith agent thus runs the following n_i dimensional iteration

$$\begin{aligned} r_{t+1}^i &= r_t^i + \gamma_t \phi(X_t) [c(X_t, X_{t+1}) - \mu_t + \phi^{Y_{t+1}^i} (X_{t+1})^T r^{Y_{t+1}^i} - \phi^i (X_t)^T r_t^i], \\ \mu_{t+1} &= \mu_t + k \gamma_t [c(X_t, X_{t+1}) - \mu_t], \end{aligned}$$

where k is an arbitrary positive constant. We show convergence of the combined iterates $[r_t^1, r_t^2, ..., r_t^n, \mu_t]$.

Rewrite the above iteration as

$$\begin{aligned} r_{t+1}^{i} &= r_{t}^{i} + \gamma_{t} [\phi(X_{t})\overline{c}(X_{t}) - \phi(X_{t})\mu_{t} + \phi(X_{t})\sum_{j=1}^{n} q(i,j)\sum_{s=1}^{m} p(X_{t},s)\phi^{j}(s)^{T}r^{j} \\ &- \phi(X_{t})\phi^{i}(X_{t})^{T}r_{t}^{i} + M_{t+1}^{i}], \\ \mu_{t+1} &= \mu_{t} + k\gamma_{t} [\overline{c}(X_{t}) - \mu_{t} + M_{t+1}]. \end{aligned}$$

Here M_{t+1}^i and M_{t+1} are martingale difference sequences given by resp.,

$$\left(\phi^{i}(X_{t})\phi^{Y_{t+1}^{i}}(X_{t+1})^{T}r_{t}^{Y_{t+1}^{i}} - \phi^{i}(X_{t})(\sum_{j=1}^{n}q(i,j)\sum_{s=1}^{m}p(X_{t},s)\phi^{j}(s)^{T}r_{t}^{j})\right) + c(X_{t},X_{t+1})\phi^{i}(X_{t}) - \bar{c}(X_{t})\phi^{i}(X_{t}).$$

and

$$k[c(X_t, X_{t+1}) - \overline{c}(X_t)].$$

Using similar matrix notation from section 2 and using the fact that $\sum_{l=1}^{m} d(l)\bar{c}(l) =$ $\mu^*,$ the o.d.e. corresponding to (4) is

$$\dot{\bar{r}}^i = \Phi^{i^T} D\bar{c} - \overline{\mu} \Phi^{i^T} D\bar{\mathbf{1}} + \Phi^{i^T} DP \sum_{j=1}^n q(i,j) \Phi^j \bar{r}^j - \Phi^{i^T} D\Phi^i \bar{r}^i,$$

$$\dot{\bar{\mu}} = k(\mu^* - \overline{\mu}).$$

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Let $r = [\bar{r}^1, \bar{r}^2,, \bar{r}^n]$, the concatentation of all \bar{r}^i 's. It satisfies the o.d.e.

$$\dot{r} = \begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} \bar{c}\\ \cdot\\ \bar{c}\\ \bar{c} \end{bmatrix} - \\ \frac{1}{\bar{c}} \begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} \bar{1}\\ \cdot\\ \vdots\\ \bar{1}\\ \vdots\\ \bar{1} \end{bmatrix} +$$

$$\begin{bmatrix} \Phi^{1^{T}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n^{T}} \end{bmatrix} \begin{bmatrix} D & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & D \end{bmatrix} \begin{bmatrix} q(1,1)P & \cdots & q(1,n)P\\ \vdots & \ddots & \vdots\\ q(n,1)P & \cdots & q(n,n)P \end{bmatrix}$$
$$\times \begin{bmatrix} \Phi^{1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Phi^{n} \end{bmatrix} \begin{bmatrix} \bar{r}^{1}\\ \cdot\\ \cdot\\ \bar{r}^{i}\\ \cdot\\ \cdot\\ \bar{r}^{n} \end{bmatrix}.$$

Consider the augmented Markov chain as in section 2 and analogous definitions for Ψ, ν, ρ and \tilde{c} . Then the ODE for $r(\cdot)$ and $\overline{\mu}$ is ,

$$\dot{r} = n \left(\Psi^T \nu \tilde{c} - \overline{\mu} \Psi^T \nu e + \Psi^T \nu \rho \Psi r - \Psi^T \nu \Psi r \right).$$

$$\dot{\overline{\mu}} = k (\mu^* - \overline{\mu}).$$
(7)

We assume A1, A2, A3 and A5 here as well. Hence Lemma 1, Lemma 2 hold in this case also. In addition we make the following key assumption.

