Feedback Refinement Relations for the Synthesis of Symbolic Controllers

Gunther Reissig, Alexander Weber, and Matthias Rungger

Abstract

We present an abstraction and refinement methodology for the automated controller synthesis to enforce general predefined specifications. The designed controllers require quantized (or symbolic) state information only and can be interfaced with the system via a static quantizer. Both features are particularly important with regard to any practical implementation of the designed controllers and, as we prove, are characterized by the existence of a feedback refinement relation between plant and abstraction. Feedback refinement relations are a novel concept introduced in this paper. Our work builds on a general notion of system with set-valued dynamics and possibly non-deterministic quantizers to permit the synthesis of controllers that robustly, and provably, enforce the specification in the presence of various types of uncertainties and disturbances. We identify a class of abstractions that is canonical in a well-defined sense, and provide a method to efficiently compute canonical abstractions. We demonstrate the practicality of our approach on two examples.

Index Terms

Discrete abstraction, symbolic model, nonlinear system, symbolic control, automated synthesis, robust synthesis; MSC: Primary, 93B51; Secondary, 93B52, 93C10, 93C30, 93C55, 93C57, 93C65

I. INTRODUCTION

A common approach to engineer reliable, robust, high-integrity hardware and software systems that are deployable in safety-critical environments, is the application of formal verification techniques to ensure the correct, error-free implementation of some given formal specifications. Typically, the verification phase is executed as a distinct step after the design phase, e.g. [1]. In case that the system fails to satisfy the specification, it is the engineer's burden to identify the fault, adjust the system accordingly and return to the verification phase. A more appealing approach, especially in the context of intricate, complex dynamical systems, is to merge the design and verification phase and utilize automated correct-by-construction formal synthesis procedures, e.g. [2]. In our treatment of controller design problems we follow the latter approach. That is, given a mathematical system description and a formal specification which expresses the desired system behavior, we seek to synthesize a controller that provably enforces the specification on the system. Subsequently, we often refer to the system that is to be controlled as the *plant*.

For finite systems, which are described by transition systems with finite state, input and output alphabets, there exist a number of automata-theoretic schemes, known under the label of *reactive synthesis*, to algorithmically synthesize controllers that enforce complex specifications, possibly formulated in some temporal logic, see e.g. [2]-[6].

Those methods have been extended to infinite systems within an abstraction and refinement framework, e.g. [2], [7]-[20], which roughly proceeds in three steps. In the first step, the concrete

G. Reissig and A. Weber are with the University of the Federal Armed Forces Munich, Dept. Aerospace Eng., Chair of Control Eng. (LRT-15), D-85577 Neubiberg (Munich), Germany, http://www.reiszig.de/gunther/

M. Rungger is with the Hybrid Control Systems Group at the Department of Electrical and Computer Engineering at the Technical University of Munich, 80333 Munich, Germany.

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infinite system (together with the specification) is lifted to an abstract domain where it is substituted by a finite system, which is often referred to as *abstraction* or *symbolic model*. In the second step, an auxiliary problem on the abstract domain ("abstract problem") is solved using one of the previously mentioned methods for finite systems. In the third step, the controller that has been synthesized for the abstraction is refined to the concrete system.

The correctness of this controller design concept is usually ensured by relating the concrete system with its abstraction in terms of a system relation. The most common approaches are based on (alternating) (bi-)simulation relations and approximate variants thereof [2]. In this work, we address two shortcomings of the abstraction and refinement process based on simulation relations and related concepts. The first shortcoming, which we refer to as the state information issue, results from the fact that the refined controller requires the exact state information of the concrete system. However, usually, the exact state is not known and only quantized (or symbolic) state information is available, which constitutes a major obstacle to the practical implementation of the synthesized controllers. The second issue refers to the huge amount of dynamics added to the abstract controller in the course of its refinement, so that, effectively, the refined controller contains the abstraction as a building block. Given the fact that an abstraction may very well comprise millions of states and billions of transitions [7], [14], an implementation of the refined controller is often too expensive to be practical. We refer to this problem as the *refinement complexity issue*. We illustrate both issues by examples in Section IV. See also [21].

In this paper, we propose a novel notion of system relation, termed *feedback refinement relation*, to resolve both issues. If the concrete system is related with the abstraction via a feedback refinement relation, then, as we shall show, the abstract controller can be connected to the plant via a static quantizer only, irrespective of the particular specification we seek to enforce on the plant. See Fig. 1. Moreover, the existence of a feedback refinement relation between plant and abstraction is not only

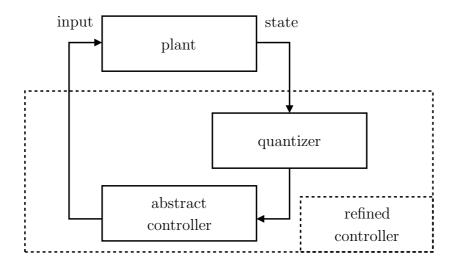


Figure 1. Closed loop resulting from the abstraction and refinement approach based on feedback refinement relations, proposed in this paper.

sufficient to ensure the simple structure of the closed loop in Fig. 1, but in fact also necessary.

Our work builds on a general notion of system with set-valued dynamics and possibly non-deterministic quantizers. This is particularly useful to model various types of disturbances, including plant uncertainties, input disturbances and state measurement errors. We demonstrate how to account for those perturbations in our framework so that the synthesized controllers robustly enforce the specification.

In general, abstractions over-approximate the plant behavior, and so their practical use will depend on the accuracy of the approximation that can be achieved by actual computational methods; see the discussion in [7, Sect. I]. In this regard, we show that the set membership relation together with an abstraction whose state alphabet is a cover of the concrete state alphabet is canonical in a well-defined sense, and provide a method to compute canonical abstractions of perturbed nonlinear sampled systems. The practicality of the approach is demonstrated on two examples – a path planning problem for an autonomous vehicle and an aircraft landing maneuver.

Related Work. Feedback refinement relations are based on the common principle of "accepting more inputs and generating fewer outputs" that is often encountered in component-based design methodologies, e.g. contract-based design [22] and interface theories [23]. Those theories are usually developed in a purely behavioral setting, see e.g. [19], [22], [23], and are therefore not immediately applicable in our framework which is based on stateful systems. This class of systems contains a great variety of system descriptions, including common models like transition systems [2], [24] as well as discrete-time control systems [25].

There exist a number of abstraction-based controller synthesis methods, based on stateful systems, that do not suffer from the state information issue nor from the refinement complexity issue [7]–[13]. However, none of those approaches offers necessary and sufficient conditions for the controller refinement procedure to be free of the mentioned issues. In addition, the majority of these works are tailored to certain types of specifications or systems. Specifically, simple safety and reachability problems are considered in [10], [12] and [7]–[10], respectively, while [10]–[12] is limited to piecewise affine, incrementally stable, and simple integrator dynamics, respectively. Moreover, plants are assumed to be non-blocking in [7]–[13]. In contrast, our framework covers stateful systems with general, set-valued dynamics, including transitions systems and discrete-time control systems as special cases. We allow systems to be blocking, and any linear time property can serve as a specification.

A class of methods known under the label of *hierarchical control* are similar in spirit to abstractionbased methods in that they synthesize discrete controllers using finite-state models derived from concrete control problems, e.g. [26]–[28]. However, the finite-state models in [27], [28] are not abstractions in the usual sense, in that they approximate the behavior of an interconnection of the plant with low-level controllers, rather than the behavior of the plant itself. In [26] one is required to derive a quantizer in accordance with the exact plant dynamics, and to verify rather complex system properties. Moreover, those hierarchical schemes require exact state information or, in the case of linear output feedback [29], require exact output information, and are unable to account for quantized or perturbed measurements. Additionally, for general nonlinear plants, all of the aforementioned approaches require the synthesis of low-level controllers to enforce a high-level plan, which is considered as an open problem [30] and current solutions exist only for rather restrictive classes of systems [29], [31], [32]. In contrast, the refinement step in our approach is completely independent of the plant dynamics and does not involve the design of low-level controllers.

For any of the aforementioned approaches, often a lack of robustness further restricts the applicability of the methods. For example, [9]–[11] do not cover uncertainties in plant dynamics, while in [8], [10], [11], [26]–[28] the quantizer is assumed to be deterministic which mandates the state measurement to be precise, without any error; see Section VI-B.

Similarly to our work, the synthesis scheme in [13] introduces a novel system relation. However, in contrast to the theory in [13], feedback refinement relations do not rely on a metric of the state alphabet, which is crucial in establishing the necessity as well as the canonicity result. Likewise, the authors of [13] consider perturbations, but assume that the effect of these perturbations is given as level sets of a metric.

In addition to a general synthesis framework, we present a method to construct abstractions of perturbed nonlinear control systems. The abstractions are based on a cover of the state alphabet by non-empty compact hyper-intervals and the over-approximation of attainable sets of those hyper-intervals under the system dynamics. While the use of attainable sets for the construction of abstractions is a well-known concept [7], [8], [14], [15], none of the aforementioned works accounts for uncertainties or perturbations. Moreover, while our method to over-approximate attainable sets is similar to those in [14], [15] in that it is based on a growth bound, we present several extensions

that render the approach more efficient.

To summarize, our contribution is threefold. First, we introduce feedback refinement relations as a novel means to synthesize symbolic controllers. We show that feedback refinement relations are necessary and sufficient for the controller refinement that solves the state information issue and the refinement complexity issue. Our theory applies to a more general class of synthesis problems than previous research that addresses the mentioned issues, and in particular, any linear time property can serve as a specification. Second, our work permits the synthesis of controllers that robustly, and provably, enforce the specification in presence of various uncertainties and disturbances. Third, we identify a class of canonical abstractions and present a method to compute such abstractions. Our construction improves known methods in several directions and thereby, as we demonstrate by some numerical examples, facilitates a more efficient computation of abstractions of perturbed nonlinear control systems.

Some of the results we present have been announced in [21].

II. NOTATION

The relative complement of the set A in the set B is denoted by $B \setminus A$. \mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ denote the sets of real numbers, non-negative real numbers, integers and non-negative integers, respectively, and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$. We adopt the convention that $\pm \infty + x = \pm \infty$ for any $x \in \mathbb{R}$. [a, b], [a, b[, and]a, b] denote closed, open and half-open, respectively, intervals with end points a and b. [a; b], [a; b[, and]a; b] stand for discrete intervals, e.g. $[a; b] = [a, b] \cap \mathbb{Z}$ and $[0; 0] = \emptyset$.

In \mathbb{R}^n , the relations $\langle \langle \langle \rangle \rangle > \rangle$ are defined component-wise, e.g. a < b iff $a_i < b_i$ for all $i \in [1; n]$. $f: A \Rightarrow B$ denotes a set-valued map of A into B, whereas $f: A \to B$ denotes an ordinary map; see [33]. If f is set-valued, then f is strict and single-valued if $f(a) \neq \emptyset$ and f(a) is a singleton, respectively, for every a. The restriction of f to a subset $M \subseteq A$ is denoted $f|_M$. Throughout the text, we denote the identity map $X \to X: x \mapsto x$ by id. The domain of definition X will always be clear form the context.

We identify set-valued maps $f: A \rightrightarrows B$ with binary relations on $A \times B$, i.e., $(a, b) \in f$ iff $b \in f(a)$. Moreover, if f is single-valued, it is identified with an ordinary map $f: A \to B$. The inverse mapping $f^{-1}: B \rightrightarrows A$ is defined by $f^{-1}(b) = \{a \in A \mid b \in f(a)\}$, and $f \circ g$ denotes the composition of f and g, $(f \circ g)(x) = f(g(x))$.

The set of maps $A \to B$ is denoted B^A , and the set of all signals that take their values in B and are defined on intervals of the form [0; T[is denoted $B^{\infty}, B^{\infty} = \bigcup_{T \in \mathbb{Z}_+ \cup \{\infty\}} B^{[0;T[}$.

III. PLANTS, CONTROLLERS, AND CLOSED LOOPS

A. Systems

We consider dynamical systems of the form

$$x(t+1) \in F(x(t), u(t))
 y(t) \in H(x(t), u(t)).
 (1)$$

The motivation to use a set-valued transition function F and a set-valued output function H in our system description, originates from the desire to describe disturbances and other kinds of nondeterminism in a unified and concise manner. This description is also sufficiently expressive to model the plant and the controller, but unfortunately leads to subtle issues with interconnected systems. Consider e.g. the serial composition in Fig. 2, where $F_i: X_i \times U_i \rightrightarrows X_i, X_1 = U_1 = \{0\}, X_2 = U_2 =$ $\{0,1\}, Y_2 = \{a,b,c\}, F_1(0,0) = \{0\}, H_1(0,0) = U_2, \text{ and } F_2 \text{ and } H_2: X_2 \times U_2 \rightrightarrows Y_2 \text{ are given as follows:}$ $F_2(1,0) = F_2(0,1) = \{0\}, F_2(0,0) = F_2(1,1) = \{1\}, H_2(0,0) = H_2(1,0) = \{a\}, H_2(0,1) = \{b\}, \text{ and}$ $H_2(1,1) = \{c\}$. To recover the behavior at the terminals u_1 and y_2 with a system of the form (1), we let $X = X_1 \times X_2, F: X \times U_1 \rightrightarrows X$ and $H: X \times U_1 \rightrightarrows Y_2$. As Y_2 contains more elements than $X \times U_1$, which can all appear in y_2 , the map H must be multi-valued, which in turn implies that the following property of the composed system in Fig. 2 cannot be retained: Between any two appearances of b in y_2 there are an even number of a's, and between any appearance of b and any appearance of c there are an odd number of a's.

It follows that the class of systems of the form (1) is not closed under interconnection, given the natural constraint that the state alphabet of the composed system equals the product of the state alphabets of the individual systems. To circumvent this problem we consider a slightly more general

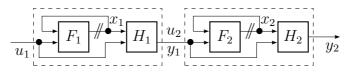


Figure 2. Serial composition of two dynamical systems of the form (1). The symbol // denotes a delay.

form of system dynamics given by

$$x(t+1) \in F(x(t), v(t)), \tag{2a}$$

$$(y(t), v(t)) \in H(x(t), u(t)), \tag{2b}$$

where v is an *internal variable*. We formalize the notion of system as follows.

III.1 Definition. A system is a septuple

$$S = (X, X_0, U, V, Y, F, H),$$
(3)

where X, X₀, U, V and Y are nonempty sets, $X_0 \subseteq X$, $H: X \times U \rightrightarrows Y \times V$ is strict, and $F: X \times V \rightrightarrows X$.

A quadruple $(u, v, x, y) \in U^{[0;T[} \times V^{[0;T[} \times X^{[0;T[} \times Y^{[0;T[} \text{ is a solution of the system (3) (on [0;T[, starting at <math>x(0))$ if $T \in \mathbb{N} \cup \{\infty\}$, (2a) holds for all $t \in [0; T - 1[$, (2b) holds for all $t \in [0;T[$, and $x(0) \in X_0$.

The internal variables allow us to introduce the constraint $u_2 = y_1$ imposed by the composition in Fig. 2 and recover the behavior of the serial composed system with a system of the form (3) given by $X = X_0 = \{0, 1\}, U = \{0\}, V = Y = \{a, b, c\}$ with $F(0, a) = F(1, c) = \{1\}, F(1, a) = F(0, b) = \{0\}$ and $H(0, 0) = \{(a, a), (b, b)\}, H(1, 0) = \{(a, a), (c, c)\}.$

We call the sets X, X_0, U, V , and Y the state, initial state, input, internal variable, and output alphabet, respectively. The functions F and H are, respectively, the transition function and the output function of (3). We call the system (3)

- (i) *autonomous* if U is a singleton;
- (ii) static if X is a singleton;
- (iii) Moore if the output does not depend on the input, i.e., $(y, v) \in H(x, u) \land u' \in U \Rightarrow \exists_{v'}(y, v') \in H(x, u'); ^1$

(iv) simple, if U = V, X = Y, H = id, and all states are admissible as initial states, i.e., $X = X_0$.

We assume throughout that the plant is given by a simple system, which restricts our theory to that class of plants.

B. System composition

In the following, we define the serial and feedback composition of two systems. We start with the serial composition.

¹The notation $\exists_s A$ reads as "there exists s such that the statement A holds".

III.2 Definition. Let $S_i = (X_i, X_{i,0}, U_i, V_i, Y_i, F_i, H_i)$ be systems, $i \in \{1, 2\}$, and assume that $Y_1 \subseteq U_2$. Then S_1 is serial composable with S_2 , and the serial composition of S_1 and S_2 , denoted $S_2 \circ S_1$, is the septuple

$$(X_{12}, X_{1,0} \times X_{2,0}, U_1, V_{12}, Y_2, F_{12}, H_{12}),$$

where $X_{12} = X_1 \times X_2$, $V_{12} = V_1 \times V_2$, $F_{12} \colon X_{12} \times V_{12} \rightrightarrows X_{12}$ and $H_{12} \colon X_{12} \times U_1 \rightrightarrows Y_2 \times V_{12}$ satisfy

$$F_{12}(x,v) = F_1(x_1,v_1) \times F_2(x_2,v_2),$$

$$H_{12}(x,u_1) = \{(y_2,v) \mid \exists_{y_1}(y_1,v_1) \in H_1(x_1,u_1) \land (y_2,v_2) \in H_2(x_2,y_1)\}.$$

We readily see that the output function H_{12} is strict which implies that $S_2 \circ S_1$ is a system. We use the serial composition mainly to describe the interconnection of an input quantizer $Q: U' \Rightarrow U$ or a state quantizer $Q: X \Rightarrow X'$ with a system S of the form (3). We assume that Q is strict and interpret the quantizer as a static system with strict transition function. Suppose that U' is a non-empty set, then the serial composition $S \circ Q$ of Q and S is defined by

$$S \circ Q = (X, X_0, U', V, Y, F, H'),$$

where $H': X \times U' \Rightarrow Y \times V$ takes the form H'(x, u') = H(x, Q(u')). Now suppose that S is simple, then we may interpret $Q: X \Rightarrow X'$ as a measurement map that yields a quantized version of the state of the system S. This situation is modeled by the serial composition $Q \circ S$ of S and Q,

$$Q \circ S = (X, X, U, U, X', F, H'),$$

where H' takes the form $H'(x, u) = Q(x) \times \{u\}$.

We turn our attention to the feedback composition of two systems as illustrated in Fig. 3.

III.3 Definition. Let $S_i = (X_i, X_{i,0}, U_i, V_i, F_i, H_i)$ be systems, $i \in \{1, 2\}$, and assume that S_2 is Moore, $Y_2 \subseteq U_1$ and $Y_1 \subseteq U_2$, and that the following condition holds:

(**Z**) If $(y_2, v_2) \in H_2(x_2, y_1), (y_1, v_1) \in H_1(x_1, y_2)$ and $F_2(x_2, v_2) = \emptyset$, then $F_1(x_1, v_1) = \emptyset$.

Then S_1 is feedback composable with S_2 , and the closed loop composed of S_1 and S_2 , denoted $S_1 \times S_2$, is the septuple

 $(X_{12}, X_{1,0} \times X_{2,0}, \{0\}, V_{12}, Y_{12}, F_{12}, H_{12}),$

where $X_{12} = X_1 \times X_2$, $V_{12} = V_1 \times V_2$, $Y_{12} = Y_1 \times Y_2$, and $F_{12} \colon X_{12} \times V_{12} \rightrightarrows X_{12}$ and $H_{12} \colon X_{12} \times \{0\} \rightrightarrows Y_{12} \times V_{12}$ satisfy

$$F_{12}(x,v) = F_1(x_1,v_1) \times F_2(x_2,v_2),$$

$$H_{12}(x,0) = \{(y,v) | (y_1,v_1) \in H_1(x_1,y_2) \land (y_2,v_2) \in H_2(x_2,y_1) \}.$$

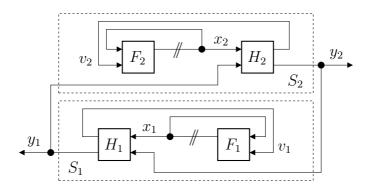


Figure 3. Closed loop $S_1 \times S_2$ of systems S_1 and S_2 according to Definition III.3, in which the system S_2 is required to be Moore.

The requirement (Z), which has its analog in the theory developed in [2], is particularly important and will be needed later to ensure that if the concrete closed loop is non-blocking, then so is the abstract closed loop. The assumption that S_2 is additionally Moore is common [34] and ensures that the closed loop does not contain a delay free cycle. We emphasize that we avoid the assumption that the controller is allowed to set the initial state of the plant, as appears e.g. in [2].

We conclude this section with a proposition that we use in several proofs throughout the paper.

III.4 Proposition. Let S_1 be feedback composable with S_2 , and let $T \in \mathbb{N} \cup \{\infty\}$. Then the closed loop $S_1 \times S_2$ is an autonomous Moore system, and (0, v, x, y) is a solution of $S_1 \times S_2$ on [0; T[iff (y_2, v_1, x_1, y_1) is a solution of S_1 on [0; T[and (y_1, v_2, x_2, y_2) is a solution of S_2 on [0; T[.

Proof. We claim that H_{12} is strict. Indeed, assume that $x \in X_{12}$ and $a \in Y_1$. Since H_1 and H_2 are both strict, there exist $(y_2, b) \in H_2(x_2, a)$ and $(y_1, v_1) \in H_1(x_1, y_2)$. Then there exists v_2 satisfying $(y_2, v_2) \in H_2(x_2, y_1)$ as S_2 is Moore, and so $(y, v) \in H_{12}(x, 0)$. This proves our claim. The remaining requirements in Definition III.1 are clearly satisfied, which shows that $S_1 \times S_2$ is a system, and that system is autonomous, and hence, Moore. The claim on the solutions of $S_1 \times S_2$ is straightforward to prove using Definitions III.1 and III.3.

IV. MOTIVATION

In this section, we provide two examples that demonstrate the state information issue and the refinement complexity issue, which have led to the development of the novel notion of feedback refinement relation. Both examples show that the drawbacks do not depend on the specific refinement technique, but are *intrinsic* to the use of alternating (bi)simulation relations, bisimulation relations and their approximate variants.

Let us consider two systems S_1 and S_2 and two controllers C_1 and C_2 ,

$$S_{i} = (X_{i}, X_{i}, U, U, Y, F_{i}, H_{i}),$$

$$C_{i} = (X_{c,i}, X_{c,i,0}, Y, V_{c,i}, U, F_{c,i}, H_{c,i}),$$

in which we assume that the transition functions of the four systems are all strict, that $X_i \subseteq Y$, and that $H_i(x, u) = \{(x, u)\}$ for all $(x, u) \in X_i \times U$. We readily see that the controller C_i is feedback composable with the system S_i , $i \in \{1, 2\}$. Subsequently, we interpret S_1 as the concrete system and S_2 as its abstraction.

Let $Q \subseteq X_1 \times X_2$ be a strict relation. Then Q is an alternating simulation relation from S_1 to S_2 if the following holds for every pair $(x_1, x_2) \in Q$:

(ASR) If $u_2 \in U$, then there exists $u_1 \in U$ such that the condition

$$\emptyset \neq Q(x_1') \cap F_2(x_2, u_2) \tag{4}$$

holds for every $x'_1 \in F_1(x_1, u_1)$.

Note that usually there is an additional condition on outputs of related states, which here would have required the notion of approximate rather than ordinary alternating simulation relation [2, Def. 9.6]. Since that subtlety is not essential to our discussion, we omit it here in favor of a clearer presentation.

As already mentioned, alternating simulation relations are often used to prove the correctness of a particular abstraction-based controller design procedure. The very center of any such argument is the reproducibility of the system behavior of the concrete closed loop $C_1 \times S_1$ by the abstract closed loop $C_2 \times S_2$, i.e., for every solution $(0, v_1, (x_{c,1}, x_{s,1}), y_1)$ of $C_1 \times S_1$ on \mathbb{Z}_+ there exists a solution $(0, v_2, (x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ on \mathbb{Z}_+ satisfying

$$(x_{s,1}(t), x_{s,2}(t)) \in Q \text{ for all } t \in \mathbb{Z}_+.$$
(5)

This reproducibility property is then used to provide evidence that certain properties that the abstract closed loop $C_2 \times S_2$ satisfies, actually also hold for the concrete closed loop $C_1 \times S_1$.

In the first example, we show that (5) cannot hold if C_1 attains state information only through Q, i.e., if C_1 takes the form $C'_1 \circ Q$. In other words, the refined controller cannot be symbolic but requires full state information.

IV.1 Example. We consider the systems S_1 and S_2 which we graphically illustrate by

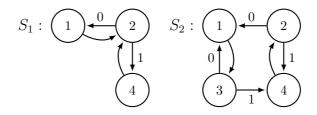
$$S_1: \underbrace{3}^{0} \underbrace{0}_{0} \underbrace{1}_{1} \underbrace{1}_{2} \underbrace{0}_{2} S_2: \underbrace{3}^{0} \underbrace{0}_{1} \underbrace{1}_{1} \underbrace{0}_{1} \underbrace{1}_{2} \underbrace{1}_{2}$$

The input and output alphabets of S_1 and S_2 are given by $U = \{0, 1\}$ and $Y = \{1, 2, 3\}$, respectively. The transition functions should be clear from the illustration, e.g. $F_1(2, 1) = \{1\}$ and $F_1(1, u) = \{1\}$ for any $u \in U$. It is also easily verified that the relation Q given by $Q = \{(1, 1), (2, 3), (3, 3)\}$ is an alternating simulation relation from S_1 to S_2 .

Let the abstract controller C_2 be static with $X_{c,2} = \{0\}$, $V_{c,2} = Y$, and $H_{c,2}(0,3) = \{(0,3)\}$, i.e., C_2 enables exactly the control letter 0 at the abstract state 3. If the concrete controller C_1 is symbolic, then, at the initial time, the sets of control letters enabled at the plant states 2 and 3 coincide. Indeed, these sets must only depend on the associated abstract states, and Q(2) = Q(3). In addition, by the symmetry of the plant S_1 , we may assume without loss of generality that the control letter 0 is enabled at the initial time, so that there exists a solution $(0, v_1, (x_{c,1}, x_{s,1}), y_1)$ of the closed loop $C_1 \times S_1$ satisfying $x_{s,1}(0) = x_{s,1}(1) = 2$. Then the condition (5) requires $x_{s,2}(0) = x_{s,2}(1) = 3$ to hold for some solution $(0, v_2, (x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ – a requirement that contradicts the dynamics of $C_2 \times S_2$. This shows that the property of reproducibility cannot be attained using a symbolic controller for the plant S_1 . The crucial point with this example is that the condition (ASR) cannot be satisfied if the choice of u_1 depends only on the abstract states associated with the plant state x_1 , but not directly on x_1 itself.

In the next example we show that a static controller C_2 for the abstraction S_2 cannot be refined to a static controller C_1 for the concrete system S_1 .

IV.2 Example. We consider the systems S_1 and S_2 with the transition functions illustrated graphically by



The input alphabet and the output alphabet is given by $U = \{0, 1\}$ and $Y = \{1, 2, 3, 4\}$, respectively. It is easily verified that the relation Q given by $Q = \{(1, 1), (2, 2), (2, 3), (4, 4)\}$ is an alternating simulation relation from S_1 to S_2 . In addition, in this example the relation Q satisfies even the more restrictive requirement that $u_1 = u_2$ holds in (ASR).