• (A6) $\Psi r \neq e$ for any $r \in \mathbb{R}^{n_1+n_2+\ldots+n_n}$.

Theorem 4 $\bar{\mu}_t, t \ge 0$ a.s. converges to μ^* . $\bar{r}_t, t \ge 0$, a.s. converges to an r^* , given as the unique solution to

$$\Psi^T \nu \tilde{c} - \mu^* \Psi^T \nu e + \Psi^T \nu \rho \Psi r - \Psi^T \nu \Psi r = 0.$$

Proof: The scalar n on the right hand side of (7) does not affect its trajectory, so can be ignored. But then (7) is exactly of the same form as (7) with the same assumptions being satisfied. Hence the same analysis applies.

4 Discussion

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1. Performance comparison for Discounted Problem:

Define Π_i to be the projection onto the range of ϕ^i w.r.t. the weighted norm $\|\cdot\|$, where the weights are the values of the stationary probability distribution η . Let T denote the Bellman operator defined by

$$(Tx)(i) := \bar{c}(i) + \alpha \sum_{j} p(i,j) x(j) \; \forall i.$$

Recall from Tsitsiklis and Van Roy (1997) that this is a contraction w.r.t. the weighted norm above. Furthermore, by triangle inequality and convexity we have

$$||J^* - \sum_{j=1}^n q(i,j)x_j||^2 \le \sum_{j=1}^n q(i,j)||J^* - x_j||^2.$$

Let,

$$\begin{array}{rcl} e_i^* & := & \|J^* - \Pi_i J^*\| \\ e_i & := & \|J^* - \phi^i r^{*i}\| \\ e^* & := & [e_1^*, ..., e_i^*, ..., e_n^*]^T \\ e^{*(2)} & := & [e_{1}^*, ..., e_i^{*2}, ..., e_n^{*2}]^T \\ e & := & [e_1, ..., e_i, ..., e_n]^T \\ e^{(2)} & := & [e_1^2, ..., e_i^2, ..., e_n^2]^T \end{array}$$

Our analysis borrows ideas from Tsitsiklis and Van Roy (1997,1999).

$$\begin{split} \|J^* - \phi^i r^{*i}\|^2 &= \|J^* - \Pi_i J^*\|^2 + \|\Pi_i J^* - \phi^i r^{*i}\|^2 \\ &= \|J^* - \Pi_i J^*\|^2 + \|\Pi_i T J^* - \Pi_i T (\sum_{j=1}^n q(i,j) \phi^j r^{*j})\|^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \|T J^* - T (\sum_{j=1}^n q(i,j) \phi^j r^{*j})\|^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \alpha^2 \|J^* - \sum_{j=1}^n q(i,j) \phi^j r^{*j}\|^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \alpha^2 \sum_{j=1}^n q(i,j) \|J^* - \phi^j r^{*j}\|^2, \end{split}$$

The first equality follows from Pythagoras theorem. The first and second inequalities follows from non-expansivity of Π_i and contraction property of T, respectively. Thus we have

$$e_i^2 \leq e_i^{*2} + \alpha^2 \sum_j q(i,j) e_j^2$$

$$\Rightarrow e^{(2)} \leq e^{*(2)} + \alpha^2 Q e^{(2)}$$

$$\Rightarrow e^{(2)} \leq (I - \alpha^2 Q)^{-1} e^{*(2)}$$

$$= (\sum_{k=0}^{\infty} \alpha^{2k} Q^k) e^{*(2)}.$$

This is justified because the last expression shows that $(I - \alpha^2 Q)^{-1}$ is a non-negative matrix. Let $\tilde{Q} := (1 - \alpha^2) \sum_{k=0}^{\infty} \alpha^{2k} Q^k$, a doubly stochastic matrix. Thus we have