Suppose that the abstract controller C_2 is static and enables exactly the control letters 0 and 1 at the abstract states 2 and 3, respectively. If the concrete controller C_1 is static, then the set of control letters enabled at the plant state 2 does not vary with time. By the symmetry of the plant S_1 , we may again assume without loss of generality that the control letter 0 is enabled at the state 2, so that there exists a solution $(0, v_1, (x_{c,1}, x_{s,1}), y_1)$ of the closed loop $C_1 \times S_1$ satisfying $x_{s,1}(0) = x_{s,1}(2) = 1$. Then the condition (5) asks for $x_{s,2}(0) = x_{s,2}(2) = 1$ for some solution $(0, v_2, (x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ – a requirement that contradicts the dynamics of $C_2 \times S_2$. This shows that the property of reproducibility cannot be attained using a static controller for the plant S_1 despite the fact that the abstract controller is static. The crucial point with this example is that the condition (4) only mandates that for each transition from x_1 to x'_1 in S_1 there exists a state $x'_2 \in Q(x'_1)$ that is a successor of x_2 in S_2 , but it is not required that every $x'_2 \in Q(x'_1)$ succeeds x_2 ; consider e.g. the case $x_1 = x_2 = 1$, $x'_1 = x'_2 = 2$. As a result, the state 1 and 4 cannot precede the state 2 and 3, respectively, in S_2 , and so, implicitly, the static controller C_2 has some access to the history of the solution. In contrast, at the state 2 the dynamics of S_1 does not encode analogous information, which in fact could here only be provided by a controller for S_1 that is dynamic rather than static.

As our examples show, alternating simulation relations are not adequate for the controller refinement, whenever i) the concrete controller has merely symbolic state information and ii) the complexity of the refined controller should not exceed the complexity of the abstract controller. Moreover, we point out that in both examples the respective relation Q is not merely an alternating simulation relation according to our definition in (ASR), but also an 1-approximate bisimulation relation and 1approximate alternating bisimulation relation according to Definitions 9.5 and 9.8 in [2], respectively. Hence, the latter concepts also suffer from both issues described in this section.

V. FEEDBACK REFINEMENT RELATIONS

In this section, we introduce *feedback refinement relations* as a novel means to compare systems in the context of controller synthesis, in which we focus on simple systems.

A. Definition and basic properties

We start by introducing the behavior of a system, where we follow the notion of *infinitary completed* trace semantics [35].

V.1 Definition. Let S denote the system (3). The set $\mathcal{B}(S)$,

$$\mathcal{B}(S) = \{(u, y) | \exists_{v, x, T}(u, v, x, y) \text{ is a solution of } S \text{ on } [0; T[, and if T < \infty, then } F(x(T-1), v(T-1)) = \emptyset\},$$
(6)

is called the behavior of S.

Note that it often occurs that a system is non-continuable for a certain state-input pair, e.g. the terminating state of a terminating program. With our notion of system behavior, which possibly consists of finite signals as well as infinite signals, such signals are naturally included as valid elements.

In our definition of system relation below, we need a notion of state dependent admissible inputs. For any simple system S of the form (3), we define the set $U_S(x)$ of admissible inputs at the state $x \in X$ by

 $U_S(x) = \{ u \in U \mid F(x, u) \neq \emptyset \},\$

and the image of a subset $\Omega \subseteq X$ under U_S is denoted $U_S(\Omega)$.

V.2 Definition. Let S_1 and S_2 be simple systems,

$$S_i = (X_i, X_i, U_i, U_i, X_i, F_i, \mathrm{id})$$

$$\tag{7}$$

for $i \in \{1, 2\}$, and assume that $U_2 \subseteq U_1$. A strict relation $Q \subseteq X_1 \times X_2$ is a feedback refinement relation from S_1 to S_2 if the following holds for all $(x_1, x_2) \in Q$: (i) $U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$;

(*ii*) $u \in U_{S_2}(x_2) \Rightarrow Q(F_1(x_1, u)) \subseteq F_2(x_2, u).$

The fact that Q is a feedback refinement relation from S_1 to S_2 will be denoted $S_1 \preccurlyeq_Q S_2$, and we write $S_1 \preccurlyeq_Q S_2$ if $S_1 \preccurlyeq_Q S_2$ holds for some Q.

Intuitively, and similarly to simulation relations and their variants, a feedback refinement relation from a system S_1 to a system S_2 associates states of S_1 with states of S_2 , and imposes certain conditions on the local dynamics of the systems in the associated states. However, while e.g. alternating simulation relations only require that for each input u_2 admissible for S_2 there exists an associated input u_1 admissible for S_1 [2], our definition above additionally mandates that $u_1 = u_2$. Moreover, the definition of (approximate) alternating simulation relation requires that for each transition from x_1 to x'_1 in S_1 there exists a state x'_2 associated with x'_1 and a transition from x_2 to x'_2 in S_2 ; see condition (4). In contrast, feedback refinement relations require the existence of the latter transition for every state x'_2 associated with x'_1 .

We next show that the relation \preccurlyeq is reflexive and transitive.

V.3 Proposition. Let S_1 , S_2 and S_3 be simple systems. Then:

- (a) $S_1 \preccurlyeq_{\text{id}} S_1$.
- (b) If $S_1 \preccurlyeq_Q S_2$ and $S_2 \preccurlyeq_R S_3$, then $S_1 \preccurlyeq_{R \circ Q} S_3$.

Proof. Suppose that S_i is of the form (7), $i \in \{1, 2, 3\}$. The requirements in Def. V.2 are satisfied with $Q = \operatorname{id}, S_1 = S_2$ and $x_1 = x_2$, which proves (a). To prove (b), assume that $S_1 \preccurlyeq_Q S_2 \preccurlyeq_R S_3$. Then $R \circ Q$ is strict since both R and Q are so, and $U_3 \subseteq U_1$. Let $(x_1, x_3) \in R \circ Q$. Then there exists $x_2 \in X_2$ satisfying $(x_1, x_2) \in Q$ and $(x_2, x_3) \in R$. Thus, $U_{S_3}(x_3) \subseteq U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$, and so the condition (i) in Def. V.2 is satisfied with $R \circ Q$ and S_3 in place of Q and S_2 , respectively. As for the condition (ii), additionally assume that $u \in U_{S_3}(x_3)$. Then $u \in U_{S_2}(x_2)$, and $S_1 \preccurlyeq_Q S_2 \preccurlyeq_R S_3$ implies $Q(F_1(x_1, u)) \subseteq F_2(x_2, u)$ and $R(F_2(x_2, u)) \subseteq F_3(x_3, u)$. Then $R(Q(F_1(x_1, u))) \subseteq F_3(x_3, u)$, and so $S_1 \preccurlyeq_{R \circ Q} S_3$.

B. Feedback composability and behavioral inclusion

In the following, we present the main result of this section. We consider three systems S_1 , S_2 and Cand assume that C is feedback composable with S_2 . We first prove that, given a feedback refinement relation Q from S_1 to S_2 , $Q \circ S_1$ and S_1 are, respectively, feedback composable with C and $C \circ Q$. Subsequently, we show that the behavior of the closed loops $C \times (Q \circ S_1)$ and $(C \circ Q) \times S_1$ are both reproducible by the closed loop $C \times S_2$.

Even though we do not assign any particular role to the systems S_1 , S_2 and C, in foresight of the next section, where we use our result to develop abstraction-based solutions of general control problems, we might regard S_1 , S_2 and C as the plant, the abstraction and controller for the abstraction, respectively. In this context, we might assume that the state of S_1 is accessible only through the measurement map Q. In that case, $Q \circ S_1$ actually represents the system for which we seek a controller and the behavior of $\mathcal{B}(C \times (Q \circ S_1))$ is of interest. Alternatively, we may start with the premise that a controller for S_1 needs to be realizable on a digital device and hence, can accept only a finite input alphabet. In that case, we may interpret Q as an input quantizer for the discrete controller C and the behavior of $\mathcal{B}((C \circ Q) \times S_1)$ is of interest. In any case, we show that both behaviors are reproduced by the abstract closed loop $C \times S_2$. In the rest of the paper, we identify $\{0\} \times (U \times Y)$ with $U \times Y$ in the obvious way.

V.4 Theorem. Let Q be a feedback refinement relation from the system S_1 to the system S_2 , and assume that the system C is feedback composable with S_2 . Then the following holds.

- (i) C is feedback composable with $Q \circ S_1$, and $C \circ Q$ is feedback composable with S_1 .
- (*ii*) $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2).$
- (iii) For every $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ there exists a map x_2 such that $(u, x_2) \in \mathcal{B}(C \times S_2)$ and $(x_1(t), x_2(t)) \in Q$ for all t in the domain of x_1 .

Proof. By our hypotheses, S_1 and S_2 are simple, so we assume that these systems are of the form (7). Moreover,

$$Q \circ S_1 = (X_1, X_1, U_1, U_1, X_2, F_1, H_1'), \tag{8}$$

where $U_2 \subseteq U_1$ and H'_1 takes the form $H'_1(x, u) = Q(x) \times \{u\}$. Let the system C be of the form

$$C = (X_c, X_{c,0}, U_c, V_c, Y_c, F_c, H_c),$$
(9)

and observe that $Y_c \subseteq U_1$ and $X_2 \subseteq U_c$ as C is feedback composable (f.c.) with S_2 . Moreover, since $X_1 \neq \emptyset$ and Q is strict, the serial composition $C \circ Q$ is well-defined,

$$C \circ Q = (X_c, X_{c,0}, X_1, V_c, Y_c, F_c, H'_c),$$

where H'_c takes the form $H'_c(x_c, x_1) = H_c(x_c, Q(x_1))$.

To prove (i), we first observe that the conditions

$$x_2 \in Q(x_1), (u, v) \in H_c(x_c, x_2), F_1(x_1, u) = \emptyset$$
(10)

together imply $F_c(x_c, v) = \emptyset$. Indeed, it follows from (10) and the requirement (i) in Definition V.2 that $F_2(x_2, u) = \emptyset$, and our claim follows as C is f.c. with S_2 . This shows that C is f.c. with $Q \circ S_1$. Similarly, let $x_1 \in X_1$, $(u, v) \in H'_c(x_c, x_1)$ and $F_1(x_1, u) = \emptyset$. Then, by the definition of H'_c , there exists $x_2 \in Q(x_1)$ such that $(u, v) \in H_c(x_c, x_2)$. Then (10) holds, and so $F_c(x_c, v) = \emptyset$ as we have already shown. Hence, $C \circ Q$ is f.c. with S_1 , which completes the proof of (i).

To prove (ii), let $(u, x_2) \in \mathcal{B}(C \times (Q \circ S_1))$ be defined on $[0; T[, T \in \mathbb{N} \cup \{\infty\}\}$. Then there exist maps x_c, x_1 and v such that $(0, (v, u), (x_c, x_1), (u, x_2))$ is a solution of $C \times (Q \circ S_1)$ on [0; T[. Moreover, if additionally $T < \infty$, then we also have

$$F_c(x_c(T-1), v(T-1)) = \emptyset \lor F_1(x_1(T-1), u(T-1)) = \emptyset.$$
(11)

By Proposition III.4, (u, u, x_1, x_2) is a solution of $Q \circ S_1$ on [0; T[, and (x_2, v, x_c, u) is a solution of C on [0; T[. The former fact implies the following:

$$\forall_{t \in [0:T]} \ x_2(t) \in Q(x_1(t)),\tag{12}$$

$$\forall_{t \in [0; T-1[} x_1(t+1) \in F_1(x_1(t), u(t)).$$
(13)

We claim that (u, u, x_2, x_2) is a solution of S_2 , so that $(0, (v, u), (x_c, x_2), (u, x_2))$ is a solution of $C \times S_2$ by Proposition III.4. First, we observe that $F_2(x_2(t), u(t)) \neq \emptyset$ for every $t \in [0; T - 1]$. Indeed, $(u(t), v(t)) \in H_c(x_c(t), x_2(t))$ for every such t since (x_2, v, x_c, u) is a solution of C on [0; T]. Hence, $F_2(x_2(t), u(t)) = \emptyset$ for some $t \in [0; T - 1]$ implies $F_c(x_c(t), v(t)) = \emptyset$ as C is f.c. with S_2 . This is a contradiction as $x_c(t+1) \in F_c(x_c(t), v(t))$, so $F_2(x_2(t), u(t)) \neq \emptyset$ for every $t \in [0; T - 1]$. Consequently, $u(t) \in U_{S_2}(x_2(t))$ for all $t \in [0; T - 1]$, so (12), (13) and the requirement (ii) in Definition V.2 imply that $x_2(t+1) \in F_2(x_2(t), u(t))$ for all $t \in [0; T - 1]$. This shows that $(0, (v, u), (x_c, x_2), (u, x_2))$ is a solution of $C \times S_2$ on [0; T].

Finally, we see that if $T < \infty$ and $u(T-1) \in U_{S_2}(x_2(T-1))$, then (12) and the requirement (i) in Definition V.2 together imply $F_1(x_1(T-1), u(T-1)) \neq \emptyset$, and in turn, (11) shows that $F_c(x_c(T-1), v(T-1)) = \emptyset$. Thus, $(u, x_2) \in \mathcal{B}(C \times S_2)$, which proves (ii).

To prove (iii), let $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ be defined on $[0; T[, T \in \mathbb{N} \cup \{\infty\}\}$. Then there exist maps x_c and v such that $(0, (v, u), (x_c, x_1), (u, x_1))$ is a solution of $(C \circ Q) \times S_1$ on [0; T[. Moreover, if additionally $T < \infty$, then we also have

$$F_c(x_c(T-1), v(T-1)) = \emptyset \lor F_1(x_1(T-1), u(T-1)) = \emptyset.$$
(14)

By Proposition III.4, (u, u, x_1, x_1) and (x_1, v, x_c, u) is a solution of S_1 and $C \circ Q$, respectively. In particular, by the definition of H'_c , there exists a map $x_2: [0; T[\to X_2 \text{ such that } x_2(t) \in Q(x_1(t)) \text{ and } (u(t), v(t)) \in H_c(x_c(t), x_2(t)) \text{ for all } t \in [0; T[$. Then (x_2, v, x_c, u) and (u, u, x_1, x_2) is a solution of C and $Q \circ S_1$, respectively, so $(0, (v, u), (x_c, x_1), (u, x_2))$ is a solution of $C \times (Q \circ S_1)$ by Proposition III.4. We next observe that if $T < \infty$ and $F_1(x_1(T-1), u(T-1)) \neq \emptyset$, then (14) implies $F_c(x_c(T-1), v(T-1)) = \emptyset$. This shows that $(u, x_2) \in \mathcal{B}(C \times (Q \circ S_1))$, and so (iii) follows from (ii).

Next we show, that feedback refinement relations are not only sufficient, but indeed necessary for the controller refinement as considered in this paper.

V.5 Theorem. Let S_1 and S_2 be simple systems of the form (7), and let $Q \subseteq X_1 \times X_2$ be a strict relation. If for every system C that is feedback composable with S_2 follows that C is feedback composable with $Q \circ S_1$ and $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2)$ holds, then Q is a feedback refinement relation from S_1 to S_2 .

Proof. In the proof we consider systems $Q \circ S_1$ of the form (8). Let C be given by $(\{0\}, \{0\}, X_2, \{0\}, U_2, F_c, H_c)$ with $F_c(0, 0) = \emptyset$ and H_c being strict. Then, C is feedback composable (f.c.) with S_2 , and in turn, C is f.c. with $Q \circ S_1$ by our hypothesis. This implies $U_2 \subseteq U_1$ as required in Def. V.2.

To prove that Q satisfies the condition (i) in Def. V.2, we let $(x_1, x_2) \in Q$ and $u \in U_{S_2}(x_2)$ and show that $F_1(x_1, u) \neq \emptyset$. Let C be given by $(\{0\}, \{0\}, X_2, X_2, U_2, F_c, H_c)$ with $H_c(0, x'_2) = \{(u, x'_2)\}$ for all $x'_2 \in X_2$ and $F_c(0, x_2) = \{0\}$ and $F_c(0, x'_2) = \emptyset$ for $x'_2 \in X_2 \setminus \{x_2\}$. Then C is f.c. with S_2 . In particular, the condition (Z) in Definition III.3 reduces to $F_2(x_2, u) \neq \emptyset$. Then C is also f.c. with $Q \circ S_1$ by our hypothesis, and here the condition (Z) implies $F_1(x_1, u) \neq \emptyset$ and the claim follows.

To prove that Q satisfies the condition (ii) in Definition V.2, we choose C by $(\{0\}, \{0\}, X_2, X_2, U_2, F_c, H_c\}$ with H_c and F_c defined by: if $U_{S_2}(x_2) = \emptyset$ we set $H_c(0, x_2) = U_2 \times \{x_2\}$ and $F_c(0, x_2) = \emptyset$; otherwise $H_c(0, x_2) = U_{S_2}(x_2) \times \{x_2\}$ and $F_c(0, x_2) = \{0\}$. With this definition of C condition (Z) holds and C is f.c. with S_2 , and by our hypothesis, C is also f.c. with $Q \circ S_1$. Suppose that condition (ii) does not hold, then there exist $(x_1, x_2) \in Q$, $u \in U_{S_2}(x_2), x'_1 \in F_1(x_1, u)$ and $x'_2 \in Q(x'_1)$ such that $x'_2 \notin F_2(x_2, u)$. Let $\bar{x}_1 = x_1x'_1$ and $\bar{u} = uu'$ with $(u', x'_2) \in H_c(0, x'_2)$. Then $(\bar{u}, \bar{u}, \bar{x}_1, \bar{x}_1)$ is a solution of S_1 on [0; 2[. Define $\bar{x}_2 = x_2x'_2$ and observe that $(\bar{u}, \bar{u}, \bar{x}_1, \bar{x}_2)$ is a solution of $Q \circ S_1$. Let $\bar{x}_c = 00$, since $F_2(x_2, u) \neq \emptyset$, we see that $(u, x_2) \in H_c(0, x_2)$ and $\{0\} = F_c(0, x_2)$. Also $(u', x'_2) \in H_c(0, x'_2)$ by our choice of u' and thus $(\bar{x}_2, \bar{x}_2, \bar{x}_c, \bar{u})$ is a solution of C. Hence by Proposition III.4 we see that $(0, (\bar{x}_2, \bar{u}), (\bar{x}_c, \bar{x}_1), (\bar{u}, \bar{x}_2))$ is a solution of $C \times (Q \circ S_1)$. Consider $(\hat{u}, \hat{x}_2) \in \mathcal{B}(C \times (Q \circ S_1))$ with $\hat{u}|_{[0;2[} = \bar{u}$ and $\hat{x}_2|_{[0;2[} = \bar{x}_2$. Since $\bar{x}_2(1) \notin F_2(\bar{x}_2(0), \bar{u}(0))$ the sequence $(0, (\bar{x}_2, \bar{u}), (\bar{x}_c, \bar{x}_2), (\bar{u}, \bar{x}_2))$ cannot be a solution of $C \times S_2$, and so $(\hat{u}, \hat{x}_2) \notin \mathcal{B}(C \times S_2)$. This is a contradiction, which establishes condition (ii) in Definition V.2.

VI. Symbolic Controller Synthesis

In this section, we propose a controller synthesis technique based on the concept of feedback refinement relations which resolves the state information and refinement complexity issues as explained and illustrated in Sections I and IV, applies to general specifications, and produces controllers that are robust with respect to various disturbances. We follow the general three step procedure of abstraction-based synthesis outlined in Section I, where we focus on the first and third steps. Our results will be complemented by the computational method presented in Section VIII, whereas the solution of the abstract control problem – the second step of the general procedure – is beyond the scope of the present paper. Indeed, large classes of these problems can be solved efficiently using standard algorithms, e.g. [2]-[6], [17].

A. Solution of control problems

We begin with the definition of the synthesis problem.

VI.1 Definition. Let S denote the system (3). Given a set Z, any subset $\Sigma \subseteq Z^{\infty}$ is called a specification on Z. A system S is said to satisfy a specification Σ on $U \times Y$ if $\mathcal{B}(S) \subseteq \Sigma$. Given a specification Σ on $U \times Y$, the system C solves the control problem (S, Σ) if C is feedback composable with S and the closed loop $C \times S$ satisfies Σ .

It is clear that we can use linear temporal logic (LTL) to define a specification for a given system S. Indeed, suppose that we are given a finite set \mathcal{P} of atomic propositions, a labeling function

 $L: U \times Y \rightrightarrows \mathcal{P}$ and an LTL formula φ defined over \mathcal{P} , see e.g. [24, Chapter 5]. Then we can formulate the control problem (S, Σ) to enforce the formula φ on S using the specification

$$\Sigma = \{ (u, y) \in (U \times Y)^{\mathbb{Z}_+} \mid L \circ (u, y) \text{ satisfies } \varphi \}.$$

Our notion of specification is not limited to LTL, e.g. "y(t) = 1 holds for all even $t \in \mathbb{Z}_+$ " is not expressible in LTL [24, Remark 5.43], but is a valid specification in our framework.

We are now going to solve control problems using Theorem V.4. As we have already discussed, the concrete control problem (S_1, Σ_1) will not be solved directly. Instead, we will consider an auxiliary problem for the abstraction ("abstract control problem"), whose solution will induce a solution of the concrete problem.

VI.2 Definition. Let S_1 and S_2 be simple systems of the form (7), let Σ_1 be a specification on $U_1 \times X_1$, and let $Q \subseteq X_1 \times X_2$ be a strict relation. A specification Σ_2 on $U_2 \times X_2$ is called an abstract specification associated with S_1 , S_2 , Q and Σ_1 , if the following condition holds.

If $(u, x_2) \in \Sigma_2$, where x_2 and u are defined on [0; T[for some $T \in \mathbb{N} \cup \{\infty\}$, and if $x_1: [0; T[\to X_1$ satisfies $(x_1(t), x_2(t)) \in Q$ for all $t \in [0; T[$, then $(u, x_1) \in \Sigma_1$.

For the sake of simplicity, we write $(S_1, \Sigma_1) \preccurlyeq_Q (S_2, \Sigma_2)$ whenever $S_1 \preccurlyeq_Q S_2$ and Σ_2 is an abstract specification associated with S_1, S_2, Q and Σ_1 . The result presented below shows how to use a solution of the abstract control problem to arrive at a solution of the concrete control problem, resulting in the closed loop in Fig. 1.

VI.3 Theorem. If $(S_1, \Sigma_1) \preccurlyeq_Q (S_2, \Sigma_2)$ and the abstract controller C solves the control problem (S_2, Σ_2) , then the refined controller $C \circ Q$ solves the control problem (S_1, Σ_1) .

Proof. As C solves (S_2, Σ_2) , C is feedback composable with S_2 , and hence, $C \circ Q$ is feedback composable with S_1 by Theorem V.4.

It remains to show that $\mathcal{B}((C \circ Q) \times S_1) \subseteq \Sigma_1$. So, let $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ be arbitrary and invoke Theorem V.4 again to see that there exists a map x_2 such that $(u, x_2) \in \mathcal{B}(C \times S_2)$ and $(x_1(t), x_2(t)) \in Q$ for all t in the domain of x_2 . Then $(u, x_2) \in \Sigma_2$ since C solves (S_2, Σ_2) , and the definition of the abstract specification Σ_2 shows that $(u, x_1) \in \Sigma_1$.

B. Uncertainties and disturbances

We next show that it is an easy task in our framework to synthesize controllers that are robust with respect to various disturbances including plant uncertainties, input disturbances and measurement errors. In particular, we demonstrate that the synthesis of a robust controller can be reduced to the solution of an auxiliary, unperturbed control problem.

Let us consider the closed loop illustrated in Fig. 4 consisting of a plant given by a simple system S_1 of the form (7), the *perturbation maps* P_i , given by strict set-valued maps with non-empty domains

$$P_1: U_1 \rightrightarrows U_1, \quad P_2: X_1 \rightrightarrows X_1, P_3: \hat{U}_1 \rightrightarrows Y_1, \quad P_4: X_1 \rightrightarrows Y_2,$$
(15)

and a strict quantizer

$$Q \colon \hat{X}_1 \rightrightarrows X_2. \tag{16}$$

We seek to synthesize a controller given as a system

$$C = (X_c, X_{c,0}, X_2, V_c, \hat{U}_1, F_c, H_c),$$
(17)

to robustly enforce a given specification Σ_1 on $Y_1 \times Y_2$.

The behavior of the closed loop in Fig. 4 is defined as the set of all sequences $(y_1, y_2) \in (Y_1 \times Y_2)^{[0;T[}, T \in \mathbb{N} \cup \{\infty\})$, for which there exist a solution (u, u, x, x) of S_1 on [0; T[and a solution (u_c, v_c, x_c, y_c) of C on [0; T[that satisfy the following two conditions:

(i) For all $t \in [0; T]$ we have

$$u(t) \in P_1(y_c(t)), \quad u_c(t) \in Q(P_2(x(t))), y_1(t) \in P_3(y_c(t)), \quad y_2(t) \in P_4(x(t)).$$
(18)

(ii) If $T < \infty$, then

$$F_1(x(T-1), u(T-1)) = \emptyset, \text{ or} F_c(x_c(T-1), v_c(T-1)) = \emptyset.$$
(19)

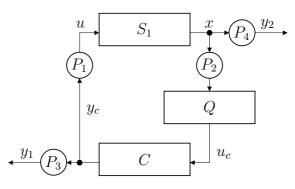


Figure 4. Various perturbations in the closed loop.

It is straightforward to observe, that the perturbations maps P_1 and P_2 may be used to model input disturbances and measurement errors, respectively. We assume that the uncertainties of the dynamics of S_1 have already been modeled by the set-valued transition function F_1 . The controller Cand the quantizer Q, which will usually be discrete, are not subject to any additional perturbations either. The maps P_3 and P_4 are useful in the presence of output disturbances. For example, the plant S_1 might represent a sampled variant of a continuous-time control system and the specification of the desired behavior is naturally formulated in continuous time, rather than in discrete time. In that context, one can use P_3 and P_4 to "robustify" the specification like in [36] such that properties of the sampled behavior carry over to the continuous-time behavior.

Given some specifications Σ_1 on $Y_1 \times Y_2$ and $\hat{\Sigma}_1$ on $\hat{U}_1 \times X_1$, we call $\hat{\Sigma}_1$ a robust specification of Σ_1 w.r.t. P_3 and P_4 if for the functions $(y_c, x, y_1, y_2) \in (\hat{U}_1 \times X_1 \times Y_1 \times Y_2)^{[0;T]}$, $T \in \mathbb{N} \cup \{\infty\}$, we have that

$$(y_c, x) \in \hat{\Sigma}_1$$
 and $\forall_{t \in [0; T[} y_1(t) \in P_3(y_c(t)), y_2(t) \in P_4(x(t))$

implies $(y_1, y_2) \in \Sigma_1$.

In the following result, we present sufficient conditions for a controller C to robustly enforce a given specification Σ_1 on the perturbed closed loop illustrated in Fig. 4, in terms of the auxiliary simple system \hat{S}_1 ,

$$\hat{S}_1 = (X_1, X_1, \hat{U}_1, \hat{U}_1, X_1, \hat{F}_1, \mathrm{id}),
\hat{F}_1(x, u) = F_1(x, P_1(u)),$$
(20)

together with a robust specification $\hat{\Sigma}_1$ of Σ_1 . We show in the subsequent corollary, which follows immediately by Theorem VI.3, how to use an abstraction (S_2, Σ_2) to synthesize such a controller C.