$$e^{(2)} \leq (1 - \alpha^2)^{-1} \tilde{Q} e^{*(2)}$$

$$\Rightarrow \max_i e_i^2 \leq \frac{\beta(e^*)}{(1 - \alpha^2)} \max_i e_i^{*2}$$

$$\Rightarrow \max_i e_i \leq \frac{\sqrt{\beta(e^*)}}{\sqrt{(1 - \alpha^2)}} \max_i e_i^*,$$

where $\beta(e^*), \sqrt{\beta(e^*)} \in (0, 1)$. The second inequality follows if we assume that an agent with the maximum e_i^* samples an agent with a lesser e_i^* with non-zero probability. This assumption in turn follows from irreducibility and an assumption that at least one e_i^* is different from the rest. Thus we get a multiplicative improvement over $\frac{\max_i e_i^*}{\sqrt{(1-\alpha^2)}}$, which would correspond to the estimate from Tsitsiklis and Van Roy (1997). Let $\Pi := \frac{1}{n} \sum_{i=1}^{n} \Pi_i$ and $\bar{J} = \frac{1}{n} \sum_{i=1}^{n} \phi^i r^{*i}$. Then

$$\begin{split} \|J^* - \bar{J}\| &\leq \|J^* - \Pi J^*\| + \|\Pi J^* - \bar{J}\| \\ &\leq \|J^* - \Pi J^*\| + \frac{1}{n} \sum_{i=1}^n \|\Pi_i J^* - \phi^i r^{*i}\| \\ &\leq \|J^* - \Pi J^*\| + \frac{1}{n} \sum_{i=1}^n \alpha \sum_{j=1}^n q(i,j) \|J^* - \phi^j r^{*j}\| \\ &\leq \|J^* - \Pi J^*\| + \frac{1}{n} \sum_{i=1}^n \alpha \|J^* - \phi^i r^{*i}\| \\ &\leq \|J^* - \Pi J^*\| + \frac{\alpha \beta(e^*)}{(1-\alpha)} \max_i e_i^* \\ &\leq \frac{(1-\alpha) \|J^* - \Pi J^*\| + \alpha \beta(e^*) \max_i e_i^*}{(1-\alpha)}. \end{split}$$

The numerator is a convex combination of $||J^* - \Pi J^*||$ and $\beta(e^*) \max_i e_i^*$, a multiplicative improvement over $\max_i e_i^*$.

This suggests that the maximum error and the variance should be less in the distributed algorithm. We do not, however, have a formal proof for this. We have instead included some simulations that support this intuition.

Specifically, we have considered the problem of calculating the average discounted sum of queue lengths over an infinite horizon. The maximum queue length is capped at 50. The arrival probability is 0.3 and the departure probability is 0.35. The discount factor is 0.9. We have considered 3 agents. The sampling probabilities, basis functions and initial values of weights are as follows:

The plot of variance vs iteration and maximum error vs iteration for both our distributed algorithm (coupled) and the uncoupled algorithm are given below. We see that the variance and maximum error is lower in the former case.