VI.4 Theorem. Consider a simple system S_1 , perturbation maps P_i , $i \in [1; 4]$, a strict quantizer Q, and a controller C as illustrated in Fig. 4 and respectively defined in (7), (15), (16) and (17), and assume that F_1 is strict. Let Σ_1 be a specification on $Y_1 \times Y_2$. Let $(\hat{S}_1, \hat{\Sigma}_1)$ be an auxiliary control problem, where \hat{S}_1 follows from S_1 according to (20) and $\hat{\Sigma}_1$ is a robust specification of Σ_1 w.r.t. P_3 and P_4 .

If $C \circ \hat{Q}$, with $\hat{Q} = Q \circ P_2$, solves the control problem $(\hat{S}_1, \hat{\Sigma}_1)$, then the behavior of the perturbed closed loop in Fig. 4 is a subset of Σ_1 .

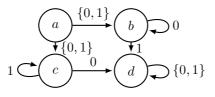
Proof. Our assumptions imply that $C \circ \hat{Q}$ is feedback composable with \hat{S}_1 . Using Definition III.3, Proposition III.4, the strictness of F_1 , and the properties (18)-(19), it is straightforward to show that (y_1, y_2) is an element of the behavior of the closed loop in Fig. 4 iff there exists $(y_c, x) \in \mathcal{B}((C \circ \hat{Q}) \times \hat{S}_1)$ satisfying $y_1(t) \in P_3(y_c(t))$ and $y_2(t) \in P_4(x(t))$ for all t. Consequently, if (y_1, y_2) is an element of the behavior of the closed loop in Fig. 4, then there exist $(y_c, x) \in \hat{\Sigma}_1$ satisfying $y_1(t) \in P_3(y_c(t))$ and $y_2(t) \in P_4(x(t))$ for all t, and so $(y_1, y_2) \in \Sigma_1$ by the definition of $\hat{\Sigma}_1$.

VI.5 Corollary. In the context of Theorem VI.4, if C solves an abstract control problem (S_2, Σ_2) with $(\hat{S}_1, \hat{\Sigma}_1) \preccurlyeq_{\hat{Q}} (S_2, \Sigma_2)$, where X_2 is the state space of S_2 , then the behavior of the closed loop in Fig. 4 is a subset of Σ_1 .

In the following example we demonstrate that it is crucial to account for the measurement errors P_2 in terms of the auxiliary quantizer $\hat{Q} = Q \circ P_2$, as opposed to accounting for those type of disturbances in terms of an alternative auxiliary system $\tilde{S}_1 = (X_1, X_1, \hat{U}_1, \hat{U}_1, X_1, \tilde{F}_1, \text{id})$ with \tilde{F}_1 given by

$$\tilde{F}_1(x_1, u) = P_2(F_1(x_1, P_1(u))).$$
 (21)

VI.6 Example. We consider the simple system S_1 of the form (7) with the transition function illustrated graphically



The state and input alphabet are given by $X_1 = \{a, b, c, d\}$ and $U_1 = \{0, 1\}$, respectively. Suppose we are given the specification Σ_1 on $U_1 \times X_1$ defined implicitly by $(u, x) \in \Sigma_1$ iff d is in the image of x. Let us consider the quantizer Q = id and the perturbation maps $P_1 = P_3 = P_4 = \text{id}$ and P_2 defined by $P_2(a) = \{a\}, P_2(b) = P_2(c) = \{b, c\}$ and $P_2(d) = \{d\}$. Let the auxiliary system \tilde{S}_1 coincide with S_1 except the transition function is given by $\tilde{F}_1(x, u) = P_2(F_1(x, u))$.

The controller $C \circ Q$, with C given as static system with strict transition function and output map H_c : $\{0\} \times X_1 \rightrightarrows U_1 \times X_1$ defined by $H_c(0, a) = H_c(0, d) = U_1 \times \{a\}, H_c(0, b) = \{(1, a)\}, H_c(0, c) = \{(0, a)\}$ solves the control problem (\tilde{S}_1, Σ_1) . However, $(u, x) = ((0, a), (1, c), (1, c), (1, c), \ldots)$ is an element of the behavior of the closed loop according to Fig. 4 and yet violates the specification Σ_1 .

As the example demonstrates, we cannot rely on the auxiliary system with transition function (21) to synthesize a robust controller but we need a quantizer that is robust with respect to disturbances. That is essentially expressed by requiring that $C \circ \hat{Q}$ with $\hat{Q} = Q \circ P_2$ solves the auxiliary control problem $(\hat{S}_1, \hat{\Sigma}_1)$. Intuitively, we require that the controller C "works" with any quantizer symbol $x_2 \in Q(P_2(x_1))$ no matter how the disturbance P_2 is acting on the state x_1 . Note that in Example VI.6, the controller $C \circ (\mathrm{id} \circ P_2)$ does not solve the control problem $(\hat{S}_1, \hat{\Sigma}_1)$ (which in this case equals (S_1, Σ_1)).

Finally, we would like to mention that in the context of control systems, any symbolic controller synthesis procedure that is based on a deterministic quantizer is bound to be *non-robust*. Indeed, consider the context of Theorem VI.4 and suppose that $X_1 = \mathbb{R}^n$, X_2 is a partition of X_1 and let $P_2(x_1)$ equal the closed Euclidean ball with radius $\varepsilon \ge 0$ centered at x_1 . Let us consider the deterministic quantizer $Q = \epsilon$. Then $\hat{Q} = Q \circ P_2$ is deterministic only in the degenerate case $\varepsilon = 0$.

VII. CANONICAL FEEDBACK REFINEMENT RELATIONS

In this section, we show that the set membership relation \in , together with an abstraction whose state alphabet is a cover of the concrete state alphabet is canonical. A *cover* of a set X is a set of

subsets of X whose union equals X.

We show that $(S_1, \Sigma_1) \preccurlyeq_Q (S_3, \Sigma_3)$ implies that there exist (S_2, Σ_2) , with X_2 being a cover of X_1 by non-empty subsets, together with a relation R such that the following holds:

$$(S_1, \Sigma_1) \preccurlyeq_{\in} (S_2, \Sigma_2) \preccurlyeq_R (S_3, \Sigma_3).$$

This implies that if we can solve the concrete control problem (S_1, Σ_1) using some abstract control problem (S_3, Σ_3) , then we can equally use an abstract control problem (S_2, Σ_2) with X_2 being to a cover of X_1 by non-empty subsets. Moreover, (S_2, Σ_2) can be derived from the problem (S_3, Σ_3) and the quantizer Q alone and is otherwise independent of (S_1, Σ_1) .

A. Canonical abstractions

VII.1 Proposition. Let S_1 and S_2 be simple systems of the form (7), in which X_2 is a cover of X_1 by non-empty subsets and $U_2 \subseteq U_1$. Then $S_1 \preccurlyeq_{\in} S_2$ iff the following conditions hold.

- (i) $x \in \Omega \in X_2$ implies $U_{S_2}(\Omega) \subseteq U_{S_1}(x)$.
- (ii) If $\Omega, \Omega' \in X_2$, $u \in U_{S_2}(\Omega)$ and $\Omega' \cap F_1(\Omega, u) \neq \emptyset$, then $\Omega' \in F_2(\Omega, u)$.

The above result, whose straightforward proof we omit, will be used in our proof of the canonicity result, Theorem VII.2. It additionally indicates constructive methods to compute a canonical abstraction S_2 of a plant S_1 if the abstract state space X_2 and the input alphabet $U_2 \subseteq U_1$ are given. From condition (ii) it follows that, if $\Omega \in X_2$, $u \in U_2$ and $F_1(x, u) \neq \emptyset$ for every $x \in \Omega$, then we may either choose $F_2(\Omega, u)$ to be empty, which is of course not desirable², or ensure that the latter set contains every cell Ω' that intersects the attainable set $F_1(\Omega, u)$ of the cell Ω under the control letter u. This can be achieved by numerically over-approximating attainable sets, for which many algorithms are available, see e.g. [7] and Section VIII.

On the other hand, condition (i) requires that $F_2(\Omega, u)$ is empty whenever $F_1(x, u)$ is so for some $x \in \Omega$. This raises the question of how to detect the phenomenon of blocking of the dynamics of the plant. If the transition function F_1 is explicitly given, we assume that its description directly facilitates the detection of blocking. In the case that the plant represents a sampled system, so that F_1 is the time- τ -map of some continuous-time control system, blocking can usually be detected in the course of over-approximating attainable sets. For example, if an over-approximation W of the attainable set $F_1(\Omega, u)$ is computed using interval arithmetic, and if $F_1(x, u) = \emptyset$ for some $x \in \Omega$, then W will be unbounded, e.g. [37, Chapter II.3], which is easily detected.

B. Canonicity result

Before we state and prove the canonicity result, we introduce a technical condition that we impose on the feedback refinement relation Q from (S_1, Σ_1) to (S_3, Σ_3) , i.e.,

(C) if
$$\emptyset \neq Q^{-1}(x) = Q^{-1}(\tilde{x}), \ \emptyset \neq Q^{-1}(x') = Q^{-1}(\tilde{x}'), \ \tilde{x}' \in F_3(\tilde{x}, u), \text{ and } u \in U_{S_3}(x), \text{ then } x' \in F_3(x, u).$$

We point out that condition (C) is not an essential restriction and it actually holds for a great variety of abstractions and relations. For example, it automatically holds if the abstraction S_3 is defined as a quotient system [2, Definition 4.17]. In that case, the elements of X_3 correspond to the equivalence classes of an equivalence relation on X_1 . Therefore, we have that $Q^{-1}(x) = Q^{-1}(\tilde{x})$ implies $x = \tilde{x}$ and condition (C) is trivially satisfied. Similarly, relations that are based on level sets of simulation functions $V: X_1 \times X_3 \to \mathbb{R}_+$ with $X_1, X_3 \subseteq \mathbb{R}^n$, see e.g. [18], for popular choices of simulation functions like $V(x_1, x_3) = \sqrt{(x_1 - x_3)^\top P(x_1 - x_3)}$ with P being a positive definite

²One should always choose $F_2(\Omega, u) \neq \emptyset$, since it enlarges the set of control letters available to any abstract controller and thereby facilitates the solution of the abstract control problem.

matrix, where x^{\top} denotes the transpose of x, satisfy (C). In this case, the relation is given by $Q = \{(x_1, x_3) \in X_1 \times X_3 \mid V(x_1, x_3) \leq \varepsilon\}$ and again $Q^{-1}(x) = Q^{-1}(\tilde{x})$ implies $x = \tilde{x}$ and we conclude that (C) holds. Lastly, the condition (C) also holds, for the case that Q is given and the abstraction S_3 is computed using a deterministic algorithm to over-approximate attainable sets. This is immediate from the following reformulation of the condition (ii) in Definition V.2: If $x_2, x'_2 \in X_2$, $u \in U_{S_2}(x_2)$, and $Q^{-1}(x'_2) \cap F_1(Q^{-1}(x_2), u) \neq \emptyset$, then $x'_2 \in F_2(x_2, u)$.

VII.2 Theorem. Let (S_3, Σ_3) be a control problem, in which S_3 is simple and of the form (7). Let X_1 be any set, and assume that $Q: X_1 \rightrightarrows X_3$ satisfies the condition (C).

Then there exist a simple system S_2 of the form (7), a relation $R \subseteq X_2 \times X_3$ and a specification Σ_2 on $U_2 \times X_2$ such that the following holds.

(*) If $(S_1, \Sigma_1) \preccurlyeq_Q (S_3, \Sigma_3)$ and the system S_1 has state space X_1 , then $(S_1, \Sigma_1) \preccurlyeq_{\in} (S_2, \Sigma_2) \preccurlyeq_R (S_3, \Sigma_3)$ and X_2 is a cover of X_1 by non-empty subsets.

Proof. We will prove that (*) holds for the following choices of S_2 , R and Σ_2 :

 $X_2 = \{\Omega \mid \emptyset \neq \Omega = Q^{-1}(x) \land x \in X_3\}, \ R(\Omega) = \{x \in X_3 \mid \Omega = Q^{-1}(x)\}, \ U_2 = U_3, \ F_2(\Omega, u) = R^{-1}(F_3(R(\Omega), u)), \text{ and } (u, \Omega) \in (U_2 \times X_2)^{\infty} \text{ is an element of } \Sigma_2 \text{ iff there exists } (u, x_3) \in \Sigma_3 \text{ satisfying } (\Omega(t), x_3(t)) \in R \text{ for all } t \text{ in the domain of } u.$

To establish (*), assume that $(S_1, \Sigma_1) \preccurlyeq_Q (S_3, \Sigma_3)$. Then Q is strict, which already proves our claim on X_2 , and S_1 is simple, and so we can assume that S_1 takes the form (7).

To prove $S_1 \preccurlyeq_{\in} S_2$, we first notice that the condition (i) in Proposition VII.1 is satisfied. Indeed, let $x_1 \in \Omega \in X_2$ and $u \in U_{S_2}(\Omega)$. By our choice of F_2 and R, there exists x_3 satisfying $(x_1, x_3) \in Q$ and $u \in U_{S_3}(x_3)$. Then $u \in U_{S_1}(x_1)$ by Def. V.2 applied to $S_1 \preccurlyeq_Q S_3$. To establish the condition (ii) in Prop. VII.1, we let $\Omega, \Omega' \in X_2$ and $u \in U_{S_2}(\Omega)$ and assume that $\Omega' \cap F_1(\Omega, u) \neq \emptyset$. By the latter fact there exist $x_1 \in \Omega$ and $x'_1 \in \Omega' \cap F_1(x_1, u)$, and $u \in U_{S_2}(\Omega)$ implies that there exists x_3 such that $\Omega = Q^{-1}(x_3)$ and $u \in U_{S_3}(x_3)$. We pick x'_3 satisfying $\Omega' = Q^{-1}(x'_3)$. Then $(x_1, x_3), (x'_1, x'_3) \in Q$, and so $S_1 \preccurlyeq_Q S_3$ implies $x'_3 \in Q(x'_1) \subseteq F_3(x_3, u)$. Hence, $\Omega' \in F_2(\Omega, u)$ by our choice of F_2 . This proves $S_1 \preccurlyeq_{\in} S_2$.