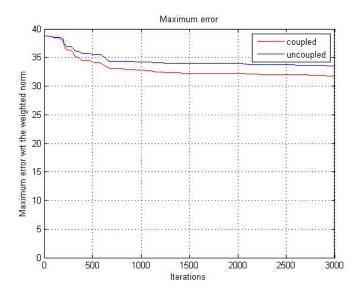


Figure 1: Maximum error vs number of iterations

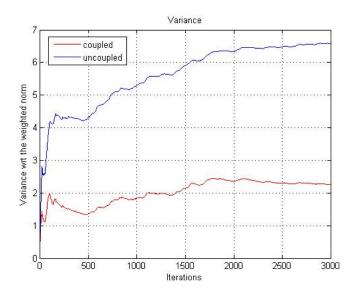


Figure 2: Variance vs Number of iterations

2. Performance comparison for Average Cost Problem:

Define the equivalence relation $i' \approx j'$ if for some finite $n \geq 1$, there is a sequence $i' = i_0, j_0, i_1, j_1, \cdots, i_{n-1}, j_{n-1}, i_n = j'$ such that $p(j_k, i_k)$, $p(j_k, i_{k+1}) > 0 \forall 0 \leq k < n$. Consider equivalence classes under \approx . We assume that the whole state space is one equivalence class. Note that this assumption will be satisfied if every node has a self loop, which is true for most queuing models. Furthermore, it does not cause any loss of generality as observed in Tsitsiklis and Van Roy (1999), because inserting a self-loop of probability $\delta \in (0, 1)$ at each state is tantamount to replacing P by $(1 - \delta)P + \delta I$, equivalently, introducing a sojourn time binomially distributed with parameter δ at each state. This does not affect either β or J^* , and amounts to a harmless time scaling for the algorithm.

Let θ denote the zero vector.

Lemma 3 Under the above assumption, $\sup_{x \neq \theta, \eta^T x = 0} \frac{\|Px\|}{\|x\|} < 1$, where $\|\cdot\|$ is the weighted norm.

Proof Clearly $\sup_{x \neq \theta, \eta^T x = 0} \frac{\|Px\|}{\|x\|} \leq 1$. If equality holds, we must have

$$\sum_{j} p(i,j)x_j^2 = (\sum_{j} p(i,j)x_j)^2 \quad \forall i,$$

which is possible only if for each i, x_j is constant for $j \in \{k : p(i, k) > 0\}$. Thus $x_{i'} = x_{j'}$, if i' and j' belong to the same equivalence class (defined above). Since we have assumed that the entire state space is a single equivalence class, x is a constant vector. This along with $\eta^T x = 0, x \neq \theta$ gives us a contradiction. Hence the claim follows.

Define Π_i is before. As before Let $\overline{\mathbf{I}} := [1, 1, ..., 1, ..., 1]^T$. Define $\Pi_{\overline{\mathbf{I}}^c}$ to be the projection on the subspace that is orthogonal to $\overline{\mathbf{I}}$ with respect to the weighted norm $\|.\|$. Let T be defined as

$$(Tx)(i) := \bar{c}(i) - \mu^* + \sum_{j=1}^n p(i,j)x(j).$$

Since $J^* + c\bar{\mathbf{1}} \ \forall c \in R$ are valid differential cost functions, we define the error of the i^{th} agent as:

$$\inf_{c \in R} \|J^* + c\bar{\mathbf{1}} - \phi^i r^{*i}\|$$

= $\|\Pi_{\bar{\mathbf{1}}^c} (J^* - \phi^i r^{*i})\|$

As $\eta^T J^* = 0$, we have $\Pi_{\bar{\mathbf{1}}^c} J^* = J^*$ and hence the error is equal to

$$||J^* - \prod_{\bar{\mathbf{1}}^c} \phi^i r^{*i}||.$$

Let $e, e^{(2)}, e^*$ and $e^{*(2)}$ be defined analogously. We assume that $\forall i \ \phi_j^i \ j \in 1, 2, ... n_i$ are orthogonal to $\overline{\mathbf{1}}$ w.r.t. the weighted norm. This gives us:

$$\eta^T J^* = 0,$$

 $\eta^T \phi^j = 0.$ (8)

This assumption is not restrictive since $\eta^T J^* = 0$ and we are approximating J^* . This leads to

$$\Pi_{i}\mu^{*}\mathbf{1} = 0$$

$$\Rightarrow \Pi_{i}\mu^{*}\bar{\mathbf{1}} = \Pi_{i}(\eta^{T}J^{*})\bar{\mathbf{1}}.$$

$$\Pi_{i}TJ^{*} = \Pi_{i}(c - \mu^{*}\bar{\mathbf{1}} + PJ^{*})$$

$$= \Pi_{i}(c - (\eta^{T}J^{*})\bar{\mathbf{1}} + PJ^{*})$$

$$= \Pi_{i}(c + (P - \bar{\mathbf{1}}\eta^{T})J^{*}).$$
Similarly,
$$\Pi_{i}\phi^{i}r^{*i} = \Pi_{i}(c + (P - \bar{\mathbf{1}}\eta^{T})\phi^{i}r^{*i}).$$