To prove $S_2 \preccurlyeq_R S_3$, let $(\Omega, x_3) \in R$ and $u \in U_{S_3}(x_3)$ and pick any $x_1 \in \Omega$. Then $(x_1, x_3) \in Q$ by our choice of R, and using $S_1 \preccurlyeq_Q S_3$ we obtain $u \in U_{S_1}(x_1)$. The latter fact implies that there exists $x'_1 \in F_1(x_1, u)$, and using $S_1 \preccurlyeq_Q S_3$ again we see that $Q(x'_1) \subseteq F_3(x_3, u)$. Since Q is strict we may pick $x'_3 \in Q(x'_1)$. Then $R^{-1}(x'_3) \neq \emptyset$, and hence, $u \in U_{S_2}(\Omega)$ by the definition of F_2 , which proves the condition (i) in Definition V.2. To prove the condition (ii) in that definition, let $(\Omega, x_3) \in R$, $u \in U_{S_3}(x_3)$ and $\Omega' \in F_2(\Omega, u)$. Then $\Omega' \in R^{-1}(F_3(\Omega, u))$, so there exist \tilde{x}_3 and $\tilde{x}'_3 \in F_3(\tilde{x}_3, u)$ satisfying $\Omega = Q^{-1}(\tilde{x}_3)$ and $\Omega' = Q^{-1}(\tilde{x}'_3)$. Then condition (C) implies $x'_3 \in F_3(x_3, u)$, and in turn, $R(\Omega') \subseteq F_3(x_3, u)$.

To complete the proof, we notice that, by the definition of Σ_2 , Σ_3 is an abstract specification associated with S_2 , S_3 , R and Σ_2 , which shows $(S_2, \Sigma_2) \preccurlyeq_R (S_3, \Sigma_3)$. Finally, to prove $(S_1, \Sigma_1) \preccurlyeq_{\in} (S_2, \Sigma_2)$, let $(u, \Omega) \in \Sigma_1$, assume that u is defined on [0; T[, and let $x_1 \colon [0; T[\to X_1 \text{ satisfy } x_1(t) \in \Omega(t)]$ for all $t \in [0; T[$. Then, by the definition of Σ_2 , there exists $(u, x_3) \in \Sigma_3$ such that $R(\Omega(t)) = \{x_3(t)\}$ for all $t \in [0; T[$. The latter condition implies $(x_1(t), x_3(t)) \in Q$, and $(S_1, \Sigma_1) \preccurlyeq_Q (S_3, \Sigma_3)$ implies $(u, x_1) \in \Sigma_1$.

VIII. COMPUTATION OF ABSTRACTIONS FOR PERTURBED SAMPLED CONTROL SYSTEMS

In the previous section we have seen that the computation of abstractions basically reduces to the over-approximation of attainable sets of the plant. A large number of over-approximation methods have been proposed which apply to different classes of systems, e.g. [2], [7], [38]–[40]. In this section, we present an approach to over-approximate attainable sets of continuous-time perturbed control systems, based on a matrix-valued Lipschitz inequality.

A. The sampled system

Let us consider a perturbed control system of the form

$$\dot{x} \in f(x, u) + W \tag{22}$$

with $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^n$. We assume throughout this section that U is non-empty, W contains the origin, and that $f(\cdot, u)$ is locally Lipschitz for all $u \in U$. We use the set W to represent various uncertainties in the dynamics of the control system (22).

For $\tau \in \mathbb{R}_+$ and an interval $I \subseteq [0, \tau]$, a solution of (22) on I with (constant) input $u \in U$ is defined as an absolutely continuous function $\xi \colon I \to \mathbb{R}^n$ that satisfies $\dot{\xi}(t) \in f(\xi(t), u) + W$ for almost every (a.e.) $t \in I$. We say that ξ is continuable to $[0, \tau]$ if there exists a solution $\bar{\xi}$ of (22) on $[0, \tau]$ with input $u \in U$ such that $\bar{\xi}|_I = \xi$.

We formulate a sampled variant of (22) as system as follows.

VIII.1 Definition. Let S_1 be a simple system of the form (7), and let $\tau > 0$. We say that S_1 is the sampled system associated with the control system (22) and the sampling time τ , if $X_1 = \mathbb{R}^n$, $U_1 = U$ and the following holds: $x_1 \in F_1(x_0, u)$ iff there exist a solution ξ of (22) on $[0, \tau]$ with input u satisfying $\xi(0) = x_0$ and $\xi(\tau) = x_1$.

In the sequel, φ denotes the general solution of the unperturbed system associated with (22) for constant inputs. That is, if $x_0 \in \mathbb{R}^n$, $u \in U$, and $f(\cdot, u)$ is locally Lipschitz, then $\varphi(\cdot, x_0, u)$ is the unique non-continuable solution of the initial value problem $\dot{x} = f(x, u), x(0) = x_0$ [37].

Similar to other approaches [14], [15] to over-approximate attainable sets that are known for *unperturbed* systems, our computation of attainable sets of the perturbed system is based on an estimate of the distance of neighboring solutions of (22).

VIII.2 Definition. Consider the sets $K \subseteq \mathbb{R}^n$, $U' \subseteq U$ and the sampling time $\tau > 0$. A map $\beta \colon \mathbb{R}^n_+ \times U' \to \mathbb{R}^n_+$ is a growth bound on K, U' associated with τ and (22) if the following conditions hold:

(i) $\beta(r, u) \ge \beta(r', u)$ whenever $r \ge r'$ and $u \in U'$,

(ii) $[0,\tau] \times K \times U' \subseteq \operatorname{dom} \varphi$ and if ξ is a solution of (22) on $[0,\tau]$ with input $u \in U'$ and $\xi(0), p \in K$ then

$$|\xi(\tau) - \varphi(\tau, p, u)| \le \beta(|\xi(0) - p|, u) \tag{23}$$

holds component-wise.

Let us emphasize some distinct features of the estimate (23). First of all, we formulate the inequality (23) component-wise, which allows to bound the difference of neighboring solutions for each state coordinate independently. Second, β is a local estimate, i.e., we require (23) to hold only for initial states in K. Moreover, β is allowed to depend on the input, but these inputs are assumed to be constant, and we do not bound the effect of different inputs on the distance of the solutions. All those properties contribute to more accurate over-approximations of the attainable sets. This, in turn, leads to less conservative abstractions; see our example in Section IX-A. Note that it is also immediate to account for extensions like time varying inputs and using different sampling times.

B. The abstraction

We continue with the construction of an abstraction S_2 of the sampled system S_1 . The state alphabet X_2 of the abstraction is defined as a cover of the state alphabet X_1 where the elements of the cover X_2 are non-empty, closed *hyper-intervals*, i.e., every element $x_2 \in X_2$ takes the form

$$\llbracket a, b \rrbracket = \mathbb{R}^n \cap ([a_1, b_1] \times \cdots \times [a_n, b_n])$$

for some $a, b \in (\mathbb{R} \cup \{\pm \infty\})^n, a \leq b$.

Our notion of hyper-intervals allows for unbounded cells in X_2 . Nevertheless, in the computation of the abstraction S_2 , we work with a subset $\bar{X}_2 \subseteq X_2$ of compact cells. We interpret the cells in \bar{X}_2 as the "real" quantizer symbols, and the remaining ones, as overflow symbols, see [7, Sect. III.A].

VIII.3 Definition. Consider two simple systems S_1 and S_2 of the form (7), a set $\overline{X}_2 \subseteq X_2$ and a function $\beta \colon \mathbb{R}^n_+ \times U_2 \to \mathbb{R}^n_+$. Given $\tau > 0$, suppose that S_1 is the sampled system associated with (22) and sampling time τ . We call S_2 an abstraction of S_1 based on \overline{X}_2 and β , if

(i) X_2 is a cover of X_1 by non-empty, closed hyper-intervals and every element $x_2 \in \bar{X}_2$ is compact; (ii) $U_2 \subseteq U_1$;

(iii) for $x_2 \in \overline{X}_2$, $x'_2 \in X_2$ and $u \in U_2$ we have

$$(\varphi(\tau, c, u) + \llbracket -r', r' \rrbracket) \cap x'_2 \neq \emptyset \Rightarrow x'_2 \in F_2(x_2, u),$$
(24)

 $\begin{array}{l} \text{where } \llbracket a,b \rrbracket = x_2, \ c = \frac{b+a}{2}, \ r = \frac{b-a}{2} \text{ and } r' = \beta(r,u);\\ (iv) \ F_2(x_2,u) = \emptyset \ \text{whenever } x_2 \in X_2 \setminus \overline{X_2}, \ u \in U_2. \end{array}$

Note that the implicit definition of the transition function F_2 according to (iii) in Definition VIII.3 is equivalently expressible as follows. Let $u \in U_2$ and $[\![a,b]\!] \in \overline{X}_2$, then $[\![a',b']\!] \in X_2$ has to be an element of $F_2([\![a,b]\!], u)$ if

$$a' - r' \le \varphi(\tau, c, u) \le b' + r'$$

holds, where c, r and r' are as in Definition VIII.3.

We illustrate the transition function $F_2(x_2, u)$ of an abstraction in Fig. 5.

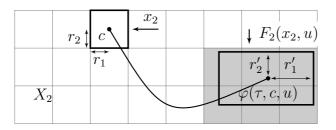


Figure 5. Illustration of the transition function of an abstraction.

VIII.4 Theorem. Consider two simple systems S_1 and S_2 of the form (7) and a set $\bar{X}_2 \subseteq X_2$, and let $\tau > 0$. Suppose that S_1 is the sampled system associated with (22) and sampling time τ . Let β be a growth bound on $\bigcup_{x_2 \in \bar{X}_2} x_2$, U_2 associated with τ and (22). If S_2 is an abstraction of S_1 based on \bar{X}_2 and β , then $S_1 \preccurlyeq_{\epsilon} S_2$.

Proof. To verify the condition (i) in Proposition VII.1 first note that $U_{S_2}(x_2) = \emptyset$ if $x_2 \in X_2 \setminus \bar{X}_2$ by our assumption on S_2 . On the other hand, if $x_1 \in x_2 \in \bar{X}_2$, then $U_2 \subseteq U_{S_1}(x_1)$ by our assumption on β , so the condition (i) in Proposition VII.1 is satisfied. To verify the requirement (ii) in Proposition VII.1, assume that $x_2, x'_2 \in X_2$ and $u \in U_{S_2}(x_2)$. Then $x_2 \in \bar{X}_2$ by our assumption on S_2 , so $x_2 = [[c - r, c + r]]$ for some c, r. Moreover, if additionally $x_1 \in x_2$ and $x'_2 \cap F_1(x_1, u) \neq \emptyset$, then by Definition VIII.1 there exists a solution $\xi : [0, \tau] \to \mathbb{R}^n$ of the system (22) with input u satisfying $\xi(0) = x_1$ and $\xi(\tau) \in x'_2$. It follows that $|\xi(0) - c| \leq r$, and hence, $|\xi(\tau) - \varphi(\tau, c, u)| \leq r'$. Then (24) implies that $x'_2 \in F_2(x_2, u)$. An application of Proposition VII.1 completes the proof.

As seen from the above proof, the set $\varphi(\tau, c, u) + [[-r', r']]$ in (24) over-approximates the attainable set $F_1([[a, b]], u)$. The approximation error, which greatly influences the accuracy of the abstraction, can be reduced by working with smaller cells [[a, b]]. However, the accuracy can also be improved without rediscretizing the state space X_1 , by covering cells $[[a, b]] \in \overline{X}_2$ by compact hyperintervals $\gamma_i + [[-\rho_i, \rho_i]]$ with $\rho_i < r, i \in I$, and then using, in place of the premise in (24), the test $\exists_{i \in I} (\varphi(\tau, \gamma_i, u) + [[-\beta(\rho_i, u), \beta(\rho_i, u)]]) \cap x'_2 \neq \emptyset$.

C. A growth bound

In this subsection we present a specific growth bound for the case that f is continuously differentiable in its first argument and the perturbations are given by $W = \llbracket -w, w \rrbracket$ for some $w \in \mathbb{R}^n_+$. In the following proposition, we use $D_j f_i$ to denote the partial derivative with respect to the *j*th component of the first argument of f_i .

VIII.5 Theorem. Let $\tau > 0$ and let f, U and W be as in (22) with $W = \llbracket -w, w \rrbracket$ for some $w \in \mathbb{R}^n_+$. Let $U' \subseteq U$ and assume in addition that $f(\cdot, u)$ is continuously differentiable for every $u \in U'$. Furthermore, let $K \subseteq K' \subseteq \mathbb{R}^n$ with K' being convex, so that for any $u \in U'$, any $\tau' \in [0, \tau]$ and any solution ξ on $[0, \tau']$ of (22) with input u and $\xi(0) \in K$, we have $\xi(t) \in K'$ for all $t \in [0, \tau']$. Lastly, let the parametrized matrix $L: U' \to \mathbb{R}^{n \times n}$ satisfy

$$L_{i,j}(u) \ge \begin{cases} D_j f_i(x, u), & \text{if } i = j, \\ |D_j f_i(x, u)|, & \text{otherwise} \end{cases}$$

for all $x \in K'$ and all $u \in U'$. Then any ξ as above is continuable to $[0, \tau]$, and the map β given by

$$\beta(r, u) = e^{L(u)\tau} r + \int_0^\tau e^{L(u)s} w \, \mathrm{d}s$$

is a growth bound on K, U' associated with τ and (22).

Theorem VIII.5 can be applied quite easily for obtaining growth bounds. Firstly, the computation of an a priori enclosure K' to solutions of (22) is standard, e.g. [41] and the references therein. Secondly, the parametrized matrix L requires bounding partial derivatives on K'. Such bounds can be computed in an automated way using, e.g., interval arithmetic [42]. Finally, given L, the evaluation of the expression for β is straightforward. We emphasize, however, that Theorem VIII.5 provides only one of several methods to over-approximate attainable sets. Any over-approximation method can be used to compute abstractions based on feedback refinement relations.

Having a growth bound at hand, the application of Theorem VIII.4 becomes a routine task. Examples are presented in the next section.

For the proof of Theorem VIII.5 we need the following auxiliary result, which appears in [43] without proof.

VIII.6 Lemma. Let $\tau > 0$ and $A \subseteq \mathbb{R}^n$. Let $\xi_i: [0, \tau] \to A$, $i \in \{1, 2\}$, be two perturbed solutions of a dynamical system with continuous right hand side $f: \mathbb{R}^n \to \mathbb{R}^n$, i.e., the maps ξ_i are absolutely continuous and satisfy

$$\xi_i(t) - f(\xi_i(t))| \le w_i(t) \quad \text{for a.e.} \quad t \in [0, \tau],$$

where $w_i: [0, \tau] \to \mathbb{R}^n_+$, $i \in \{1, 2\}$, are integrable. Consider a matrix $L \in \mathbb{R}^{n \times n}$ with $L_{i,j} \ge 0$ for $i \neq j$ and suppose that for all $x, y \in A$ we have

$$x_i \ge y_i \Rightarrow f_i(x) - f_i(y) \le \sum_{j=1}^n L_{i,j} |x_j - y_j|.$$

$$(25)$$

Let $\rho: [0,\tau] \to \mathbb{R}^n_+$ be absolutely continuous and satisfying

$$\dot{\phi}(t) = L\rho(t) + w_1(t) + w_2(t)$$

for a.e. $t \in [0, \tau]$. Then $|\xi_1(0) - \xi_2(0)| \le \rho(0)$ implies $|\xi_1(t) - \xi_2(t)| \le \rho(t)$ for every $t \in [0, \tau]$.

Proof. Let $\tilde{\rho}$: $[0, \tau] \to \mathbb{R}^n_+$ be absolutely continuous such that $\tilde{\rho}(0) = \rho(0)$ and $\tilde{\rho}'(t) = L\tilde{\rho}(t) + w_1(t) + w_2(t) + \varepsilon$ for some $\varepsilon \in (\mathbb{R}_+ \setminus \{0\})^n$ and a.e. $t \in [0, \tau]$. We shall prove that

$$|\xi_1(t) - \xi_2(t)| \le \tilde{\rho}(t) \tag{26}$$

holds for all $t \in [0, \tau]$, so that the lemma follows from a limit argument. To this end, denote the function $\xi_1 - \xi_2 - \tilde{\rho}$ on $[0, \tau]$ by z and let $t_0 = \sup\{t \in [0, \tau] \mid \forall_{s \in [0, t]} z(s) \leq 0\}$. Then $t_0 \geq 0$ as $|\xi_1(0) - \xi_2(0)| \leq \rho(0)$, and since we can interchange the roles of ξ_1 and ξ_2 if necessary, we may assume without loss of generality that (26) holds for all $t \in [0, t_0]$. It remains to show that $t_0 = \tau$.

Assume that $t_0 < \tau$. Using (26), a continuity argument shows that we may choose $t_2 \in [t_0, \tau]$ and $i \in [1; n]$ such that $z_i(t_2) > 0$, $z_i(t_0) = 0$ and

$$\varepsilon_i + \sum_{j=1}^n L_{i,j} \tilde{\rho}_j(t) \ge \sum_{j=1}^n L_{i,j} |\xi_{1,j}(t) - \xi_{2,j}(t)|$$
(27)

for all $t \in [t_0, t_2]$. Define $t_1 = \sup\{t \in [t_0, t_2] \mid z_i(t) \leq 0\}$ and note that $z_i(t_1) = 0$ as z_i is continuous. The inequality $z'_i(t) \leq f_i(\xi_1(t)) - f_i(\xi_2(t)) + w_{1,i}(t) + w_{2,i}(t) - \tilde{\rho}'_i(t)$ for a.e. $t \in [t_1, t_2]$ and the definition of $\tilde{\rho}$ then imply that

$$z_i(t_2) \le \int_{t_1}^{t_2} \left(f_i(\xi_1(t)) - f_i(\xi_2(t)) - \sum_{j=1}^n L_{i,j} \tilde{\rho}_j(t) - \varepsilon_i \right) \mathrm{d}t.$$

Thus, $z_i(t_2) \leq 0$ by (27) and (25). This contradicts our choice of t_2 , and so $t_0 = \tau$.

Proof of Theorem VIII.5. Fix $p \in K$, $u \in U'$ and note that $\beta(r, u) \geq \beta(r', u)$ if $r \geq r'$ as all entries of $e^{L(u)\tau}$ are non-negative [44, Th. 7.7]. Next, we show that condition (ii) in Definition VIII.2 holds. In order to apply Lemma VIII.6 we shall establish (25) for K', $f(\cdot, u)$ and L(u) in place of A, fand L. Indeed, by the mean value theorem, there exists $z \in \{x + t(y - x) | t \in [0, 1]\}$ such that $f_i(x, u) - f_i(y, u) = \sum_{j=1}^n D_j f_i(z, u)(x_j - y_j)$. Hence, by the definition of L, we obtain (25). Now, let ξ be a solution on $[0, \tau]$ of (22) with input u such that $\xi(0) \in K$. By Filippov's Lemma [45], there exists an integrable map $s \colon [0, \tau] \to W$ such that $\dot{\xi}(t) = f(\xi(t), u) + s(t)$ for a.e. $t \in [0, \tau]$. So, apply Lemma VIII.6 to $f(\cdot, u), K', \varphi(\cdot, p, u), \xi, 0, w$ and L(u) in place of $f, A, \xi_1, \xi_2, w_1, w_2$ and L, respectively, to obtain $|\xi(\tau) - \varphi(\tau, p, u)| \leq \beta(|\xi(0) - p|, u)$.

Finally, suppose there exists $\xi \colon [0, \tau'] \to K'$ as in the statement of the theorem that is not continuable to $[0, \tau]$. Then, there exist $t_0 \in [0, \tau]$ and a solution $\bar{\xi} \colon [0, t_0[\to \mathbb{R}^n \text{ of } (22) \text{ with input } u \text{ such that}$ $\bar{\xi}|_{[0,\tau']} = \xi$ and $\bar{\xi}(t)$ becomes unbounded as $t \in [0, t_0[$ approaches t_0 [46]. On the other hand, applying Lemma VIII.6 to $f(\cdot, u), K', \bar{\xi}|_{[0,t]}, \xi(0), w, |f(\xi(0), u)|, L(u) \text{ and } t \text{ in place of } f, A, \xi_1, \xi_2, w_1, w_2, L$ and τ we conclude that $|\bar{\xi}(t) - \xi(0)|$ is uniformly bounded for $t \in [0, t_0[$, which is a contradiction. \Box

D. The Case of Periodic Dynamics

Occasionally we will have to consider continuous-time control systems of the form (22) whose dynamics are periodic, i.e., $f(\xi + p, \cdot) = f(\xi, \cdot)$ for some *period* $p \in \mathbb{R}^n \setminus \{0\}$ and all $\xi \in \mathbb{R}^n$. Our result below shows how to exploit periodicity to obtain abstractions that are finite and yet are capable of reproducing solutions that are unbounded in the direction of the period. This is useful, e.g. when one of the components of the state represents an angle and the number of full loops is potentially unbounded; see Section IX-A for an example.

VIII.7 Theorem. Let $p_1, \ldots, p_\ell \in \mathbb{R}^n$, $\ell \in \mathbb{N}$, be such that f in (22) satisfies $f(x + p_i, u) = f(x, u)$ for all $i \in [1; \ell]$, $x \in \mathbb{R}^n$ and $u \in U$. Consider systems S_1 and S_2 of the form (7), where $U_2 \subseteq U_1$ and S_1 is the sampled system associated with (22) and sampling time $\tau > 0$. Define the map $\pi : X_1 \rightrightarrows X_1$ by $\pi(x) = \left\{ x + \sum_{i=1}^{\ell} k_i p_i \mid k \in \mathbb{Z}^{\ell} \right\}$, and let R be a set of non-empty subsets of X_1 such that $X_2 = \{\pi(\Omega) \mid \Omega \in R\}$ and X_2 is a cover of X_1 .

- Then $S_1 \preccurlyeq_{\in} S_2$ iff the following conditions hold:
 - (a) $x \in \Omega \in R$ implies $U_{S_2}(\pi(\Omega)) \subseteq U_{S_1}(x)$.
 - (b) If $\Omega, \Omega' \in \mathbb{R}$, $u \in U_{S_2}(\pi(\Omega))$ and $\pi(\Omega') \cap F_1(\Omega, u) \neq \emptyset$, then $\pi(\Omega') \in F_2(\pi(\Omega), u)$.

Obviously, the transition function F_2 of the system S_2 can be computed by over-approximating attainable sets $F_1(\Omega, u)$ as detailed in Sections VII-A, VIII-B and VIII-C, and by verifying the

condition $(\Omega' + \sum_{i=1}^{\ell} k_i p_i) \cap F_1(\Omega, u) \neq \emptyset$, for $\Omega, \Omega' \in \mathbb{R}$ with Ω being compact, and finitely many $k \in \mathbb{Z}^{\ell}$.

Proof. First observe that $F_1(x, u) + \langle k | p \rangle = F_1(x + \langle k | p \rangle, u)$ for all $k \in \mathbb{Z}^{\ell}$, $x \in X_1$ and $u \in U_1$, where $\langle k | p \rangle = \sum_{i=1}^{\ell} k_i p_i$. Then $U_{S_1}(x + \langle k | p \rangle) = U_{S_1}(x)$ for all $x \in X_1$ and all $k \in \mathbb{Z}^{\ell}$, which shows that the condition (i) in Proposition VII.1 is equivalent to (a). We shall show that the condition (ii) is equivalent to (b), which proves the theorem.

If $\Omega, \Omega' \in R$, $u \in U_{S_2}(\pi(\Omega))$ and $\pi(\Omega') \cap F_1(\Omega, u) \neq \emptyset$, then $\pi(\Omega), \pi(\Omega') \in X_2$ and $\Omega \subseteq \pi(\Omega)$, and so (ii) shows that $\pi(\Omega') \in F_2(\pi(\Omega), u)$. Conversely, if $\Omega, \Omega' \in X_2$, $u \in U_{S_2}(\Omega)$ and $\Omega' \cap F_1(\Omega, u) \neq \emptyset$, then there exist $\Omega_0, \Omega'_0 \in R$ satisfying $\Omega = \pi(\Omega_0)$ and $\Omega' = \pi(\Omega'_0)$. Hence, $\pi(\Omega'_0) \cap (\langle k | p \rangle + F_1(\Omega_0, u)) \neq \emptyset$ for some $k \in \mathbb{Z}^\ell$, and since $\pi(\Omega'_0) = \pi(\Omega'_0) + \langle k | p \rangle$ we have $\pi(\Omega'_0) \cap F_1(\Omega_0, u) \neq \emptyset$. Then (b) shows that $\Omega' \in F_2(\Omega, u)$, which completes the proof.

IX. EXAMPLES

In this section, we demonstrate the practicality of our approach on control problems for nonlinear plants.

A. A path planning problem for an autonomous vehicle

We consider an autonomous vehicle whose dynamics we assume to be given by the bicycle model in [47, Ch. 2.4]. More concretely, the dynamics of the system are of the form (22), where $f : \mathbb{R}^3 \times U \to \mathbb{R}^3$ is given by

$$f(x, (u_1, u_2)) = \begin{pmatrix} u_1 \cos(\alpha + x_3) \cos(\alpha)^{-1} \\ u_1 \sin(\alpha + x_3) \cos(\alpha)^{-1} \\ u_1 \tan(u_2) \end{pmatrix}$$

with $U = [-1, 1] \times [-1, 1]$ and $\alpha = \arctan(\tan(u_2)/2)$. Here, (x_1, x_2) is the position and x_3 is the orientation of the vehicle in the 2-dimensional plane. The control inputs u_1 and u_2 are the rear wheel velocity and the steering angle. Perturbations are not acting on the system dynamics, i.e., $W = \{(0, 0, 0)\}$.

The concrete control problem is formulated with respect to the sampled system S_1 associated with (22) and sampling time $\tau = 0.3$. The control objective is to enforce a certain patrolling behavior on the vehicle which is situated in a maze; see Fig. 6. Specifically, the vehicle, whose initial state is $A_{1,0} = \{(0.4, 0.4, 0)\}$, should patrol infinitely often between the target regions $A_{1,r_1} = [0, 0.5] \times [0, 0.5] \times \mathbb{R}$ and $A_{1,r_2} = (9, 0, 0) + A_{1,r_1}$, while avoiding the obstacles $A_{1,a}$. The third component of $A_{1,a}$ equals \mathbb{R} . We formalize our concrete control problem through the pair (S_1, Σ_1) with the specification Σ_1 defined as

$$\{(u, x) \in (U_1 \times X_1)^{\mathbb{Z}_+} \mid x(0) \in A_{1,0} \Rightarrow \\ \forall_{t \in \mathbb{Z}_+} (x(t) \notin A_{1,a} \land \forall_{i \in \{1,2\}} \exists_{t' \in [t;\infty[} x(t') \in A_{1,r_i}) \},$$
(28)

where $U_1 = U$ and $X_1 = \mathbb{R}^3$. To solve (S_1, Σ_1) we solve an abstract control problem (S_2, Σ_2) as detailed below.