Hence

$$\begin{split} \|J^* - \Pi_{\bar{\mathbf{1}}^c} \phi^i r^{*i} \|^2 &= \|J^* - \phi^i r^{*i} \|^2 \\ &= \|J^* - \Pi_i J^* \|^2 + \|\Pi_i J^* - \phi^i r^{*i} \|^2 \\ &= \|J^* - \Pi_i J^* \|^2 + \|\Pi_i T J^* - \Pi_i T \sum_{j=1}^n q(i,j) \phi^j r^{*j} \|^2 \end{split}$$

For the sake of brevity, denote $(J^* - \sum_{j=1}^n q(i,j)\phi^j r^{*j})$ by E_i . By (8) $\eta^T E_i = 0$. Thus,

$$\begin{split} |J^* - \Pi_{\bar{\mathbf{1}}^c} \phi^i r^{*i}||^2 &= \|J^* - \Pi_i J^*\|^2 + \|\Pi_i (P - \bar{\mathbf{1}} \eta^T) E_i\|^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \|(P - \bar{\mathbf{1}} \eta^T) E_i\|^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \sum_{l=1}^m \eta(l) (\sum_{j=1}^m (p(l,j) - \eta(j)) E_i(j))^2 \\ &= \|J^* - \Pi_i J^*\|^2 + \sum_{l=1}^m \eta(l) (\sum_{j=1}^m (p(l,j) (E_i(j) - \eta^T E_i))^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \alpha^2 \sum_{l=1}^m \eta(l) (E_i(l) - \eta^T E_i)^2 \\ &\leq \|J^* - \Pi_i J^*\|^2 + \alpha^2 \|E_i - \eta^T E_i\|^2 \\ &= \|J^* - \Pi_i J^*\|^2 + \alpha^2 \|J^* - \sum_{j=1}^n q(i,j) \phi^j r^{*j}\|^2, \end{split}$$

where $\alpha := \sup_{\eta^T x = 0, x \neq 0} \frac{\|Px\|}{\|x\|}$. From Lemma 3 we know that $\alpha < 1$. Following the steps in the preceding section we get,

$$\max_{i} e_{i} \leq \sqrt{\frac{\beta(e^{*})}{1-\alpha^{2}}} \max_{i} e_{i}^{*} \|J^{*}-\bar{J}\| \leq \frac{(1-\alpha)\|J^{*}-\Pi J^{*}\|+\alpha\beta(e^{*})\max_{i} e_{i}^{*}}{(1-\alpha)}$$

where \bar{J} and Π are defined as before and $\beta(e^*) \in [0, 1)$. Thus we get multiplicative improvements in the bound over the uncoupled case. It is worth noting that the bound derived in Tsitsiklis and Van Roy (1999) does not seem to extend easily to the distributed set-up. As before, we do expect that the variance should be less in the distributed algorithm with gossip as opposed to the uncoupled case. Again we do not have a formal proof, but we have included simulations to support our intuition. We simulate with the same parameters in section 4.1 except that the feature vectors are projected on $\bar{\mathbf{1}}^c$. The simulations showed significant reduction in variance, however the maximum error was approximately same for both. We have included the graph for variance here.

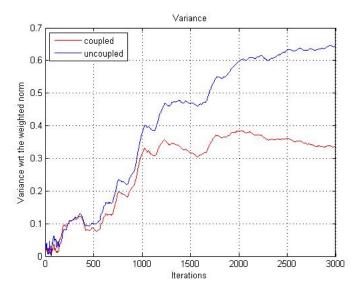


Figure 3: Variance vs number of iteration

3. Interestingly, the simple convergence proof above fails for $TD(\lambda)$ for $\lambda \neq 0$. It will be interesting to see whether our scheme can be modified to suit general λ .

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