As f possesses the period $p = (0, 0, 2\pi)$ we construct a canonical abstraction S_2 of the form (7) using Theorem VIII.7, where R consist of the shifted copies of the hyper-interval

$$\left[-\frac{1}{10},\frac{1}{10}\right]\times\left[-\frac{1}{10},\frac{1}{10}\right]\times\left[-\frac{\pi}{35},\frac{\pi}{35}\right],$$

whose centers form the set $\frac{2}{10}[0;50] \times \frac{2}{10}[0;50] \times \frac{2\pi}{35}[-17;17]$, and of the hyper-intervals $\{x \in \mathbb{R}^3 \mid x_j \geq 10.1\}$, $\{x \in \mathbb{R}^3 \mid x_j \leq -0.1\}$, $j \in \{1,2\}$. Set $U_2 = \{0,\pm0.3,\pm0.6,\pm0.9\} \times \{0,\pm0.3,\pm0.6,\pm0.9\}$, and let X_2 be as in Theorem VIII.7. The transition function F_2 is computed according to the remark following Theorem VIII.7, in which $F_2(x_2, u) = \emptyset$ if $(x_2, u) \in X_2 \times U_2$, $x_2 \cap A_{1,a} \neq \emptyset$. The required growth bound β on \mathbb{R}^3 , U_2 associated with τ and (22) is obtained using Theorem VIII.5. In

particular, $\beta(r, u) = e^{L(u)\tau}r$, where L is given by $L_{1,3}(u_1, u_2) = L_{2,3}(u_1, u_2) = |u_1\sqrt{\tan^2(u_2)/4 + 1}|$, and $L_{i,j}(u_1, u_2) = 0$ for $(i, j) \notin \{(1, 3), (2, 3)\}$.

The computation of F_2 takes 2.25 seconds (Intel Core i7 2.9 GHz) resulting in an abstraction having 37266181 transitions.

To construct the abstract specification Σ_2 we let $A_{2,0} = \{x_2 \in X_2 \mid x_2 \cap A_{1,0} \neq \emptyset\}$, $A_{2,r_i} = \{x_2 \in X_2 \mid x_2 \subseteq A_{1,r_i}\}$, $i \in \{1,2\}$ and $A_{2,a} = \{x_2 \in X_2 \mid x_2 \cap A_{1,a} \neq \emptyset\}$; see Fig. 6. We define Σ_2 by (28), where we substitute U_1 , X_1 , $A_{1,0}$, A_{1,r_1} , A_{1,r_2} , $A_{1,a}$ with U_2 , X_2 , $A_{2,0}$, A_{2,r_1} , A_{2,r_2} , $A_{2,a}$, respectively. It is straightforward to verify that Σ_2 is an abstract specification associated with S_1 , S_2 , \in and Σ_1 .

The abstract problem (S_2, Σ_2) can be solved using the algorithm in [6, Fig. 1], which simplifies here to two rather than three nested fixed-point iterations since for our problem the general reactivity (1) specification in [6] reduces to $\nu Z. \cap_{i \in \{1,2\}} \mu Y. (\oslash Y \cup (A_{2,r_i} \cap \bigotimes Z))$, where $\bigotimes A = \{x \in X_2 \mid \exists_{u \in U_2} \emptyset \neq F_2(x, u) \subseteq A\}$. We actually use a Dijkstra-like algorithm [48] for the inner fixed-point to successfully solve (S_2, Σ_2) within 0.54 seconds. The solution is refined to a solution of (S_1, Σ_1) by adding a static quantizer; see Theorem VI.3. A similar problem with considerably less complex specification is solved in [14], where the run times in seconds are 13509 (abstraction) and 535 (synthesis) on Intel Core 2 Duo 2.4 GHz.

We would like to discuss two of the advantages of the growth bounds we have introduced in Section VIII. As we already mentioned, β bounds each component of neighboring solutions separately, which can be directly seen by the formula $\beta(r, u) = r + r_3 \cdot L_{1,3}(u_1, u_2) \cdot (\tau, \tau, 0)^{\top}$. This distinguishes β from an estimate based on a norm. Moreover, β depends on the input, which is crucial for the present example. Indeed, the function $e^{(\sup L)\tau}r$, where $\sup L \in \mathbb{R}^{3\times3}$ is given by $(\sup L)_{i,j} = \sup_{u \in U_2} L_{i,j}(u)$, is also a growth bound on \mathbb{R}^3 , U_2 associated with τ and (22), which leads to an abstraction with 43288873 transitions. However, due to the poor approximation quality of this growth bound we obtain an unsolvable abstract control problem.

B. An aircraft landing maneuver

We consider an aircraft DC9-30 whose dynamics we model according to [49]. We use x_1, x_2, x_3 to denote the state variables, which respectively correspond to the velocity, the flight path angle and the altitude of the aircraft. The input alphabet is given by $U = [0, 160 \cdot 10^3] \times [0^\circ, 10^\circ]$ and represents the thrust of the engines (in Newton) and the angle of attack. The dynamics are given by $f : \mathbb{R}^3 \times U \to \mathbb{R}^3$,

$$f(x,u) = \begin{pmatrix} \frac{1}{m}(u_1 \cos u_2 - D(u_2, x_1) - mg \sin x_2) \\ \frac{1}{mx_1}(u_1 \sin u_2 + L(u_2, x_1) - mg \cos x_2) \\ x_1 \sin x_2 \end{pmatrix}$$

where $D(u_2, x_1) = (2.7 + 3.08 \cdot (1.25 + 4.2 \cdot u_2)^2) \cdot x_1^2$, $L(u_2, x_1) = (68.6 \cdot (1.25 + 4.2 \cdot u_2)) \cdot x_1^2$ and $mg = 60 \cdot 10^3 \cdot 9.81$ account for the drag, lift and gravity, respectively [49].

We consider the input disturbance $P_1: U \rightrightarrows U$ given by $P_1(u) = (u + [-5 \cdot 10^3, 5 \cdot 10^3] \times [-0.25^\circ, 0.25^\circ]) \cap U$ and measurement errors of the form $P_2: \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ given by $P_2(x) = x + \frac{1}{20} [-0.25, 0.25] \times \frac{1}{20} [-0.05^\circ, 0.05^\circ] \times \frac{1}{20} [-1, 1]$. We do not consider any further disturbances, i.e., we let $W = \{(0, 0, 0)\}, P_3 = \mathrm{id}, \mathrm{and} P_4 = \mathrm{id}.$

The concrete control problem is formulated with respect to the sampled system $S_1 = (X_1, X_1, U_1, U_1, X_1, F_1, T_1, T_1, T_2)$ associated with (22) and the sampling time $\tau = 0.25$. We aim at steering the aircraft from an altitude of 55 meters close to the ground with an appropriate total and horizontal touchdown velocity. More formally, the specification Σ_1 is given by

$$\Sigma_{1} = \left\{ (u, x) \in (U_{1} \times X_{1})^{\mathbb{Z}_{+}} \mid x(0) \in A_{0} \Rightarrow (\exists_{s \in \mathbb{Z}_{+}} x(s) \in A_{r} \land \forall_{t \in [0;s[} x(t) \notin A_{a}) \right\},$$
where $I = [-3^{\circ}, 0^{\circ}], A_{0} = [80, 82] \times [-2^{\circ}, -1^{\circ}] \times \{55\},$

$$A_{a} = \mathbb{R}^{3} \setminus ([58, 83] \times I \times [0, 56]),$$

$$A_{r} = ([63, 75] \times I \times [0, 2.5]) \cap \{x \in \mathbb{R}^{3} | x_{1} \sin x_{2} \ge -0.91\}.$$
(29)

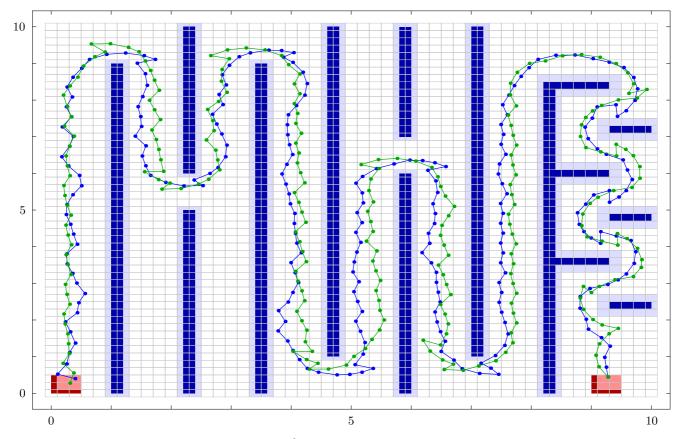


Figure 6. Projection of the states of S_1 and S_2 to $\mathbb{R}^2 \times \{0\}$. The sets $A_{1,a}$ and A_{1,r_1} , A_{1,r_2} are indicated in dark blue and in red, respectively. The states in $A_{2,a}$ and A_{2,r_1}, A_{2,r_2} are indicated in blue and in light red, respectively. A closed-loop trajectory of the concrete control problem is shown evolving from A_{1,r_1} to A_{1,r_2} in the blue part and vice versa in the green part.

As detailed in Section VI-B, the perturbed control problem is solved through an auxiliary unperturbed control problem. To begin with, define the simple system \hat{S}_1 by (20) with $\hat{U}_1 = U$. Next, let X be a cover of \mathbb{R}^3 formed by subdividing $\mathbb{R}^3 \setminus A_a$ into $210 \cdot 210 \cdot 210$ hyper-intervals, and suitable unbounded hyper-intervals. Define $X_2 = \{P_2^{-1}(\Omega) \mid \Omega \in X\}$ and let \bar{X}_2 be the subset of compact elements of X_2 that do not intersect A_a . Define the abstraction for \hat{S}_1 as the simple system S_2 given by (7), where $U_2 = \{0, 32000\} \times U', U'$ contains precisely 10 inputs equally spaced in $[0^\circ, 8^\circ]$. We apply Theorem VIII.5 with $w = M(5000, 0.25^\circ)^\top \leq (0.108, 0.002, 0)^\top$ and a suitable a priori enclosure K' to obtain a growth bound, where $M \in \mathbb{R}^{2\times 3}_+$ satisfies $M_{i,j} \geq |D_{j,2}f_i(x, u)|$ for all $x \in K'$ and $u \in P_1(U_2)$. Here, $D_{j,2}f_i$ stands for the partial derivative with respect to the *j*th component of the second argument of f_i . Note that w accounts for the perturbation P_1 . Then, we use Theorem VIII.4 to compute F_2 such that $\hat{S}_1 \preccurlyeq S_2$. The computation takes 674 seconds resulting in an abstraction with about 9.38 $\cdot 10^9$ transitions (Intel Xeon E5 3.1 GHz).

To construct the abstract specification Σ_2 for S_2 we let $A_{2,0} = \{x_2 \in X_2 \mid x_2 \cap A_{1,0} \neq \emptyset\}$, $A_{2,a} = \{x_2 \in X_2 \mid x_2 \cap A_{1,a} \neq \emptyset\}$, $A_{2,r} = \{x_2 \in X_2 \mid x_2 \subseteq A_{1,r}\}$ and define the specification Σ_2 by (29) with $U_2, X_2, A_{2,0}, A_{2,r}, A_{2,a}$ in place of U_1, X_1, A_0, A_r, A_a . It is easy to verify that Σ_2 is an abstract specification associated with \hat{S}_1, S_2, \in and Σ_1 . Note that Σ_2 (as well as Σ_1) is a particular instance of a *reach-avoid* specification. Using a standard technique [48], the abstract control problem (S_2, Σ_2) is successfully solved within 26 seconds. By Corollary VI.5 the behavior of the perturbed closed loop is a subset of Σ_1 . See Fig. 7.

We proceed to make some comments on solving perturbed control problems. At first, Theorem VIII.5 allows to deal with time-varying input perturbations, when the theorem is applied as in this example. Second, accounting for measurement errors only requires inflating the cells that would have

been used if measurement errors were not present. To conclude, perturbed control problems can be solved in our framework by using canonical abstractions.

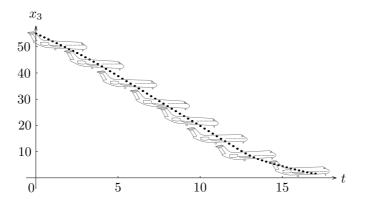


Figure 7. Time evolution of the altitude of the aircraft in the closed loop. The aircraft pitch $u_2 + x_2$ is indicated for 8 instants of time.

X. CONCLUSIONS

We have presented a novel approach to abstraction-based controller synthesis which builds on the concept of feedback refinement relation introduced in the present paper. Our framework incorporates several distinct features. Foremost, the designed controllers require quantized (or symbolic) state information only and are connected to the plant via a static quantizer, which is particularly important for any practical implementation of the controller. Our work permits the synthesis of robust correct-by-design controllers in the presence of various uncertainties and disturbances, and more generally, applies to a broader class of synthesis problems than previous research addressing the state information and refinement complexity issues as explained and illustrated in Sections I and IV. Moreover, we do not assume that the controller is able to set the initial state of the plant, which is also important in the context of practical control systems.

We have additionally identified a class of canonical abstractions, and have presented a method to compute such abstractions for perturbed nonlinear control systems. We utilized numerical examples to demonstrate the applicability and efficiency of our synthesis framework. We emphasize, however, that the computational effort is still expected to grow rapidly with the dimension of the state space of the plant, a problem that is shared by all grid based methods for the computation of abstractions.

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